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Properties of Stochastic Arrangement Increasing and Their Applications in Allocation Problems

Wei Wei

Department of Mathematical Sciences, University of Wisconsin-Milwaukee, PO Box 413, Milwaukee, WI 53201, USA; weiw@uwm.edu

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Abstract: There are extensive studies on the allocation problems in the field of insurance and finance. We observe that these studies, although involving different methodologies, share some inherent commonalities. In this paper, we develop a new framework for these studies with the tool of arrangement increasing functions. This framework unifies many existing studies and provides shortcuts to developing new results.

Keywords: arrangement increasing function; majorization; stochastic arrangement increasing; stochastic orders; optimal allocations

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1. Introduction

Allocation problems widely exist in insurance and finance. The study of allocation problems usually involves the comparison of different random variables. In the early literature, traditional stochastic orders, such as likelihood ratio order and hazard rate ratio order (see Section 2 for definitions and more discussions), were frequently employed to compare risks or stochastic returns of risky assets. Later on, these traditional stochastic orders were criticized for not capturing dependence structures among the stochastic components under consideration. To overcome this restriction, Shanthikumar and Yao (1991) introduced the notion of joint likelihood ratio orders, which incorporates dependence structure into stochastic comparison. Following this pioneering work, Cai and Wei (2014) and Cai and Wei (2015) generalized the notion of joint likelihood ratio order and developed the concepts of stochastic arrangement increasing (SAI) and weakly stochastically arrangement increasing through right and left tails (RWSAI; LWSAI). They further explored the applications of these concepts in the study of the allocations of deductibles and policy limits as well as portfolio selection problems. Similar studies can be also found in Cheung (2007), Hennessy and Lapan (2002), Hua and Cheung (2008), Kijima and Ohnishi (1996), Li and You (2012), Zhuang et al. (2009), and Pan and Li (2017), among many others. Recently, Wei (2017) extended these concepts to higher degree cases and studied their applications in portfolio selections. It is worth noting that, in addition to insurance and finance, the notions of SAI and RWSAI have been also used in the field of operations research, see for example, Belzunce et al. (2013).

The aforementioned papers, although studying different types of problems, exhibit inherent commonalities in nature. Generally, these studies concern how to allocate insurance deductibles/policy limits/investment weights to different risks/assets. In those cases, the objective functions can be viewed as a function with two (or more) vector inputs, either deterministic or random, and the study of allocation problems boils down to investigating the impact of different relative arrangements of the input vectors on the value of the objective function. This natural brings out the concept of an arrangement increasing function (see Marshall et al. (2010)).

In this paper, we shall employ the concept of arrangement increasing function to establish useful properties of SAI random vectors, and these properties will be used to unify and extend existing studies on allocation problems. The rest of the paper is organized as follows: Section 2 introduces the concept of the arrangement increasing function, the majorization order, stochastic orders, as well as some relevant results. Section 3 establishes the main results of SAI random vectors. Specifically, the behaviors of two independent SAI random vectors are characterized by arrangement increasing functions and Schur-convex and -concave functions. Section 4 demonstrates the applications of properties established in Section 3 in the study of allocation problems. Section 5 concludes the paper and outlines some future research topics.

2. Preliminaries

Use $\mathbf{x} = (x_1, \ldots, x_n)$ to denote a real vector and $\mathbf{X} = (X_1, \ldots, X_n)$ to denote a random vector. Use Π to denote a permutation matrix. For example, $\mathbf{x}\Pi$ returns a permutation of the vector $\mathbf{x}$. The class of permutations that exchange the elements in two positions is of particular interest, denoted as Π_{ij}, for $1 \le i, j \le n$. For example, $(x_1, \ldots, x_n)\Pi_{ij} = (x_1, \ldots, x_{i-1}, x_j, x_{i+1}, \ldots, x_{j-1}, x_i, x_{j+1}, \ldots, x_n)$. Furthermore, define $\mathcal{D}_n = \{\mathbf{x} \in \mathbb{R}^n : x_1 \ge \ldots \ge x_n\}$ and $\mathcal{I}_n = \{\mathbf{x} \in \mathbb{R}^n : x_1 \le \ldots \le x_n\}$.

2.1. Majorization Order

We first introduce the concept of a majorization order and some related notions. The definitions and results in this subsection are all taken from Marshall et al. (2010), to which the reader is referred for more detail.

Definition 1 (Definition A.1 of Chapter 1 of Marshall et al. (2010))**.** *Let $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ be two real vectors. $\mathbf{x}$ is said to be majorized by $\mathbf{y}$, denoted as $\mathbf{x} \le_m \mathbf{y}$, if*

$$\begin{cases} \sum_{i=1}^{k} x_{[i]} \le \sum_{i=1}^{k} y_{[i]}, & k = 1, \ldots, n-1 \\ \sum_{i=1}^{n} x_{[i]} = \sum_{i=1}^{n} y_{[i]} \end{cases}$$

where $x_{[i]}$ denotes the i^{th} largest element of $\{x_1, \ldots, x_n\}$.

Clearly, the majorization order does not concern how elements of a vector are ordered. Specifically, if $\mathbf{x} \le_m \mathbf{y}$, then $\mathbf{x}\Pi_1 \le_m \mathbf{y}\Pi_2$ for any permutation matrices Π_1 and Π_2. According to Marshall et al. (2010), if $\mathbf{x} \le_m \mathbf{y}$, there exists $\mathbf{z}_1, \ldots, \mathbf{z}_n$ such that $\mathbf{x} \le_m \mathbf{z}_1 \le_m \cdots \le_m \mathbf{z}_n \le_m \mathbf{y}$ and $\mathbf{z}_{i+1}$ differs from $\mathbf{z}_i$ by only two elements for all $i = 0, 1, \ldots, n$ (with the convention of $\mathbf{z}_0 = \mathbf{x}$ and $\mathbf{z}_{n+1} = \mathbf{y}$). This means that $\mathbf{y}$ can be reached by $\mathbf{x}$ through a sequence of operations that preserve the majorization order and only modify two elements each time. In this sense, most proofs involving the majorization order in this paper can be reduced to a bivariate case. In the bivariate case, $(x_1, x_2) \le_m (y_1, y_2)$ if and only if $x_1 + x_2 = y_1 + y_2$ and $\max\{y_1, y_2\} \ge \max\{x_1, x_2\}$.

Definition 2 (Definition A.1 of Chapter 3 of Marshall et al. (2010))**.** *Let ϕ be a real-valued function defined on $\mathcal{F} \subset \mathbb{R}^n$. ϕ is said to be Schur-convex (or Schur-concave) if $\phi(\mathbf{x}) \le (\text{or} \ge)\phi(\mathbf{y})$ for any $\mathbf{x}, \mathbf{y} \in \mathcal{F}$ such that $\mathbf{x} \le_m \mathbf{y}$.*

Clearly, ϕ is Schur-concave if and only if $-\phi$ is Schur-convex.

The following lemma provides a useful tool to justify the Schur-convexity and Schur-concavity of a function.

Lemma 1 (Theorem A.3 of Chapter 3 of Marshall et al. (2010))**.** *Let ϕ be a real-valued function defined on $\mathcal{D}_n$ and continuously differentiable on the interior of $\mathcal{D}_n$. ϕ is then Schur-convex (or Schur-concave) on $\mathcal{D}_n$ if and only if $\phi_{(k)}(\mathbf{z})$ is decreasing (or increasing) in k, for any $\mathbf{z} \in \mathcal{D}_n$.*

To avoid technical discussions, when justifying Schur-convexity or -concavity using Lemma 1, we allow $\phi_{(k)}(\mathbf{z})$ to represent the one-sided derivative when $\phi(\mathbf{z})$ is continuous and both left- and right-differentiable in each argument.

Definition 3 (Definition C.2 of Chapter 6 of Marshall et al. (2010))**.** *A bivariate function ϕ is said to be L-superadditive (or L-subadditive) if it satisfies*

$$\phi(\alpha_1+\delta_1,\alpha_2+\delta_2)+\phi(\alpha_1-\delta_1,\alpha_2-\delta_2)$$
$$\geq (or \leq)\, \phi(\alpha_1+\delta_1,\alpha_2-\delta_2)+\phi(\alpha_1-\delta_1,\alpha_2+\delta_2)$$

for any $\delta_1, \delta_2 \geq 0$.

L-superadditive is also referred to as *supermodular* in the literature. If ϕ is twice differentiable, then $\phi(x, y)$ is L-superadditive (or L-subadditive) if and only if $\frac{\partial^2}{\partial x \partial y}\phi \geq (or \leq) 0$. Readers are referred to Chapter 6 of Marshall et al. (2010) for more discussion about this concept.

2.2. Arrangement Increasing

Definition 4. *Let $g(\mathbf{x};\mathbf{y}) : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ be a multivariate function. g is said to be arrangement increasing (or decreasing) if*

(i) *g is permutation invariant, i.e., $g(\mathbf{x}\Pi;\mathbf{y}\Pi) = g(\mathbf{x};\mathbf{y})$ for any permutation matrix Π and real vectors $\mathbf{x}$ and $\mathbf{y}$* [1]*; and*
(ii) *$g(\mathbf{x};\mathbf{y}) \geq (or \leq) g(\mathbf{x};\mathbf{y}\Pi_{12})$ for any $x_1 \leq x_2$ and $y_1 \leq y_2$.*

Remark 1. *Clearly, $g(\mathbf{x};\mathbf{y})$ is arrangement decreasing if and only if $-g(\mathbf{x};\mathbf{y})$ is arrangement increasing, if and only if $g(\mathbf{x};-\mathbf{y})$ is arrangement increasing. Furthermore, if g is arrangement increasing (or decreasing), then $u \circ g$ is arrangement increasing (or decreasing) for any univariate increasing function u.*

Definition 4 is taken from Proposition F.7 in Chapter 6 of Marshall et al. (2010). The original definition of *arrangement increasing* involves some technical concepts and thus is not used here. Proposition F.7 is an equivalent characterization of the original definition. Note that an arrangement increasing function is permutation-invariant, meaning that its value depends only on the relative arrangement (but not the absolute arrangement) of the two input vectors. For example, $g(x_1, x_2; y_1, y_2) = x_1y_1 + x_2y_2$ is an arrangement increasing function, and the inputs of $(x_1, x_2; y_1, y_2)$ and $(x_2, x_1; y_2, y_1)$ return the same function value.

In the rest of the paper, it is usually necessary to justify the arrangement increasing properties of given functions. For this purpose, we cite some useful results from Chapter 6 of Marshall et al. (2010).

Lemma 2. *(i)* *If g has the form $g(\mathbf{u};\mathbf{v}) = \phi(\mathbf{u}+\mathbf{v})$ for $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$, then g is arrangement increasing if and only if ϕ is Schur-convex on $\mathbb{R}^n$.*
(ii) *If g has the form $g(\mathbf{u};\mathbf{v}) = \sum_{i=1}^n \phi(u_i, v_i)$ for $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$, then g is arrangement increasing if and only if ϕ is L-supperadditive.*

Below we establish the arrangement increasing property of several functions, which will be used frequently in this paper.

Lemma 3. *(i)* *The function $f_D(\mathbf{x};\mathbf{y}) = \sum_{i=1}^n x_i \wedge y_i$ is arrangement increasing.*

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[1] The permutation invariance implies the domain of the function is also permutation invariant, that is for any $(\mathbf{x};\mathbf{y})$ in the domain, so is $(\mathbf{x}\Pi;\mathbf{y}\Pi)$ for any permutation matrix Π. Throughout out this paper, we assume this is true whenever we consider an arrangement increasing function.

(ii) *The function* $f_L(\mathbf{x};\mathbf{y}) = \sum_{i=1}^n (x_i - y_i)_+$ *is arrangement decreasing.*

Proof. (i) Denote $\eta(x,y) = x \wedge y$. It is easy to verify that η is L-superadditive. Therefore, $f_D(\mathbf{x};\mathbf{y}) = \sum_{i=1}^n \eta(x_i, y_i)$ is arrangement increasing from Lemma 2 (ii).

(ii) Consider function $\psi(\mathbf{x}) = \psi(\mathbf{x}) = \sum_{i=1}^n (x_i)_+$. Noting that x_+ is convex, it is easy to verify that $\psi(\mathbf{x})$ is Schur-convex. Following Lemma 2 (i), $g(\mathbf{x};\mathbf{y}) \triangleq \sum_{i=1}^n (x_i + y_i)_+ = \psi(\mathbf{x}+\mathbf{y})$ is arrangement increasing. Therefore, $f_L(\mathbf{x},\mathbf{y}) = g(\mathbf{x};-\mathbf{y})$ is arrangement decreasing from Remark 1. □

2.3. Stochastic Orders

Stochastic orders are used to compare random variables. Below, we state the definitions of three commonly used stochastic orders. Their definitions can be found in the standard literature (see Shaked and Shanthikumar (2007)).

Definition 5. *(i) Assume random variables X and Y have survival functions $\bar{F}_X(x)$ and $\bar{F}_Y(x)$. X is said to be smaller than Y in hazard rate order, denoted as $X \leq_{hr} Y$, if $\frac{\bar{F}_Y(x)}{\bar{F}_X(x)}$ is increasing in x such that $\bar{F}_X(x) > 0$.*

(ii) Assume X and Y have probability density functions $f_X(x)$ and $f_Y(x)$. X is said to be smaller than Y in the likelihood ratio order, denoted as $X \leq_{lr} Y$, if $\frac{f_Y(x)}{f_X(x)}$ is increasing in x such that $f_X(x) > 0$.

Definition 6. *A random variable X is said to be smaller than Y in the sense of the usual stochastic order (respectively, increasing convex order and increasing concave order), denoted as $X \leq_{st} Y$ (respectively, $X \leq_{icx} Y$ and $X \leq_{icv} Y$), if $\mathbb{E}[u(X)] \leq \mathbb{E}[u(Y)]$ for any increasing (respectively, increasing convex and increasing concave) function $u(x)$ such that the expectations exist.*

It has been well established (see, for example, Shaked and Shanthikumar (2007)) that $X \leq_{lr} Y \Rightarrow X \leq_{hr} Y \Rightarrow X \leq_{st} Y$, and $X \leq_{st} Y$ implies $X \leq_{icx} Y$ and $X \leq_{icv} Y$. It is worth mentioning that, in the literature of finance and economics, the usual stochastic order ($\leq_{st}$) and increasing concave order ($\leq_{icv}$) are respectively referred to as the first order and second order stochastic dominance. They are both implied by the likelihood ratio order ($\leq_{lr}$).

3. Properties of SAI Characterized by Arrangement Increasing Functions

Definition 7. *A random vector $\mathbf{X} = (X_1, \ldots, X_n)$ is said to be stochastic arrangement increasing (SAI), if $\mathbb{E}[h(\mathbf{X})] \geq \mathbb{E}[h(\mathbf{X}\Pi_{ij})]$ for any $i < j$ and $h : \mathbb{R}^n \to \mathbb{R}$ such that*

$$h(\mathbf{x}) \geq h(\mathbf{x}\Pi_{ij}) \qquad \text{for any } x_i \leq x_j. \tag{1}$$

It is worth pointing out that, in the literature, Condition (1) is sometimes referred to as arrangement increasing, which has a different meaning from Definition 4. Evidently, Condition (1) concerns a function with the input of a single vector, while Definition 4 concerns a function with the input of two vectors. As a matter of fact, the notion defined by Condition (1) can be viewed as a degenerated case of the notion of arrangement increasing defined in Definition 4. It is not difficult to verify that a function $h : \mathbb{R}^n \to \mathbb{R}$ satisfies Condition (1) if and only if $g(\mathbf{x};\mathbf{y}) \triangleq h(\mathbf{x}\Pi(\mathbf{y}_\uparrow;\mathbf{y}))$ is arrangement increasing in the sense of Definition 4, where $\Pi(\mathbf{y}_\uparrow;\mathbf{y})$ denotes the permutation matrix that transfers the ascending version of $\mathbf{y}$, that is $\mathbf{y}_\uparrow$, to $\mathbf{y}$ itself. Here and henceforth, whenever the term "arrangement increasing" is used, it refers to Definition 4.

According to Proposition 5.2 of Cai and Wei (2014), assuming X and Y are independent, (X,Y) is SAI if and only if $X \leq_{lr} Y$. In this sense, the notion of SAI incorporates a dependence structure in the comparison of X and Y. We remark that, without the assumption of independence, the SAI notion does not necessarily imply the likelihood ratio order, and thus the stochastic dominance, between the marginal distributions. However, it is possible to extend those notions of stochastic dominance in a similar way. That is, to incorporate dependence while comparing random variables according to stochastic dominance. Related discussions can be found in Wei (2017) and You and Li (2016). Some remarks are also given in Section 5.

The following lemma provides a useful characterization of the notion of SAI. It is taken from Cai and Wei (2014).

Lemma 4 (Theorem 6.1 of Cai and Wei (2014))**.** *Bivariate random vector (X, Y) is SAI if and only if*

$$\mathbb{E}[h_1(X, Y)] \geq \mathbb{E}[h_2(X, Y)]$$

for any bivariate functions g_1, g_2 such that

(i) $h_1(x, y) \geq h_2(x, y)$ *for any* $x \leq y$*; and*
(ii) $h_1(x, y) + h_1(y, x) \geq h_2(x, y) + h_2(y, x)$ *for any* $x \leq y$.

Theorem 1. *Let $\mathbf{X} = (X_1, \ldots, X_n), \mathbf{Y} = (Y_1, \ldots, Y_n)$ be two independent random vectors. If $\mathbf{X}, \mathbf{Y}$ are both SAI, then*

$$\mathbb{E}[g(\mathbf{X}; \mathbf{Y})] \geq \mathbb{E}[g(\mathbf{X}\Pi_1; \mathbf{Y}\Pi_2)] \tag{2}$$

for any arrangement increasing function $g : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$, and any permutation matrices Π_1 and Π_2.

Proof. For any arrangement increasing function g, it is permutation-invariant. Therefore, $g(\mathbf{x}; \mathbf{y}) = g(\mathbf{X}\Pi; \mathbf{Y}\Pi)$ for any permutation matrix Π. Therefore, it suffices to show that $\mathbb{E}[g(\mathbf{X}; \mathbf{Y})] \geq \mathbb{E}[g(\mathbf{X}; \mathbf{Y}\Pi_2)]$ for any permutation matrix Π_2.

We start by proving the case of $n = 2$. Consider any arrangement increasing function $g(x_1, x_2; y_1, y_2)$. For any $x_1 \leq x_2$ and $y_1 \leq y_2$, we have $g(x_1, x_2; y_1, y_2) \geq g(x_2, x_1; y_1, y_2)$. Since (X_1, X_2) is SAI, then

$$h(y_1, y_2) = \mathbb{E}[g(X_1, X_2; y_1, y_2)] \geq \mathbb{E}[g(X_2, X_1; y_1, y_2)] = \mathbb{E}[g(X_1, X_2; y_2, y_1)] = h(y_2, y_1)$$

for any $y_1 \leq y_2$. Since (Y_1, Y_2) is SAI, then $\mathbb{E}[h(Y_1, Y_2)] \geq \mathbb{E}[h(Y_2, Y_1)]$, which in turn implies that $\mathbb{E}[g(X_1, X_2; Y_1, Y_2)] \geq \mathbb{E}[g(X_1, X_2; Y_2, Y_1)]$.

For any $n \geq 3$, consider any $1 \leq i < j \leq n$. According to Proposition 3.4 of Cai and Wei (2014), $(X_i, X_j)|\mathbf{X}_{\overline{ij}} = \mathbf{x}_{\overline{ij}}$ and $(Y_i, Y_j)|\mathbf{Y}_{\overline{ij}} = \mathbf{y}_{\overline{ij}}$ are SAI. It follows from the result derived for the case $n = 2$ that

$$\mathbb{E}[g(\mathbf{x}; \mathbf{Y})] = \mathbb{E}[\mathbb{E}[g(\mathbf{X}; \mathbf{Y})|\mathbf{X}_{\overline{ij}}, \mathbf{Y}_{\overline{ij}}]] \geq \mathbb{E}[\mathbb{E}[g(\mathbf{x}; \mathbf{Y}\Pi_{ij})|\mathbf{X}_{\overline{ij}}, \mathbf{Y}_{\overline{ij}}]] = \mathbb{E}[g(\mathbf{x}; \mathbf{Y}\Pi_{ij})]$$

for any arrangement increasing function g and any $i < j$. For a general permutation matrix Π_2, it can be decomposed to the product of a sequence of Π_{ij} values. Therefore, the desired conclusion can be reached by iteration in a finite number of steps. □

For independent SAI random vectors $\mathbf{X}, \mathbf{Y}$, Theorem 1 implies that $\mathbb{E}[g(\mathbf{X}\Pi_1; \mathbf{Y}\Pi_2)]$ achieve its maximum when the components of $\mathbf{X}$ and those of $\mathbf{Y}$ are similarly ordered. This result provides a useful shortcut to solve some optimal allocation problems, as seen in Section 4.

Theorem 2. *Let $\mathbf{X} = (X_1, X_2), \mathbf{Y} = (Y_1, Y_2)$ be two independent random vectors. If $\mathbf{X}, \mathbf{Y}$ are both SAI, then*

$$\mathbb{E}[g_1(\mathbf{X}; \mathbf{Y})] \geq \mathbb{E}[g_2(\mathbf{X}; \mathbf{Y})] \tag{3}$$

for any $g_1, g_2 : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ such that, for any $x_1 \leq x_2$ and $y_1 \leq y_2$,

(i) $g_1(\mathbf{x}; \mathbf{y}) \geq g_2(\mathbf{x}; \mathbf{y})$*;*
(ii) $g_1(\mathbf{x}; \mathbf{y}) + g_1(\mathbf{x}\Pi_{12}; \mathbf{y}) \geq g_2(\mathbf{x}; \mathbf{y}) + g_2(\mathbf{x}\Pi_{12}; \mathbf{y})$*;*
(iii) $g_1(\mathbf{x}; \mathbf{y}) + g_1(\mathbf{x}; \mathbf{y}\Pi_{12}) \geq g_2(\mathbf{x}; \mathbf{y}) + g_2(\mathbf{x}; \mathbf{y}\Pi_{12})$*; and*

(iv)

$$g_1(\mathbf{x};\mathbf{y}) + g_1(\mathbf{x};\mathbf{y}\Pi_{12}) + g_1(\mathbf{x}\Pi_{12};\mathbf{y}) + g_1(\mathbf{x}\Pi_{12};\mathbf{y}\Pi_{12})$$
$$\geq g_2(\mathbf{x};\mathbf{y}) + g_2(\mathbf{x};\mathbf{y}\Pi_{12}) + g_2(\mathbf{x}\Pi_{12};\mathbf{y}) + g_2(\mathbf{x}\Pi_{12};\mathbf{y}\Pi_{12}).$$

Proof. Define $h_1(y_1, y_2) = \mathbb{E}[g_1(X_1, X_2; y_1, y_2)]$ and $h_2(y_1, y_2) = \mathbb{E}[g_2(X_1, X_2; y_1, y_2)]$. Since (X_1, X_2) is SAI, Conditions (i) and (ii) imply that $h_1(y_1, y_2) \geq h_2(y_1, y_2)$ for any $y_1 \leq y_2$ according to Lemma 4. Similarly, Conditions (iii) and (iv) imply that $h_1(y_1, y_2) + h_1(y_2, y_1) \geq h_2(y_1, y_2) + h_2(y_2, y_1)$ for any $y_1 \leq y_2$. Applying Lemma 4 on (Y_1, Y_2), we have $\mathbb{E}[g_1(\mathbf{X}, \mathbf{Y})] = \mathbb{E}[h_1(Y_1, Y_2)] \geq \mathbb{E}[h_2(Y_1, Y_2)] = \mathbb{E}[g_2(\mathbf{X}, \mathbf{Y})]$. □

Proposition 1. *Consider function $G(\mathbf{w};\mathbf{x};\mathbf{d}) = u(w_1\phi(x_1, d_1) + w_2\phi(x_2, d_2))$. Define $g_1(\mathbf{w};\mathbf{x}) = G(\mathbf{w};\mathbf{x};\mathbf{d})$ and $g_2(\mathbf{w};\mathbf{x}) = G(\mathbf{w};\mathbf{x};\mathbf{d}\Pi_{12})$ with $d_1 \leq d_2$. If u is increasing convex and $\phi(x, d)$ is increasing in x, d and L-superadditive, then g_1, g_2 satisfies Conditions (i)-(iv) in Theorem 2.*

Proof. Noting that $g_1(\mathbf{w};\mathbf{x}) = g_2(\mathbf{w}\Pi_{12};\mathbf{x}\Pi_{12})$, Condition (iv) holds with equality.

Consider any $w_1 \leq w_2$, $x_1 \leq x_2$, and $d_1 \leq d_2$. Since $\phi(x, d)$ is L-superadditive, we have

$$\phi(x_1, d_1) + \phi(x_2, d_2) \geq \phi(x_1, d_2) + \phi(x_2, d_1). \tag{4}$$

Therefore, $\phi(x_2, d_2) - \phi(x_2, d_1) \geq \phi(x_1, d_2) - \phi(x_1, d_1) \geq 0$, so $w_2(\phi(x_2, d_2) - \phi(x_2, d_1)) \geq w_1(\phi(x_1, d_2) - \phi(x_1, d_1))$ for any $w_1 \leq w_2$. This further implies that $w_1\phi(x_1, d_1) + w_2\phi(x_2, d_2)) \geq w_1\phi(x_1, d_2) + w_2\phi(x_2, d_1)$, and thus $u(w_1\phi(x_1, d_1) + w_2\phi(x_2, d_2))) \geq u(w_1\phi(x_1, d_2) + w_2\phi(x_2, d_1))$ for any increasing function u, which verifies Condition (i).

Following from Condition (4), we have $\phi(x_1, d_1) + \phi(x_2, d_2) - \phi(x_1, d_2) - \phi(x_2, d_1) \geq 0$; thus,

$$w_2(\phi(x_1, d_1) + \phi(x_2, d_2) - \phi(x_1, d_2) - \phi(x_2, d_1))$$
$$\geq w_1(\phi(x_1, d_1) + \phi(x_2, d_2) - \phi(x_1, d_2) - \phi(x_2, d_1)),$$

or, equivalently,

$$\begin{aligned} & w_1\phi(x_1, d_1) + w_2\phi(x_2, d_2) + w_1\phi(x_2, d_1) + w_2\phi(x_1, d_2) \\ \geq \quad & w_1\phi(x_1, d_2) + w_2\phi(x_2, d_1) + w_1\phi(x_2, d_2) + w_2\phi(x_1, d_1). \end{aligned}$$

Note that

$$w_1\phi(x_1, d_1) + w_2\phi(x_2, d_2) \geq \max\{w_1\phi(x_1, d_2) + w_2\phi(x_2, d_1), w_1\phi(x_2, d_2) + w_2\phi(x_1, d_1)\}.$$

Therefore, for any increasing convex function u, we have

$$\begin{aligned} & u(w_1\phi(x_1, d_1) + w_2\phi(x_2, d_2)) + u(w_1\phi(x_2, d_1) + w_2\phi(x_1, d_2)) \\ \geq \quad & u(w_1\phi(x_1, d_2) + w_2\phi(x_2, d_1)) + u(w_1\phi(x_2, d_2) + w_2\phi(x_1, d_1)). \end{aligned}$$

This verifies Condition (iii).

Condition (ii) can be verified similarly. □

Theorem 3. *Let $g : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ be an arrangement increasing function and $\mathbf{X} = (X_1, X_2)$ be an SAI random vector. As a function of $\mathbf{a} = (a_1, a_2)$, $\mathbb{E}[g(\mathbf{X};\mathbf{a})]$ is Schur-convex (or Schur-concave) in $\mathbf{a} \in \mathcal{F}$ if*

(i) $g(\mathbf{x};\mathbf{a})$ *is Schur-convex (or Schur-concave) in $\mathbf{a} \in \mathcal{F}$ for any $x_1 \leq x_2$; and*
(ii) $g(\mathbf{x};\mathbf{a}) + g(\mathbf{x}\Pi_{12};\mathbf{a})$ *is Schur-convex (or Schur-concave) in $\mathbf{a} \in \mathcal{F}$ for any $\mathbf{x}$.*

Proof. Let $\mathbf{a}, \mathbf{b} \in \mathcal{F}$ such that $\mathbf{a} \leq_m \mathbf{b}$. Define $h_{\mathbf{a}}(\mathbf{x}) = g(\mathbf{x}; \mathbf{a})$ and $h_{\mathbf{b}}(\mathbf{x}) = g(\mathbf{x}; \mathbf{b})$. It suffices to show that $\mathbb{E}[h_{\mathbf{a}}(\mathbf{X})] \leq \mathbb{E}[h_{\mathbf{b}}(\mathbf{X})]$. Since $\mathbf{X}$ is SAI, it suffices to verify that $h_{\mathbf{a}}$ and $h_{\mathbf{b}}$ satisfy the two conditions of Lemma 4, which are immediately implied by Conditions (i) and (ii). □

In Theorem 3, the domain $\mathcal{F}$ can be specified as needed. Typical choices include $\mathbb{R}^2$, $\mathcal{D}_2$, and $\mathcal{I}_2$.

Theorem 4. *Define $g(\mathbf{x}, \mathbf{a}) = u(\sum_{i=1}^n \phi(x_i, a_i))$. For any SAI random vector $\mathbf{X}$, $h(\mathbf{a}) \triangleq \mathbb{E}[g(\mathbf{x}; \mathbf{a})]$ is*

(i) Schur-concave in $\mathbf{a} \in \mathcal{D}_n$ if u is increasing concave and if ϕ is L-superadditive and concave in a;
(ii) Schur-convex in $\mathbf{a} \in \mathcal{D}_n$ if u is increasing convex and if ϕ is L-subadditive and convex in a.

Proof. (i) According to the remarks following Definition 1, it suffices to prove the desired conclusion for the case $n = 2$. Consider $g(\mathbf{x}; \mathbf{a}) = u(\phi(x_1, a_1) + \phi(x_2, a_2))$. According to Theorem 3, it suffices to verify

(a) $g(\mathbf{x}; \mathbf{a})$ is Schur-concave in $\mathbf{a} \in \mathcal{D}_2$ for any $x_1 \leq x_2$; and
(b) $g(x_1, x_2; \mathbf{a}) + g(x_2, x_1; \mathbf{a})$ is Schur-concave in $\mathbf{a} \in \mathcal{D}_2$.

Denote $\phi_{(2)}(x, a) = \frac{\partial}{\partial a}\phi(x, a)$. Note that

$$\frac{\partial}{\partial a_1} g(\mathbf{x}; \mathbf{a}) = u'(\phi(x_1, a_1) + \phi(x_2, a_2))\,\phi_{(2)}(x_1, a_1); \quad \frac{\partial}{\partial a_2} g(\mathbf{x}; \mathbf{a}) = u'(\phi(x_1, a_1) + \phi(x_2, a_2))\,\phi_{(2)}(x_2, a_2).$$

Noting that ϕ is L-superadditive and concave in a, $\phi_{(2)}(x, a)$ is increasing in x and decreasing in a. Then, for any $x_1 \leq x_2$ and $a_2 \leq a_1$, we have $\phi_{(2)}(x_1, a_1) \leq \phi_{(2)}(x_2, a_2)$, so $\frac{\partial}{\partial a_1} g(\mathbf{x}; \mathbf{a}) \leq \frac{\partial}{\partial a_2} g(\mathbf{x}; \mathbf{a})$, which verifies (a) according to Lemma 1.

From Lemma 2 (ii), $\phi(x_1, a_1) + \phi(x_2, a_2)$ is arrangement increasing. For any $x_1 \leq x_2$ and $a_2 \leq a_1$, it follows that $\phi(x_1, a_1) + \phi(x_2, a_2) \leq \phi(x_1, a_2) + \phi(x_2, a_1)$, so $u'(\phi(x_1, a_1) + \phi(x_2, a_2)) \geq u'(\phi(x_1, a_2) + \phi(x_2, a_1))$ since u is concave. Therefore,

$$\begin{aligned}
& \frac{\partial}{\partial a_1}(g(x_1, x_2; a_1, a_2) + g(x_2, x_1; a_1, a_2)) \\
= \; & u'(\phi(x_1, a_1) + \phi(x_2, a_2))\phi_{(2)}(x_1, a_1) + u'(\phi(x_1, a_2) + \phi(x_2, a_1))\phi_{(2)}(x_2, a_1) \\
\leq \; & u'(\phi(x_1, a_1) + \phi(x_2, a_2))\phi_{(2)}(x_2, a_1) + u'(\phi(x_1, a_2) + \phi(x_2, a_1))\phi_{(2)}(x_1, a_1) \\
\leq \; & u'(\phi(x_1, a_1) + \phi(x_2, a_2))\phi_{(2)}(x_2, a_2) + u'(\phi(x_1, a_2) + \phi(x_2, a_1))\phi_{(2)}(x_1, a_2) \\
= \; & \frac{\partial}{\partial a_2}(g(x_1, x_2; a_1, a_2) + g(x_2, x_1; a_1, a_2))
\end{aligned}$$

where the first inequality follows from $\phi_{(2)}(x_1, a_1) \leq \phi_{(2)}(x_2, a_2)$ and the second one follows from $\phi_{(2)}(x_2, a_1) \leq \phi_{(2)}(x_2, a_2)$ and $\phi_{(2)}(x_1, a_1) \leq \phi_{(2)}(x_1, a_2)$. This verifies (b) according to Lemma 1.

(ii) Note that

$$-h(\mathbf{a}) = \mathbb{E}\left[-u\left(\sum_{i=1}^n \phi(X_i, a_i)\right)\right] = \mathbb{E}\left[u^*\left(\sum_{i=1}^n \phi^*(X_i, a_i)\right)\right]$$

where $u^*(z) = -u(-z)$ is increasing concave, and $\phi^*(x, a) = -\phi(x, a)$ is L-superadditive and concave in a. Following the conclusion of (i), $-h(\mathbf{a})$ is Schur-concave in $\mathbf{a} \in \mathcal{D}_n$, which implies that $h(\mathbf{a})$ is Schur-convex in $\mathbf{a} \in \mathcal{D}_n$. □

4. Applications in Insurance and Finance

In this section, we shall study two typical allocation problems, namely allocations of policy limits and deductibles and portfolio selections. We remark that some of the results have been already derived, but this section shows how those studies can be unified in one framework using the results derived in Section 3.

4.1. Allocation of Policy Limits and Deductibles

We first consider the optimal allocation of policy limits and deductibles. For the convenience of comparison, we follow the notations of Zhuang et al. (2009). Let $\mathbf{X} = (X_1, \ldots, X_n)$ denote n random losses of a decision-maker and $\mathbf{S} = (S_1, \ldots, S_n)$ denote the random occurrence times of these losses. Without any insurance arrangement, the total discounted risk of the decision-maker is $\sum_{i=1}^{n} X_i e^{-\delta S_i}$, where δ denotes the force of interest.

Suppose the decision-maker enters into an insurance agreement (and thus becomes insured), which grants a total amount of policy limit ℓ and allows the insured to freely allocate the deductible to each risk. Assume the insured allocates the total policy limit according to the vector $\boldsymbol{\ell} = (\ell_1, \ldots, \ell_n)$, where $\ell_i \geq 0$ and $\sum_{i=1}^{n} l_i = \ell$. With such an insurance arrangement, the retained risk of the insured becomes

$$T_{\mathbf{x};\mathbf{S}}(\boldsymbol{\ell}) = \sum_{i=1}^{n} (X_i - \ell_i)_+ e^{-\delta S_i}.$$

The goal of the insured is to find the optimal allocation so as to minimize the total risk in a certain stochastic sense. Note that, for a fixed total policy limit, the premium is fixed and thus is not taken into consideration. Mathematically, the problem is formulated as

$$\min_{\boldsymbol{\ell} \in A(\boldsymbol{\ell})} \mathbb{E}[u(T_{\mathbf{x};\mathbf{S}}(\boldsymbol{\ell}))] \tag{5}$$

where $A(\boldsymbol{\ell}) = \{\boldsymbol{\ell} : \sum_{i=1}^{n} \ell_i = \ell, \text{ and } \ell_i \geq 0 \text{ for all } i = 1, \ldots, n\}$, and u is a utility function.

There is a similar type of insurance contract that grants a total amount of deductible and allows the insured to arbitrarily allocate deductibles to different risks. Assume the total deductible d is allocated according to $\mathbf{d} = (d_1, \ldots, d_n)$. The total retained risk is then $R_{\mathbf{x};\mathbf{S}}(\mathbf{d}) = \sum_{i=1}^{n} (X_i \wedge d_i)\, e^{-\delta S_i}$, and the goal is to minimize this risk in the following sense:

$$\min_{\ell \in A(\mathbf{d})} \mathbb{E}[u(R_{\mathbf{x};\mathbf{S}}(\mathbf{d}))] \tag{6}$$

where $A(\mathbf{d}) = \{\mathbf{d} : \sum_{i=1}^{n} d_i = d, \text{ and } d_i \geq 0 \text{ for all } i = 1, \ldots, n\}$, and u is a utility function.

Since initially proposed by Cheung (2007), these two problems have been extensively studied by many authors, see, for example, Li and You (2012), Zhuang et al. (2009), Lu and Meng (2011), and Manesh et al. (2016), and rich results have been established. Below, we reprove some of those results from a simplified approach, which illustrates the convenience of using the properties of the SAI notion derived in Section 3. Furthermore, we shall compare the retained risks under different allocation vectors subject to majorization order.

Proposition 2 (Corollaries 3.3(b) and 3.6(b) of Zhuang et al. (2009))**.** *Consider the case $\delta = 0$ and assume* $\mathbf{X}$ *is SAI.*

(i) The optimal solutions to Problem (5), $\boldsymbol{\ell}^* = (\ell_1^*, \ldots, \ell_n^*)$, should satisfy $\ell_1^* \leq \cdots \leq \ell_n^*$, for any increasing u.
(ii) The optimal solutions to Problem (6), $\mathbf{d}^* = (d_1^*, \ldots, d_n^*)$, should satisfy $d_1^* \geq \cdots \geq d_n^*$, for any increasing u.

Proof. (i) Define $g(\mathbf{x}; \mathbf{y}) = u(\sum_{i=1}^{n} (x_i - y_i)_+)$. The desired conclusion is restated as

$$\mathbb{E}[g(\mathbf{x}; \boldsymbol{\ell}^*)] \leq \mathbb{E}[g(\mathbf{x}; \boldsymbol{\ell}^* \Pi)],$$

for any permutation matrix Π.

Note that $\boldsymbol{\ell}^* = (\ell_1^*, \ldots, \ell_n^*)$ with $\ell_1^* \le \cdots \le \ell_n^*$ is a degenerated SAI random vector and is independent of the SAI random vector $\mathbf{X}$. According to Theorem 1, it suffices to verify that $g(\mathbf{x}; \mathbf{y})$ is arrangement decreasing, which is true due to Lemma 3 (ii) and Remark 1.

(ii) Similar to the proof of (i), it suffices to verify that $g(\mathbf{x}, \mathbf{y}) = u(\sum_{i=1}^n (x_i \wedge y_i))$ is arrangement increasing, which follows from Lemma 3 (i) and Remark 1.

□

Proposition 3. *Consider Problems* (5) *and* (6) *with* $\delta > 0$. *Assume* $\mathbf{X} = (X_1, \ldots, X_n)$ *and* $\mathbf{W} = (e^{-\delta S_1}, \ldots, e^{-\delta S_n})$ *are independent and both are SAI.*

(i) The optimal solutions to Problem (6), $\mathbf{d}^* = (d_1^*, \ldots, d_n^*)$, *should satisfy* $d_1^* \ge \cdots \ge d_n^*$, *for any increasing convex function* u.
(ii) The optimal solutions to Problem (5), $\boldsymbol{\ell}^* = (\ell_1^*, \ldots, \ell_n^*)$, *should satisfy* $\ell_1^* \le \cdots \le \ell_n^*$, *for any increasing convex function* u.

Proof. We focus on the proof of the case $n = 2$.

(i) It suffices to show that

$$\mathbb{E}[u(W_1(X_1 \wedge d_1) + W_2(X_2 \wedge d_2))] \ge \mathbb{E}[u(W_1(X_1 \wedge d_2) + W_2(X_2 \wedge d_1))] \tag{7}$$

for any $d_1 \le d_2$.

Define $G(\mathbf{w}; \mathbf{x}; \mathbf{d}) = u(w_1\phi(x_1, d_1) + w_2\phi(x_2, d_2))$ with $\phi(x, d) = x \wedge d$, $g_1(\mathbf{w}; \mathbf{x}) = G(\mathbf{w}; \mathbf{x}; d_1, d_2)$ and $g_2(\mathbf{w}; \mathbf{x}) = G(\mathbf{w}; \mathbf{x}; d_2, d_1)$. With these notations, Condition (7) is equivalent to $\mathbb{E}[g_1(\mathbf{W}; \mathbf{X})] \ge \mathbb{E}[g_2(\mathbf{W}; \mathbf{X})]$ for $d_1 \le d_2$. It is easy to verify that $\phi(x, d) = x \wedge d$ is L-superadditive and increasing in d. According to Proposition 1, g_1 and g_2 satisfy Conditions (i)–(iv) of Theorem 2, which immediately implies that $\mathbb{E}[g_1(\mathbf{W}; \mathbf{X})] \ge \mathbb{E}[g_2(\mathbf{W}; \mathbf{X})]$.

(ii) It suffices to show that

$$\mathbb{E}[u(W_1(X_1 - \ell_1)_+ + W_2(X_2 - \ell_2)_+)] \le \mathbb{E}[u(W_1(X_1 - \ell_2)_+ + W_2(X_2 - \ell_1)_+)]$$

for any $\ell_1 \le \ell_2$, which can be rewritten as (by denoting $t_1 = -\ell_2$ and $t_2 = -\ell_1$),

$$\mathbb{E}[u(W_1(X_1 + t_2)_+ + W_2(X_2 + t_1)_+)] \le \mathbb{E}[u(W_1(X_1 + t_1)_+ + W_2(X_2 + t_2)_+)]$$

for any $t_1 \le t_2$. The rest can be proved similarly as Condition (i). □

Proposition 3 recovers the results of Theorems 4.3 and 4.7 of Zhuang et al. (2009), Theorems 6.3 and 6.5 of Cai and Wei (2014), and Theorem 4.2 of Pan and Li (2017). Although this is not a new result, we remark that the proof is significantly simplified. Furthermore, it is worth noting that Pan and Li (2017) studied a more general version of Problems (5) and (6) and their Theorem 4.1 can be also implied by Theorem 2 and Proposition 1.

Proposition 4. *Assume* $\mathbf{X}$ *is SAI. Let* $\mathbf{a} = (a_1, \ldots, a_n)$ *and* $\mathbf{b} = (b_1, \ldots, b_n)$ *be two real vectors.*

(i) If $\mathbf{a}, \mathbf{b} \in \mathcal{D}_n$ *and* $\mathbf{a} \ge_m \mathbf{b}$, *then* $\sum_{i=1}^n (X_i \wedge a_i) \le_{icv} \sum_{i=1}^n (X_i \wedge b_i)$.
(ii) If $\mathbf{a}, \mathbf{b} \in \mathcal{D}_n$ *and* $\mathbf{a} \ge_m \mathbf{b}$, *then* $\sum_{i=1}^n (X_i - a_i)_+ \ge_{icx} \sum_{i=1}^n (X_i - b_i)_+$.

Proof. Note that the bivariate function $x \wedge d$ is L-superadditive and concave in a, (i) immediately follows from Theorem 4 (i). Similarly, noting that $(x - a)_+$ is L-subadditive and convex in a, (ii) follows from Theorem 4 (ii). □

Proposition 4 (i) implies that the deductible allocation vector $(d, 0, \ldots, 0)$ minimizes the insured's retained risk in the sense of the increasing concave order, and thus solves Problem (6). This makes sense because, when the total deductible is assigned to X_1, the "smallest" risk, it is least likely to be reached and thus produces the "smallest" retained risk. On the other hand, when it comes to Problem (5), it is not obvious what allocation strategy is optimal. However, Proposition 4 (ii) implies that the policy limit allocation vector $(l, 0, \ldots, 0)$ is the worst strategy, which would maximize the retained risk in the sense of increasing convex order. Recalling the results derived in Proposition 2, we remark that the assumption of $\mathbf{b} \in \mathcal{D}_n$ can be removed in both statements of Proposition 4.

Similar studies have been conducted, for example, by Lu and Meng (2011) and Manesh et al. (2016). In their papers, the retained risk is minimized in the sense of the usual stochastic order, and the logconcavity of the marginal distribution function or exchangeability is assumed. Proposition 4 shows that, when the minimization criterion is reduced to the sense of increasing convex order, those additional assumptions are no longer needed.

4.2. Portfolio Selections

Let $\mathbf{X} = (X_1, \ldots, X_n)$ denote the stochastic return rates of n risky assets. For an investor endowed with initial wealth w, the concern is how to allocate the investment so as to maximize the final return. With allocation vector $\mathbf{a} = (a_1, \ldots, a_n)$ where $\sum_{i=1}^n a_i = w$, the investor's final wealth is $\mathbf{a} \cdot \mathbf{X} = \sum_{i=1}^n a_i X_i$. The optimization problem is formulated as

$$\max_{\mathbf{a} \in A(w)} \mathbb{E}[u(\sum_{i=1}^n a_i X_i)] \tag{8}$$

where $A(w) = \{\mathbf{a} \geq 0 : \sum_{i=1}^n a_i = w\}$, and u is a utility function.

Proposition 5. *Assume $\mathbf{X} = (X_1, \ldots, X_n)$ is SAI. The solution to Problem (8), $(a_1^*, \ldots, a_n^*)$, should satisfy $a_1^* \leq \cdots \leq a_n^*$ for any increasing utility function $u(x)$.*

Proof. Like the proof of Proposition 2, the desired conclusion follows from the fact that $g(\mathbf{x}; \mathbf{a}) = u(\sum_{i=1}^n a_i x_i)$ is arrangement increasing for any increasing function u. □

Proposition 6. *Assume $\mathbf{X}$ is SAI. Let $\mathbf{a} = (a_1, \ldots, a_n)$ and $\mathbf{b} = (b_1, \ldots, b_n)$ be two real vectors.*

(i) If $\mathbf{a}, \mathbf{b} \in \mathcal{I}_n$ and $\mathbf{a} \geq_m \mathbf{b}$, then $\sum_{i=1}^n b_i X_i \leq_{icx} \sum_{i=1}^n a_i X_i$.
(ii) If $\mathbf{a}, \mathbf{b} \in \mathcal{D}_n$ and $\mathbf{a} \geq_m \mathbf{b}$, then $\sum_{i=1}^n b_i X_i \geq_{icv} \sum_{i=1}^n a_i X_i$.

Proof. As before, we focus on the proof for the bivariate case.

(i) Define $g(\mathbf{x}; \mathbf{a}) = g(x_1, x_2; a_1, a_2) = u(a_1 x_1 + a_2 x_2)$. It suffices to show that $\mathbb{E}[g(X_1, X_2; a_1, a_2)]$ is Schur-convex in $(a_1, a_2) \in \mathcal{I}_n$ for any increasing convex function u. According to Theorem 3,

(a) $g(x_1, x_2; a_1, a_2)$ is Schur-convex in $(a_1, a_2) \in \mathcal{I}_2$ for any $x_1 \leq x_2$; and
(b) $h(a_1, a_2) \triangleq g(x_1, x_2; a_1, a_2) + g(x_2, x_1; a_1, a_2)$ is Schur-convex in $(a_1, a_2) \in \mathcal{I}_2$ for any $x_1 \leq x_2$.

For any $x_1 \leq x_2$, it is easy to verify that $a_1 x_1 + a_2 x_2$ is Schur-convex in $(a_1, a_2) \in \mathcal{I}_2$. $g(x_1, x_2; a_1, a_2) = u(a_1 x_1 + a_2 x_2)$ is also Schur-convex since u is increasing. This verifies (a).

For any $x_1 \leq x_2$ and $a_1 \geq a_2$, $a_1 x_1 + a_2 x_2 \leq a_1 x_1 + a_2 x_2$, so $u'(a_1 x_1 + a_2 x_2) \leq u'(a_2 x_1 + a_1 x_2)$ for any convex u. Therefore,

$$\begin{aligned} \frac{\partial}{\partial a_1} h(a_1, a_2) &= x_1 u'(a_1 x_1 + a_2 x_2) + x_2 u'(a_2 x_1 + a_1 x_2) \\ &\geq x_2 u'(a_1 x_1 + a_2 x_2) + x_1 u'(a_2 x_1 + a_1 x_2) = \frac{\partial}{\partial a_2} h(a_1, a_2), \end{aligned}$$

which implies that h is Schur-convex in $(a_1, a_2) \in \mathcal{D}_2$ according to Lemma 1. By noting that $h(a_1, a_2)$ is symmetric, i.e., $h(a_1, a_2) = h(a_2, a_1)$, (b) is verified.

(ii) can be proved similarly. □

It is worth pointing out that the results of Propositions 5 and 6 are not new. Specifically, the conclusion of Propositions 5 has been derived by Theorem 5.2 by Cai and Wei (2015). The result of Proposition 6 is implied by Theorem 1 in the work by You and Li (2016). The purpose of stating these two propositions is to show that they can be proved by an innovative yet simplified approach.

5. Concluding Remarks

In this paper, we derive several properties of SAI random vectors by using arrangement increasing functions. With these properties, we aim to set up a unified framework for the study of different types of allocation problems. As evidenced in Section 4, the establishment of such a framework significantly facilitates solving the allocation problems in insurance and finance.

Another advantage of this framework is that it allows the potential of introducing dependence between random vectors. Recall that, whenever two random vectors are involved, they are assumed to be independent. However, this is not essentially necessary. Note that most results concerning two random vectors are derived based on Theorems 1 and 2, or more specifically, based on the properties described by Inequalities (2) and (3). If a new dependence structure can be developed by the characterizations of Inequalities (2) or (3), it would be possible to get rid of the assumption of independence between two random vectors. Admittedly, there are still technical difficulties and we leave it for future research.

Throughout this paper, we focus only on the SAI structure, while other dependence structures such as RWSAI, LWSAI, and WSAI are also considered in the existing studies of allocation problems. It would be interesting to set up a similar framework using arrangement increasing or other relevant functions, which can be done in future research.

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