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Joint Insolvency Analysis of a Shared MAP Risk Process: A Capital Allocation Application

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In recent years, multivariate insurance risk processes have received increasing attention in risk theory. First-passage-time problems in the context of these insurance risk processes are of primary interest for risk management purposes. In this article we study joint-ruin problems of two risk undertakers in a proportionally shared Markovian claim arrival process. Building on the existing work in the literature, joint-ruin–related quantities are thoroughly analyzed by capitalizing on existing results in certain univariate insurance surplus processes. Finally, an application is considered where the finite-time and infinite-time joint-ruin probabilities are used as risk measures to allocate risk capital among different business lines. The proposed joint-ruin allocation principle enables us to not only capture the risk dynamics over a given time horizon, but also overcome the "cross-subsidizing" effect of many existing allocation principles.

1. INTRODUCTION

In recent years, multivariate insurance risk models have received increasing attention in risk theory. The inherent flexibility to simultaneously capture risk characteristics of multiple business lines is one of the main reasons for this accrued interest. The analysis of these multivariate risk models is also a primary interest for risk management purposes, because it is vital for insurers to accurately assess the risks inherent to their overall insurance business, as well as to find efficient risk management mechanisms to mitigate them. A common mathematical framework in this context is the multivariate insurance risk process defined as

$$\mathbf{U}(t) = (U_1(t), \ldots, U_m(t)) = (u_1 + c_1 t - S_1(t), \ldots, u_m + c_m t - S_m(t)), \tag{1}$$

for $t \geq 0$, where $U_i(t)$ represents the time-t surplus level of the ith business line (with $U_i(0) = u_i \geq 0$). Also, let $c_i \geq 0$ and $\{S_i(t), t \geq 0\}$ be the level premium rate and the aggregate claim process of the ith business line, respectively. A potential challenge in analyzing the multivariate surplus process (1) resides in the dependence structure among the aggregate claim processes $\{S_i(t), t \geq 0\}$ for $i = 1, 2, \ldots, m$. In the multivariate setting, three definitions of ruin have mainly been considered (see, e.g., Cai and Li 2007):

$$\tau_{or} = \inf\{t \geq 0 : \min_{1\leq i\leq m} U_i(t) < 0\}, \tag{2}$$

$$\tau_{sim} = \inf\{t \geq 0 : \max_{1\leq i\leq m} U_i(t) < 0\}, \tag{3}$$

and

$$\tau_{and} = \inf\{t \geq 0 : \inf_{0\leq s\leq t} U_i(s) < 0 \text{ for all } i = 1, \ldots, m\}, \tag{4}$$

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where $\inf \emptyset = \infty$. For notational convenience, we sometimes denote (2)–(4) as $\tau_\star$, where $\star$ can be any of "*or*," "*and*," or "*sim*." Accordingly, let $\psi_\star(u_1, \dots, u_m, t) = \mathbb{P}\{\tau_\star \leq t\}$ be the finite-time ruin probability, where its infinite-time counterpart is $\psi_\star(u_1, \dots, u_m) = \lim_{t\to\infty} \psi_\star(u_1, \dots, u_m, t)$. Note that the ruin times (2)–(4) have different trigger sensitivities. For instance, it is not difficult to show that $\tau_{or} \leq \tau_{and} \leq \tau_{sim}$ almost surely, which implies that τ_{or} has the fastest trigger time, and τ_{sim} is triggered last.

As alluded to earlier, the multivariate insurance risk model has been the subject matter of analysis under various model setups. Cai and Li (2005, 2007) studied the surplus process (1), when the aggregate claim processes $\{S_i(t), t \geq 0\}$ $(i = 1, \dots, m)$ follow a multivariate compound Poisson risk process with phase-type jumps. Using multivariate stochastic-order arguments, bounds for the infinite-time ruin probabilities ψ_{or}, ψ_{and}, and ψ_{sim} are derived. This was later followed by the work of Gong et al. (2012), in which certain ruin related quantities for the ruin time τ_{or} are examined under a different multivariate compound Poisson setup for the claim arrival dynamic. Given the complexity of the research problem, most of the other contributions in the literature have focused on the bivariate case (i.e., $m = 2$). Chan et al. (2003), Yuen et al. (2006), and Dang et al. (2009) are notable contributions in the bivariate setting. More closely related to the present work is the contribution of Avram et al. (2008a) in a proportional reinsurance setup, where the aggregate claim process of an insurance portfolio is proportionally shared between two risk undertakers. Explicit expressions and asymptotic results for the ruin probabilities are obtained. Badescu et al. (2011) later extended the ruin analysis by adding an independent compound Poisson process to one of the two risk undertakers. All of the aforementioned papers work within the confines of the multivariate compound Poisson risk model. A notable exception in the literature is the work of Elliott et al. (2012b) on general hitting times of some hidden Markovian-modulated diffusion processes, which could in turn be used to approximate the classical compound Poisson process. In the multivariate insurance setting, Elliott et al. (2012a) utilized a partial differential equation approach to obtain the ruin probabilities when the risk processes are described by a multivariate diffusion process. See also Elliott et al. (2011) for the treatment of hitting times of a discrete Markov chain.

In this article we propose to generalize the claim-counting process to the Markovian arrival process (MAP) (e.g., Ahn and Badescu 2007) and study the bivariate risk model under a proportional reinsurance setting (e.g., Avram et al. 2008a), where an insurance portfolio is proportionally shared between two risk undertakers. The present study has three layers of significance. First, it proposes a methodology to deal with the multivariate insurance risk model with MAP claim-counting processes. Second, according to Cai and Li (2007), the multivariate risk model with comonotonic aggregate claim processes is of central importance because it provides bounds for both the finite-time and infinite-time ruin probabilities. Indeed, this article studies the bivariate risk model with a special class of comonotonicity and thus provides insights for future studies of models with general comonotonic properties. Third, the joint-ruin probabilities can be used as an alternative tool to tackle the classical capital allocation problem, as illustrated in Section 5.

In relation to the alluded capital allocation problem, a variety of capital allocation principles have been proposed. Cummins (2000) provided an overview of common capital allocation methods suitable for insurers and pointed out the importance of solvency risk in the insurance industry. By considering the marginal contribution of each business line to an insurer's default value, Myers and Read (2001) also showed how option-pricing methods can be used to allocate the required capital among business lines. For more risk-measure–based capital allocation principles, readers are referred to, e.g., Dhaene et al. (2003, 2012), Panjer (2001), Tsanakas (2009), and Xu and Mao (2013) and references therein. Numerous allocation methods model the risks via the terminal value of a position over a given time horizon (see Principles 1–3 of Section 5, for instance). A potential drawback of these allocation methods is that no path-dependent information on the underlying position is incorporated into the allocation problem. This comment does not apply to *ruin-based allocation principles* (see, e.g., Dhaene et al. 2003; Frostig and Denuit 2009; and Mitric and Trufin 2015 and references therein), where path-dependent information is incorporated in the decision-making exercise. Also, many of the allocation principles aim to minimize the aggregation of the loss deviation from capital (based on a certain risk measure) of each individual business line and implicitly allow "cross-subsidization" among different business lines (see, e.g., Dhaene et al. 2012); that is, a "ruin" event of an individual business line may be compensated by other outperforming business lines. As pointed out by Erel et al. (2015), cross-subsidization could potentially distort investment decisions, performance appraisals, incentives, and pricing. In a regulated setting, companies may also "push" riskier lines of business if cross-subsidization is allowed (see Myers and Read 2001). To minimize this "cross-subsidization" effect, we propose to examine a ruin-based capital allocation problem involving the finite-time ruin probability of τ_{or} (with the fastest trigger time among (2)–(4)). Hence, we formulate the optimal capital allocation problem as

$$\inf_{\substack{u_1, \dots, u_m \geq 0 \\ u_1 + \cdots + u_m = K}} \psi_{or}(u_1, \dots, u_m, t), \tag{5}$$

for a given total capital level $K \geq 0$ and a time horizon $t \geq 0$.

The rest of the article is organized as follows. In Section 2 the bivariate MAP risk model (1) with proportionally shared risks under study is formally defined. Some preliminary results are also reviewed. In Section 3 explicit expressions for a quintuple Laplace transform (LT) involving ruin-related quantities of interest and for the finite-time joint-ruin probabilities are given in terms of ruin quantities in the associated univariate risk model. Specific observations in the context of the compound Poisson risk model are made in Section 4. In Section 5 we consider the compound Poisson risk model with a mixture of exponential claim size distribution to illustrate the joint-ruin capital allocation principle (5) when $m = 2$.

2. PRELIMINARIES

2.1. Review of the One-Dimensional MAP Risk Model

In the univariate MAP risk model, the insurance surplus process $\{U(t), t \geq 0\}$ is modeled as

$$U(t) = u + ct - S(t), \tag{6}$$

where $u > 0$ is the initial surplus level, $c > 0$ is the level premium rate, and the aggregate claim amount process $\underline{S} = \{S(t), t \geq 0\}$ is defined as

$$S(t) = \begin{cases} \sum_{k=1}^{N(t)} X_k, & N(t) > 0, \\ 0, & N(t) = 0. \end{cases}$$

It is assumed that the claim number process $\{N(t), t \geq 0\}$ is a Markovian arrival process with representation MAP($\boldsymbol{\alpha}$, $\mathbf{G_0}$, $\mathbf{G_1}$) of order n (see, e.g., Ahn and Badescu 2007). For such a process, an irreducible underlying continuous time Markov chain (CTMC) $\underline{J} = \{J(t), t \geq 0\}$ with initial probability vector $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ on the state space $E = \{1, 2, \ldots, n\}$ is introduced: A matrix generator $\mathbf{G_0} = [g_{0,ij}]_{n\times n}$ with $g_{0,ij} \geq 0$ for $i \neq j$ is given to govern the transitions of the CTMC from state i to j ($i \neq j$) without an accompanying claim, while another matrix generator $\mathbf{G_1} = [g_{1,ij}]_{n\times n}$ with $g_{1,ij} \geq 0$ is used to define the transitions of the CTMC from state i to j with an accompanying claim. Note that $g_{0,ii}$ ($i = 1, 2, \ldots, n$) are negative such that the sum of the elements on each row of the matrix $\mathbf{G_0} + \mathbf{G_1}$ is zero.

We further assume that a claim size X_k ($k \geq 1$) accompanying a transition of $\underline{J}$ from state i to j has density p_{ij}, LT $\widetilde{p}_{ij}(s) = \int_0^\infty e^{-sx} p_{ij}(x)dx$, and mean μ_{ij}. Conditional on $\underline{J}$, the claim sizes $\{X_k\}_{k=1}^\infty$ are mutually independent, also independent of $\{N(t), t \geq 0\}$. As usual, we impose a positive security loading assumption on the process $\{U(t), t \geq 0\}$,

$$\sum_{i=1}^{n} \pi_i \sum_{j=1}^{n} g_{1,ij}\mu_{ij} < c, \tag{7}$$

where $\boldsymbol{\pi} = (\pi_1, \pi_2, \ldots, \pi_n)$ contains the stationary probabilities of the CTMC $\underline{J}$. In the following, we consider some ruin quantities of interest in the one-dimensional MAP risk model (6), which are essential components in the characterization of ruin results pertaining to $\tau_\star$ in the later sections.

Let $\tau = \inf\{t \geq 0 : U(t) < 0\}$ be the time of ruin, and define the ruin probability matrix $\boldsymbol{\Psi}(u, t; c) = [\psi_{ij}(u, t; c)]_{n\times n}$ as

$$\psi_{ij}(u, t; c) = \mathbb{P}\left(\tau \leq t, J(\tau) = j \,|U(0) = u, J(0) = i\right). \tag{8}$$

The dependence of $\boldsymbol{\Psi}$ on c is explicitly emphasized for notational convenience in the subsequent analysis. Note that the finite-time ruin probability $\boldsymbol{\Psi}(u, t; c)$ can be evaluated by inverting analytically or numerically the LT of the time of ruin. Also, let $\boldsymbol{\Psi}(u; c) = \lim_{t\to\infty} \boldsymbol{\Psi}(u, t; c)$ be the ultimate ruin probability. Of general interest in the analysis of the hitting time τ is the triple LT $\boldsymbol{\Phi}_{\delta,s,r}(u; c) = [\phi_{\delta,s,r,ij}(u; c)]_{n\times n}$, defined as

$$\phi_{\delta,s,r,ij}(u; c) = E[e^{-\delta\tau - sU(\tau-) - r|U(\tau)|}\mathbb{I}\{\tau < \infty, J(\tau) = j\}|U(0) = u, J(0) = i], \tag{9}$$

for $\delta, s, r \geq 0$. The triple LT $\boldsymbol{\Phi}_{\delta,s,r}$ is a special case of the so-called Gerber-Shiu function (see, e.g., Gerber and Shiu 1998). Readers are referred to, e.g., Ahn and Badescu (2007) and Landriault and Shi (2015) for an expression of $\boldsymbol{\Phi}_{\delta,s,r}(u; c)$. We remark that the LT of the time to ruin and the deficit at ruin (i.e., a special case of (9)) for the more general class of spectrally negative Markov additive processes can be found in Ivanovs and Palmowski (2012, Corollary 4).

An auxiliary ruin quantity of interest in the later analysis is the matrix distribution function $\mathbf{H}(u, y, t; c) = [H_{ij}(u, y, t; c)]_{n\times n}$ for a given $t \geq 0$ defined as

$$H_{ij}(u, y, t; c) = \mathbb{P}\left(U(t) \leq y, \tau > t, J(t) = j \,|U(0) = u, J(0) = i\right),$$

for $y \geq 0$ and $i, j \in E$. From Ivanovs (2014, Theorem 1), we have

$$\int_0^\infty e^{-\delta t}\mathbf{H}(u, dy, t; c)dt = \left(e^{\mathbf{R}y}\mathbf{W}_\delta(u) - \mathbf{W}_\delta(u-y)\mathbb{I}(u > y)\right) dy, \tag{10}$$

where the matrix scale function $\{\mathbf{W}_\delta(x), x > 0\}$ is defined in terms of the matrix exponent $\mathbf{L}_\delta(s)$:

$$\widetilde{\mathbf{W}}_\delta(s) = \int_0^\infty e^{-sx}\mathbf{W}_\delta(x)dx = \mathbf{L}_\delta^{-1}(s)$$

with

$$\mathbf{L}_\delta(s) = (cs - \delta)\mathbf{I} + \mathbf{G_0} + \widetilde{\mathbf{G}}_p(s). \tag{11}$$

In (11), $\mathbf{I}$ is an $n \times n$ identity matrix and $\widetilde{\mathbf{G}}_p(s) = [g_{1,ij}\widetilde{p}_{ij}(s)]_{n\times n}$. Furthermore, the matrix $\mathbf{R}$ in (10), whose nonzero eigenvalues are the zeros of $\det(\mathbf{L}_\delta(-z))$ with negative real parts, is the left solution (associated with the corresponding left eigenvectors) of $\mathbf{L}_\delta(-\mathbf{z}) = \mathbf{0}$. Readers are referred to Ivanovs (2014, Remark 2) and references therein for more details concerning the matrix $\mathbf{R}$.

An alternative expression for $\mathbf{H}(u, y, t; c)$ can be found by drawing a connection with a time-reversed process. First, we define the distribution function (df) of the aggregate claim $S(t)$ as $\mathbf{F}(y, t) = [F_{ij}(y, t)]_{n\times n}$, with

$$F_{ij}(y, t) = 1 - \overline{F}_{ij}(y, t) = \mathbb{P}\left(S(t) \leq y, J(t) = j \,|J(0) = i\right), \tag{12}$$

a quantity that has been extensively analyzed by Ren (2008) and others. For convenience, we also rewrite (12) as

$$\mathbf{F}(y, t) = \mathbf{F}(0, t) + \int_0^y \mathbf{f}(z, t)dz, \qquad y \geq 0, \tag{13}$$

where $\mathbf{f}(y, t) = [f_{ij}(y, t)]_{n\times n}$ for $y > 0$ is the aggregate claim density. Furthermore, for $\tau_x^+ = \inf\{t \geq 0 : U(t) > x\}$, it is well known (e.g., Ivanovs and Palmowski 2012) that

$$E[e^{-\delta\tau_x^+}] = \int_0^\infty e^{-\delta t}\mathbf{Z}(dt, x; c) = e^{\mathbf{Q}x},$$

where $\mathbf{Z}(t, x; c) = [Z_{ij}(t, x; c)]_{n\times n}$ is defined as

$$Z_{ij}(t, x; c) = \mathbb{P}\left(\tau_x^+ \leq t, J\left(\tau_x^+\right) = j \,|U(0) = 0, J(0) = i\right), \tag{14}$$

and $\mathbf{Q}$ is a certain transition rate matrix which has the same eigenvalues as $\mathbf{R}$. For the detailed expression of $\mathbf{Q}$, as well as the relations between $\mathbf{Q}$ and $\mathbf{R}$, see D'Auria et al. (2010, Theorem 1) and Ivanovs (2014, Section 4.2).

For a given $t > 0$, the time reversed process $\{(\widehat{U}(s), \widehat{J}(s)), 0 \leq s < t\}$ is defined as

$$\widehat{U}(s) = u + U(t) - U((t-s)-), \qquad \widehat{J}(s) = J((t-s)-),$$

for $0 \leq s < t$ (see, e.g., Ivanovs 2014). Note that the process $\{(\widehat{U}(s), \widehat{J}(s)), 0 \leq s < t\}$ is also a MAP risk process (with matrix exponent $\widehat{\mathbf{L}}_\delta(s) = [\mathbf{L}_\delta(s)]^\top = (cs - \delta)\mathbf{I} + \mathbf{G_0}^\top + [\widetilde{\mathbf{G}}_p(s)]^\top$, where $\top$ denotes the transpose of a matrix). For the process $(\widehat{U}(s), \widehat{J}(s))$, let $\widehat{\mathbf{F}}(y, t)$, $\widehat{\mathbf{f}}(y, t)$ and $\widehat{\mathbf{Z}}(s, x; c)$ be the time-reversed equivalent of (12), (13), and (14), respectively. To this end, an alternative expression for $\mathbf{H}(u, y, t; c)$ is provided in Proposition 2.1, where the proof is given in Appendix A.

Proposition 2.1. *For the MAP risk process* $\{(U(t), J(t)), t \geq 0\}$,

$$\mathbf{H}(u, dy, t; c) = \begin{cases} \mathbf{F}(0, t), & y = u + ct, \\ \mathbf{f}\,(u + ct - y, t)\, dy & \\ \quad - \left(\int_{\frac{y}{c} \wedge t}^{t} \left[\widehat{\mathbf{Z}}\,(ds, y; c)\, \widehat{\mathbf{f}}\,(u + c(t - s), t - s) \right]^{\top} \right) dy, & 0 \leq y < u + ct, \end{cases} \tag{15}$$

where $x \wedge y = \min(x, y)$.

2.2. Two-Dimensional MAP Risk Model with Proportional Reinsurance

In this article, we study the ruin times τ_{or}, τ_{and}, and τ_{sim} under a bivariate MAP risk model, where the aggregate risk $\underline{S}$ is proportionally shared by two risk undertakers. More precisely, we define the surplus level of the kth undertaker ($k = 1, 2$) at time t by

$$U_k^0(t) = u_k^0 + c_k^0 t - \omega_k S(t), \qquad k = 1, 2, \tag{16}$$

where $u_k^0 > 0$ is the initial surplus level of the kth undertaker, $c_k^0 > 0$ is its level premium rate, and $\omega_1, \omega_2 > 0$ are the quota shares with $\omega_1 = 1 - \omega_2$. We impose the positive security loading constraint on each process,

$$\sum_{i=1}^{n} \pi_i \sum_{j=1}^{n} g_{1,ij} \mu_{ij} < \left(\frac{c_1^0}{\omega_1} \wedge \frac{c_2^0}{\omega_2} \right).$$

Here, for simplicity and without loss of generality, we propose to work with the rescaled surplus process $\{U_k(t), t \geq 0\}$, where

$$U_k(t) \equiv \frac{U_k^0(t)}{\omega_k} = u_k + c_k t - S(t), \qquad k = 1, 2, \tag{17}$$

where $u_k = u_k^0/\omega_k$ and $c_k = c_k^0/\omega_k$. To avoid triviality, we assume $u_2 > u_1$ and $c_1 > c_2$.[1] Also, let $T = (u_2 - u_1)/(c_1 - c_2)$ and $\Delta(t) = U_2(t) - U_1(t) = u_2 - u_1 - (c_1 - c_2)t$ with $\Delta(T) = 0$.

Avram et al. (2008a) show that the ruin time $\tau_{or}(\tau_{sim})$ associated to the insurance surplus processes (17) can be viewed as a one-dimensional first-crossing problem of $\underline{S}$ above a piecewise linear boundary given by $b_{min}(t) = \min_{k=1,2}\{u_k + c_k t\}$ ($b_{max}(t) = \max_{k=1,2}\{u_k + c_k t\}$). Here, we consider the ruin times (2)–(4) for the bivariate process $\mathbf{U}(t)$ defined in (17). For convenience, let

$$\tau_k(t) = \inf\{s \geq t : U_k(s) < 0\}, \tag{18}$$

for $k = 1, 2$, with the abbreviated notation that $\tau_k = \tau_k(0)$. It is not difficult to conclude that

$$\tau_{or} = \begin{cases} \tau_1, & \text{if } \tau_1 \leq T \\ \tau_2, & \text{if } \tau_1 > T \end{cases}, \quad \tau_{and} = \begin{cases} \tau_2, & \text{if } \tau_1 \leq T \\ \tau_1, & \text{if } \tau_1 > T \end{cases}, \quad \tau_{sim} = \begin{cases} \tau_2, & \text{if } \tau_2 \leq T \\ \tau_1(T), & \text{if } \tau_2 > T \end{cases}. \tag{19}$$

3. QUINTUPLE LT ANALYSIS OF A SHARED MAP RISK PROCESS

In this section, we jointly consider the ruin time, the surplus prior to ruin, and the deficit at ruin of both risk undertakers in the shared MAP risk processes (17) by analyzing their quintuple LT $\mathbf{m}_\delta(u_1, u_2; \tau_\star) = [m_{\delta,ij}(u_1, u_2; \tau_\star)]_{n \times n}$ defined as

$$\begin{aligned} &m_{\delta,ij}(u_1, u_2; \tau_\star) \\ &\quad \equiv E\left[e^{-\delta\tau_\star - \eta_1 U_1(\tau_\star -) - \gamma_1 |U_1(\tau_\star)| - \eta_2 U_2(\tau_\star -) - \gamma_2 |U_2(\tau_\star)|} \mathbb{I}\{\tau_\star < \infty, J(\tau_\star) = j\} | \mathbf{U}(0) = (u_1, u_2), J(0) = i \right], \end{aligned} \tag{20}$$

[1] If both the premium rate and the initial surplus of one process are no less than their counterparts in the second process, the second process will ruin first with probability 1. In this case, the bivariate ruin problem reduces to a univariate ruin problem.

for $\delta, \eta_1, \gamma_1, \eta_2, \gamma_2 \geq 0$. As in the univariate case, when the LT parameters $\delta, \eta_1, \gamma_1, \eta_2, \gamma_2$ are all zeros, $\mathbf{m}_\delta(u_1, u_2; \tau_\star)$ reduces to the infinite-time ruin probability $\mathbf{\Psi}_\star(u_1, u_2) = \lim_{t\to\infty} \mathbf{\Psi}_\star(u_1, u_2, t)$, where $\mathbf{\Psi}_\star(u_1, u_2, t) = [\psi_{\star,ij}(u_1, u_2, t)]_{n\times n}$ is the finite-time ruin probability matrix defined as

$$\psi_{\star,ij}(u_1, u_2, t) = \mathbb{P}(\tau_\star \leq t, J(\tau_\star) = j|\mathbf{U}(0) = (u_1, u_2), J(0) = i). \tag{21}$$

Note that the quintuple LT $\mathbf{m}_\delta$ is informatively as general as the Gerber-Shiu function with an arbitrary penalty function. More importantly, the analysis of $\mathbf{m}_\delta$ in the two-dimensional MAP risk model (17) can be conducted through the triple LT $\mathbf{\Phi}_{\delta,s,r}$ in the univariate risk model. Indeed, given that $\Delta(t) = U_2(t) - U_1(t) = u_2 - u_1 + (c_2 - c_1)t$, we can reduce the dimension of $\mathbf{m}_\delta$, and arrive at

$$\begin{aligned} &m_{\delta,ij}(u_1, u_2; \tau_\star) \\ &= E[e^{-\delta\tau_\star - \xi_2\Delta(\tau_\star)}\, e^{-\eta U_1(\tau_\star-) - \gamma|U_1(\tau_\star)|}\mathbb{I}\{\tau_\star < \infty, J(\tau_\star) = j\}|U_1(0) = u_1, J(0) = i] \\ &= E[e^{-\delta\tau_\star + \xi_1\Delta(\tau_\star)}\, e^{-\eta U_2(\tau_\star-) - \gamma|U_2(\tau_\star)|}\mathbb{I}\{\tau_\star < \infty, J(\tau_\star) = j\}|U_2(0) = u_2, J(0) = i], \end{aligned}$$

where $\xi_i = \eta_i + \gamma_i$ $(i = 1, 2)$, $\eta = \eta_1 + \eta_2$, and $\gamma = \gamma_1 + \gamma_2$. Explicit expressions for (20) under the three ruin cases are stated in Proposition 3.1. The proof of these results can be found in Appendix B.

Proposition 3.1. *For the shared MAP risk process (17), if* $\delta \geq \xi_2(c_1 - c_2)$, $\mathbf{m}_\delta(u_1, u_2; \tau_\star)$ *admits the following decomposition:*

$$\begin{aligned} \mathbf{m}_\delta(u_1, u_2; \tau_{or}) &= e^{-\xi_2(u_2-u_1)}\mathbf{\Phi}_{\delta_1,\eta,\gamma}(u_1; c_1) - e^{-\delta T}\int_0^\infty \mathbf{H}(u_1, dy, T; c_1)\mathbf{\Phi}_{\delta_1,\eta,\gamma}(y; c_1) \\ &\quad + e^{-\delta T}\int_0^\infty \mathbf{H}(u_1, dy, T; c_1)\mathbf{\Phi}_{\delta_2,\eta,\gamma}(y; c_2), \end{aligned} \tag{22}$$

$$\begin{aligned} \mathbf{m}_\delta(u_1, u_2; \tau_{and}) &= e^{\xi_1(u_2-u_1)}\mathbf{\Phi}_{\delta_2,\eta,\gamma}(u_2; c_2) - e^{-\delta T}\int_0^\infty \mathbf{H}(u_1, dy, T; c_1)\mathbf{\Phi}_{\delta_2,\eta,\gamma}(y; c_2) \\ &\quad + e^{-\delta T}\int_0^\infty \mathbf{H}(u_1, dy, T; c_1)\mathbf{\Phi}_{\delta_1,\eta,\gamma}(y; c_1), \end{aligned} \tag{23}$$

$$\begin{aligned} \mathbf{m}_\delta(u_1, u_2; \tau_{sim}) &= e^{\xi_1(u_2-u_1)}\mathbf{\Phi}_{\delta_2,\eta,\gamma}(u_2; c_2) - e^{-\delta T}\int_0^\infty \mathbf{H}(u_2, dy, T; c_2)\mathbf{\Phi}_{\delta_2,\eta,\gamma}(y; c_2) \\ &\quad + e^{-\delta T}\int_0^\infty \mathbf{H}(u_2, dy, T; c_2)\mathbf{\Phi}_{\delta_1,\eta,\gamma}(y; c_1), \end{aligned} \tag{24}$$

where $\delta_1 = \delta - \xi_2(c_1 - c_2)$ *and* $\delta_2 = \delta + \xi_1(c_1 - c_2)$.

Also, expressions for the finite-time ruin probabilities $\mathbf{\Psi}_\star(u_1, u_2, t)$ can be obtained in a similar fashion. These results are presented in Proposition 3.2 (its proof can be found in Appendix C).

Proposition 3.2. *The finite-time ruin probabilities* $\mathbf{\Psi}_\star(u_1, u_2, t)$ *are given by*

$$\mathbf{\Psi}_{or}(u_1, u_2, t) = \mathbf{\Psi}(u_1, t \wedge T; c_1) + \mathbb{I}\{t > T\}\int_0^\infty \mathbf{H}(u_1, dy, T; c_1)\mathbf{\Psi}(y, t - T; c_2), \tag{25}$$

$$\begin{aligned} \mathbf{\Psi}_{and}(u_1, u_2, t) &= \mathbf{\Psi}(u_2, t; c_2) - \mathbb{I}\{t > T\}\int_0^\infty \mathbf{H}(u_1, dy, T; c_1)\mathbf{\Psi}(y, t - T; c_2) \\ &\quad + \mathbb{I}\{t > T\}\int_0^\infty \mathbf{H}(u_1, dy, T; c_1)\mathbf{\Psi}(y, t - T; c_1), \end{aligned} \tag{26}$$

$$\mathbf{\Psi}_{sim}(u_1, u_2, t) = \mathbf{\Psi}(u_2, t \wedge T; c_2) + \mathbb{I}\{t > T\}\int_0^\infty \mathbf{H}(u_2, dy, T; c_2)\mathbf{\Psi}(y, t - T; c_1). \tag{27}$$

Note that Equation (25) is a generalization of Avram et al. (2008b, Proposition 1) from the compound Poisson (CP) risk model to the MAP risk model. In the next section, we draw some additional conclusions for the CP risk model.

4. COMPOUND POISSON RISK MODEL REVISITED

In a CP risk model, the aggregate claim process $\{S(t), t \geq 0\}$ is assumed to be a compound Poisson process, that is, $\{N(t), t \geq 0\}$ is a Poisson process with arrival rate $\lambda > 0$, and the claim sizes $\{X_i\}_{i=1}^{\infty}$ (independent of $\{N(t), t \geq 0\}$) form a sequence of iid random variables with density p and mean μ. Note that the CP risk model is a special case of the MAP risk model, where the underlying CTMC J has only one state. For this insurance risk model, an explicit form for the density of the time to ruin (and hence, its finite time ruin probability $\psi(u, t; c)$) can be found in Dickson and Willmot (2005). Also, the hitting time distribution (14) is known to be

$$Z(t, x; c) = \begin{cases} 0, & t < \frac{x}{c}, \\ e^{-\lambda \frac{x}{c}} + \int_{\frac{x}{c}}^{t} \frac{x}{s} f(cs - x, s) ds, & t \geq \frac{x}{c}, \end{cases}$$

where $f(x, t) = \sum_{k=1}^{\infty} \frac{(\lambda t)^k}{k!} e^{-\lambda t} p^{*k}(x)$ is the time-t aggregate claim density (Gerber and Shiu 1998, Eq. 5.15) and p^{*k} denotes the k-fold convolution of p.[2] Moreover, Proposition 2.1 can be simplified to

$$\begin{aligned} H(u, dy, t; c) &\equiv \mathbb{P}(U(t) \in dy, t < \tau \,|U(0) = u) \\ &= \begin{cases} e^{-\lambda t}, & y = u + ct, \\ f(u + ct - y, t)\, dy, & ct \leq y < u + ct, \\ f(u + ct - y, t) dy - e^{-\lambda \frac{y}{c}} f\left(u + ct - y, t - \frac{y}{c}\right) dy & \\ \quad - \left(\int_{\frac{y}{c}}^{t} \frac{y}{s} f(cs - y, s) f(u + c(t - s), t - s) ds \right) dy, & 0 \leq y < ct. \end{cases} \end{aligned} \tag{28}$$

Equipped with explicit expressions for $\psi(u, t; c)$ and $H(u, y, t; c)$, the joint ruin probabilities $\psi_{\star}(u_1, u_2, t)$ can be computed through (25)–(27). Likewise, the quintuple LTs (22)–(24) can be restated as follows.

Proposition 4.1. *For the CP risk model, the quintuple LT $m_\delta(u_1, u_2; \tau_\star)$ admits the following decomposition:*

$$\begin{aligned} m_\delta(u_1, u_2; \tau_{or}) = e^{-\xi_2(u_2 - u_1)} \phi_{\delta_1,\eta,\gamma}(u_1; c_1) - e^{-\delta T} E[\phi_{\delta_1,\eta,\gamma}(U_1(T); c_1) \mathbb{I}\{\tau_1 > T\} | U_1(0) = u_1] \\ + e^{-\delta T} E[\phi_{\delta_2,\eta,\gamma}(U_1(T); c_2) \mathbb{I}\{\tau_1 > T\} | U_1(0) = u_1], \end{aligned} \tag{29}$$

$$\begin{aligned} m_\delta(u_1, u_2; \tau_{and}) = e^{\xi_1(u_2 - u_1)} \phi_{\delta_2,\eta,\gamma}(u_2; c_2) - e^{-\delta T} E[\phi_{\delta_2,\eta,\gamma}(U_1(T); c_2) \mathbb{I}\{\tau_1 > T\} | U_1(0) = u_1] \\ + e^{-\delta T} E[\phi_{\delta_1,\eta,\gamma}(U_1(T); c_1) \mathbb{I}\{\tau_1 > T\} | U_1(0) = u_1], \end{aligned} \tag{30}$$

$$\begin{aligned} m_\delta(u_1, u_2; \tau_{sim}) = e^{\xi_1(u_2 - u_1)} \phi_{\delta_2,\eta,\gamma}(u_2; c_2) - e^{-\delta T} E[\phi_{\delta_2,\eta,\gamma}(U_2(T); c_2) \mathbb{I}\{\tau_2 > T\} | U_2(0) = u_2] \\ + e^{-\delta T} E[\phi_{\delta_1,\eta,\gamma}(U_2(T); c_1) \mathbb{I}\{\tau_2 > T\} | U_2(0) = u_2]. \end{aligned} \tag{31}$$

In (29)–(31), the expectation

$$E[\phi_{\delta,\eta,\gamma}(U_j(T); c_k) \mathbb{I}\{\tau_j > T\} | U_j(0) = u] = \int_0^{\infty} H(u, dy, T; c_j) \phi_{\delta,\eta,\gamma}(y; c_k), \tag{32}$$

for $j, k = 1, 2$, can be evaluated from the defective distribution H defined in (28) and the knowledge of the triple LT $\phi_{\delta,\eta,\gamma}(u; c)$ in Landriault and Willmot (2009), although the calculations to evaluate m_δ may be quite intensive.

[2]In this section, we silently change all the *mathbold* matrix symbols to their normal forms, since only a single state space is involved in the CP risk model.

However, it is worth pointing out that the representation (32) can be evaluated more simply in some special cases, including when $\phi_{\delta,\eta,\gamma}(u;c)$ can be expressed as a combination of exponential functions in u. We refer the reader to Landriault and Willmot (2008), where some cases are highlighted under distributional assumptions on the claim size density p. For illustrative purposes, we focus the rest of the discussion on the joint LT of the ruin time and the deficits at ruin ($\eta = 0$), when the claim sizes $\{X_i\}_{i=1}^{\infty}$ are exponentially distributed with mean $\mu = 1/\beta$. In this context, Landriault and Willmot (2008, Corollary 7) have shown that

$$\phi_{\delta,0,\gamma}(u;c) = \nu_\delta(-R)e^{Ru}, \tag{33}$$

where ν_δ is a certain coefficient, and R is the negative root of the Lundberg equation $cs - \lambda - \delta + \lambda\beta/(\beta + s) = 0$. By substituting (33) into (32), it becomes crucial to find an efficient way to evaluate expectations of the form $E[e^{\theta U(T)}\mathbb{I}\{\tau > T\}|U(0) = u]$. For $\theta > -\beta$, let

$$\left.\frac{d\mathbb{P}^{(\theta)}}{d\mathbb{P}}\right| = e^{-\theta S(t)+\lambda t\frac{\theta}{\beta+\theta}} \triangleq L^{(\theta)}(t). \tag{34}$$

It is easy to verify that $\{L^{(\theta)}(t), t \geq 0\}$ is a martingale, and that

$$E[e^{\theta U(t)}\mathbb{I}\{\tau > t\}\,|U(0) = u] = e^{\theta u+(c-\frac{\lambda}{\beta+\theta})\theta t}\,\mathbb{P}^{(\theta)}\{\tau > t\,|U(0) = u\}, \tag{35}$$

where $\mathbb{P}^{(\theta)}\{\tau > t|U(0) = u\}$ denotes the t-year survival probability under $\mathbb{P}^{(\theta)}$. From Asmussen (2000, Section III.4), under the probability measure $\mathbb{P}^{(\theta)}$ with $\theta > -\beta$, the surplus process (6) is known to be another compound Poisson risk model with premium rate c, where the aggregate risk process $\underline{S}$ is a compound Poisson process with arrival rate $\lambda\beta/(\beta + \theta)$, and exponential claim sizes with mean $1/(\beta + \theta)$. Moreover, the t-year survival probability $\mathbb{P}^{(\theta)}$ in (35) under this risk model can be evaluated through the density of the time to ruin given by Drekic (2009),

$$\begin{aligned}&\mathbb{P}^{(\theta)}\{\tau > t|U(0) = u\}\\&= 1 - \int_0^t \frac{\lambda\beta}{\beta+\theta}e^{-\frac{\lambda\beta}{\beta+\theta}s}e^{-(\beta+\theta)(u+cs)}\left[I_0\left(2\sqrt{\lambda\beta c}\sqrt{s\,(s+u/c)}\right) - \frac{s}{s+u/c}I_2\left(2\sqrt{\lambda\beta c}\sqrt{s\,(s+u/c)}\right)\right]ds,\end{aligned}$$

where $I_k(z) = \sum_{j=0}^{\infty}\frac{(z/2)^{2j+k}}{j!(j+k)!}$ is the modified Bessel function of the first kind of order k.

Therefore, combining Proposition 4.1 and Eqs. (33) and (35), we can obtain simpler expressions for the quantities $m_\delta(u_1, u_2; \tau_{or})$, $m_\delta(u_1, u_2; \tau_{and})$ and $m_\delta(u_1, u_2; \tau_{sim})$. We remark that according to (4.6) in Gerber and Shiu (2005), it holds that $R > -\beta$, which is a necessary condition for the probability measure $\mathbb{P}^{(\theta)}$ to be well defined.

5. CAPITAL ALLOCATION APPLICATION

In this section, a capital allocation application is considered in relation to the joint ruin probabilities $\psi_{or}(u_1, u_2, t)$ in the proportional reinsurance framework (16). We aim to identify how to best allocate the total capital u among u_1 and u_2 so as to minimize the finite-time ruin probability $\psi_{or}(u_1, u_2, t)$. For simplicity, we perform the numerical analysis under the CP risk model.

Various allocation principles have been proposed over the years. A good review of the literature on this topic can be found in Dhaene et al. (2012). We mention a few capital allocation principles below related to the aggregate risk $Y(t) = \sum_{i=1}^{m} Y_i(t)$, where $\{Y_i(t), t > 0\}$ is the risk process of the ith business line.

1. *The covariance allocation principle*:

$$K_i = K \cdot \frac{Cov\,(Y_i(t), Y(t))}{Var\,(Y(t))},$$

2. *The VaR (haircut) allocation principle*:

$$K_i = K \cdot \frac{VaR_\alpha\,(Y_i(t))}{\sum_{j=1}^{m} VaR_\alpha\left(Y_j(t)\right)}, \qquad \text{where } 0 < \alpha < 1,$$

3. *The CTE allocation principle*:

$$K_i = K \cdot \frac{E\left[Y_i(t)\,|Y(t) > VaR_\alpha(Y(t))\right]}{E\left[Y(t)\,|Y(t) > VaR_\alpha(Y(t))\right]}.$$

As for ruin-based allocation methods, we also mention:

4. *The finite-time ruin allocation principle*:

$$K_i = K \cdot \frac{\psi\,(0, t; c_i)}{\sum_{j=1}^{m} \psi\left(0, t; c_j\right)},$$

where $\psi(0, t; c_i)$ is the finite-time ruin probability of the ith business line with zero initial surplus.

One may also look at an infinite-time ruin allocation principle by letting $t \to \infty$ in Principle 4. In general, the finite-time and infinite-time ruin allocation principles are not likely to minimize the joint-ruin probabilities of the multivariate surplus process. Yet they provide simple allocation methods that incorporate the likelihood of ruin for each business line into the capital allocation decision making.

In what follows, we compare the joint-ruin allocation principle (5) with the aforementioned four principles under the CP risk model.

Example 1. For the two surplus processes defined in (16), we assume that $\underline{S}$ is a compound Poisson process with claim arrival rate $\lambda = 0.15$ and claim sizes $\{X_i\}_{i=1}^{\infty}$ with density

$$p(y) = \frac{0.4}{15} e^{-\frac{y}{15}} + \frac{0.6}{10} e^{-\frac{y}{10}}, \qquad y > 0,$$

and mean $\mu = 12$. The level premium rates are $c_1^0 = 1.2$ and $c_2^0 = 1$ for the first and second business lines, respectively. With an equal quota share arrangement (i.e., $\omega_1 = \omega_2 = 0.5$), it follows that $c_1 = 2.4$ and $c_2 = 2$. We aim to numerically determine the optimal capital allocation between each business line when the total capital level is $u_1^0 + u_2^0 = K$ (or equivalently $u_1 + u_2 = 2K$ given that $u_k = u_k^0/\omega_k$ for $k = 1, 2$).

The numerical analysis below is performed using Mathematica. First, we illustrate in Figure 1 the behavior of $\psi_{or}(u_1, u_2, \infty)$ in u_1 when $2K = 40$. In Figure 2 the behavior of $\psi_{or}(u_1, u_2, 80)$ in u_1 is depicted when $2K = 60$. In both cases, we observe that the ruin probability first decreases as more capital is allocated to the first business line. The reversed trend is later observed where more capital allocated to the first business line leads to an increase in ruin probability.

Tables 1, 2, and 3 present the optimal allocation pairs (u_1^*, u_2^*) that minimize $\psi_{or}(u_1, u_2, 40)$, $\psi_{or}(u_1, u_2, 80)$, and $\psi_{or}(u_1, u_2, \infty)$, respectively, when the total capital level is set at $2K$ ($2K = 40$, 60, 80, 100, or 160). The optimal allocation pairs (u_1^*, u_2^*) are found

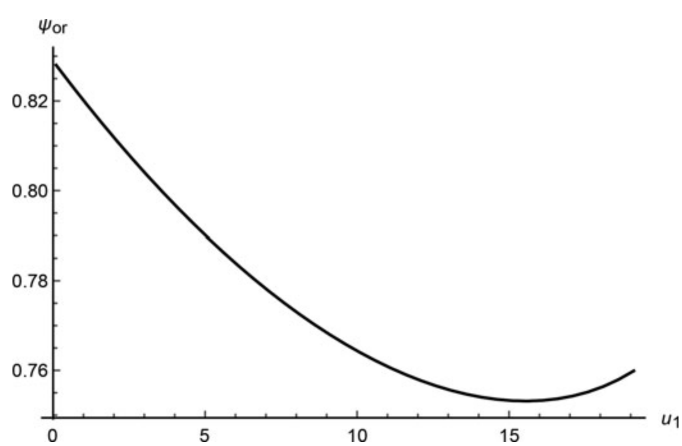

FIGURE 1. Joint-Ruin Probability $\psi_{or}(u_1, u_2, \infty)$ with $2K = 40$.

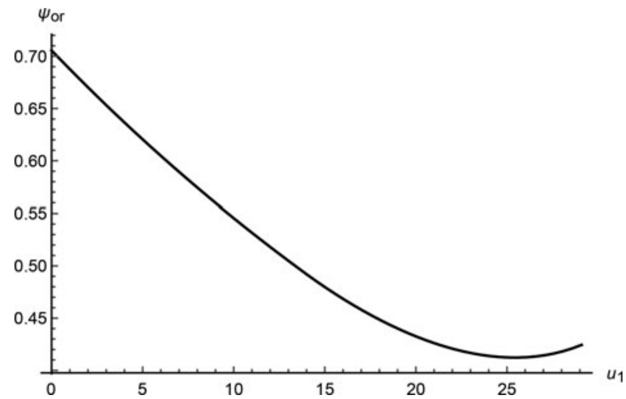

FIGURE 2. Joint-Ruin Probability $\psi_{or}(u_1, u_2, 80)$ with $2K = 60$.

numerically through a grid search method. As expected, we can see from all three tables that the minimum joint-ruin probability decreases as a function of K. Also, for a given K, we observe that more capital is allocated to the second business line as t moves from 40 to 80 to ∞. Intuitively, the impact of the safety loading on the joint-ruin probability tends to dominate in the long run. Given that $c_1 > c_2$ (and hence, the safety loading of business line 1 is greater than business line 2), it is more optimal to move capital from the first business line to the second as the time horizon increases.

Second, we compare the above allocation results for the joint-ruin probability $\psi_{or}(u_1, u_2, t)$ to the standard capital allocation methods described under Principles 1–4. The results are shown in Tables 4–9 for a time horizon $t = 40$, 80, and a total capital $2K = 40$, 100, and 160, respectively. Note that the net loss for each business line is defined as $Y_k(t) = \omega_k S(t) - c_k^0 t$ for $t \geq 0$ and $k = 1, 2$. For Principles 2 and 3, we chose $\alpha = 0.95$. For a given time horizon t and a total capital $2K$, the capital allocation results based on Principles 1–4 are presented in columns 2 and 3 of each table. Column 4 shows the percentage of capital allocated to each business line. We note that, for a given t, the percentage of total capital allocated to each business line does not vary with the total capital $2K$. Columns 5 and 6 provide the values of joint ruin probabilities using the capital allocation pairs in column 2. Since the premium rate does not affect the variance and covariance calculations in the proportional model, the capital allocation weights under the covariance principle are always equal to the proportional reinsurance weights $(\omega_1, \omega_2) = (0.5, 0.5)$, which yields the highest joint ruin probability. Except for this principle, it seems that all other allocation principles acknowledge the lower safety loading for the second business line and, thus, tend to allocate more capital to business line 2.

TABLE 1
Optimal Capital Allocations Minimizing $\psi_{or}(u_1, u_2, 40)$

$2K$	(u_1^*, u_2^*)	(u_1^0, u_2^0)	% Weight	$\psi_{or}(u_1^*, u_2^*, 40)$
40	(17.47, 22.53)	(8.74, 11.26)	(0.4368, 0.5632)	0.4130
60	(26.76, 33.24)	(13.38, 16.62)	(0.4460, 0.5540)	0.3051
80	(36.18, 43.82)	(18.09, 21.91)	(0.4523, 0.5477)	0.2227
100	(45.71, 54.29)	(22.85, 27.15)	(0.4571, 0.5429)	0.1607
160	(74.73, 85.27)	(37.37, 42.63)	(0.4671, 0.5329)	0.0570

TABLE 2
Optimal Capital Allocations Minimizing $\psi_{or}(u_1, u_2, 80)$

$2K$	(u_1^*, u_2^*)	(u_1^0, u_2^0)	% Weight	$\psi_{or}(u_1^*, u_2^*, 80)$
40	(16.67, 23.33)	(8.34, 11.66)	(0.4168, 0.5832)	0.5170
60	(25.44, 34.56)	(12.72, 17.28)	(0.4240, 0.5760)	0.4123
80	(34.31, 45.69)	(17.16, 22.84)	(0.4289, 0.5711)	0.3253
100	(43.30, 56.70)	(21.65, 28.35)	(0.4330, 0.5670)	0.2543
160	(70.96, 89.04)	(35.48, 44.52)	(0.4435, 0.5565)	0.1154

TABLE 3
Optimal Capital Allocations Minimizing $\psi_{or}(u_1, u_2, \infty)$

$2K$	(u_1^*, u_2^*)	(u_1^0, u_2^0)	% Weight	$\psi_{or}(u_1^*, u_2^*, \infty)$
40	(15.54, 24.46)	(7.77, 12.23)	(0.3885, 0.6115)	0.7531
60	(23.20, 36.80)	(11.60, 18.40)	(0.3867, 0.6133)	0.6887
80	(30.59, 49.41)	(15.30, 24.70)	(0.3824, 0.6176)	0.6289
100	(37.78, 62.22)	(18.89, 31.11)	(0.3778, 0.6222)	0.5733
160	(57.91, 92.09)	(33.96, 46.04)	(0.3619, 0.6381)	0.4311

TABLE 4
Optimal Capital Allocations under Various Principles with $(t, 2K) = (40, 40)$

Principles	(u_1^*, u_2^*)	(u_1^0, u_2^0)	% Weight	$\psi_{or}(u_1^*, u_2^*, 40)$	$\psi_{or}(u_1^*, u_2^*, \infty)$
P.1	(20, 20)	(10, 10)	(0.5, 0.5)	0.4266	0.7646
P.2	(17.53, 22.47)	(8.77, 11.23)	(0.4383, 0.5617)	0.4130	0.7550
P.3	(18.10, 21.90)	(9.05, 10.95)	(0.4525, 0.5475)	0.4137	0.7564
P.4	(18.99, 21.01)	(9.49, 10.51)	(0.4746, 0.5254)	0.4175	0.7594

Overall, there seems to be no consistent pattern to report among the allocation principles (except that the various principles lead to quite different capital allocation schemes). In this particular example, we can see that none of the allocation principles can be viewed as dominating the others to achieve the lowest finite-time joint-ruin probability. For instance, the haircut principle yields the lowest $\psi_{or}(u_1, u_2, t)$ when $t = 40$ and $2K = 40$ (which is relatively close to the global minimum of 0.4130). The CTE principle provides a capital allocation with the lowest $\psi_{or}(u_1, u_2, t)$ whenever $t = 80$ and when $(t, 2K) = (40, 100)$. Finally, the finite-time ruin allocation principle provides the lowest $\psi_{or}(u_1, u_2, t)$ $(= 0.0574)$ in the case of $(t, 2K) = (40, 160)$. Yet in this example, the allocations under the CTE principle are typically close to the ones that minimize the finite-time joint-ruin probability. These interesting observations should motivate the risk community to further explore the joint-ruin–based allocation principles and in particular their connections with other existing principles.

TABLE 5
Optimal Capital Allocations under Various Principles with $(t, 2K) = (40, 100)$

Principles	(u_1^*, u_2^*)	(u_1^0, u_2^0)	% Weight	$\psi_{or}(u_1^*, u_2^*, 40)$	$\psi_{or}(u_1^*, u_2^*, \infty)$
P.1	(50, 50)	(25, 25)	(0.5, 0.5)	0.1734	0.6015
P.2	(43.83, 56.17)	(21.91, 28.09)	(0.4383, 0.5617)	0.1633	0.5797
P.3	(45.25, 54.75)	(22.63, 27.37)	(0.4525, 0.5475)	0.1609	0.5833
P.4	(47.46, 52.54)	(23.73, 26.27)	(0.4746, 0.5254)	0.1629	0.5905

TABLE 6
Optimal Capital Allocations under Various Principles with $(t, 2K) = (40, 160)$

Principles	(u_1^*, u_2^*)	(u_1^0, u_2^0)	% Weight	$\psi_{or}(u_1^*, u_2^*, 40)$	$\psi_{or}(u_1^*, u_2^*, \infty)$
P.1	(80, 80)	(40, 40)	(0.5, 0.5)	0.0639	0.4913
P.2	(70.13, 89.87)	(35.06, 44.94)	(0.4383, 0.5617)	0.0638	0.4435
P.3	(72.40, 87.60)	(36.20, 43.80)	(0.4525, 0.5475)	0.0589	0.4489
P.4	(75.94, 84.06)	(37.97, 42.03)	(0.4746, 0.5254)	0.0574	0.4592

TABLE 7
Optimal Capital Allocations under Various Principles with $(t, 2K) = (80, 40)$

Principles	(u_1^*, u_2^*)	(u_1^0, u_2^0)	% Weight	$\psi_{or}(u_1^*, u_2^*, 80)$	$\psi_{or}(u_1^*, u_2^*, \infty)$
P.1	(20, 20)	(10, 10)	(0.5, 0.5)	0.5329	0.7646
P.2	(15.82, 24.18)	(7.91, 12.09)	(0.3956, 0.6044)	0.5178	0.7532
P.3	(17.16, 22.84)	(8.58, 11.42)	(0.4289, 0.5711)	0.5173	0.7544
P.4	(18.85, 21.15)	(9.42, 10.58)	(0.4711, 0.5288)	0.5232	0.7589

TABLE 8
Optimal Capital Allocations under Various Principles with $(t, 2K) = (80, 100)$

Principles	(u_1^*, u_2^*)	(u_1^0, u_2^0)	% Weight	$\psi_{or}(u_1^*, u_2^*, 80)$	$\psi_{or}(u_1^*, u_2^*, \infty)$
P.1	(50, 50)	(25, 25)	(0.5, 0.5)	0.2762	0.6015
P.2	(39.56, 60.44)	(19.78, 30.22)	(0.3956, 0.6044)	0.2605	0.5739
P.3	(42.89, 57.11)	(21.45, 28.55)	(0.4289, 0.5711)	0.2543	0.5779
P.4	(47.11, 52.88)	(23.56, 26.44)	(0.4711, 0.5288)	0.2613	0.5892

TABLE 9
Optimal Capital Allocations under Various Principles with $(t, 2K) = (80, 160)$

Principles	(u_1^*, u_2^*)	(u_1^0, u_2^0)	% Weight	$\psi_{or}(u_1^*, u_2^*, 80)$	$\psi_{or}(u_1^*, u_2^*, \infty)$
P.1	(80, 80)	(40, 40)	(0.5, 0.5)	0.1327	0.4913
P.2	(63.30, 96.70)	(31.65, 48.35)	(0.3956, 0.6044)	0.1297	0.4333
P.3	(68.62, 91.38)	(34.31, 45.69)	(0.4289, 0.5711)	0.1166	0.4405
P.4	(75.38, 84.62)	(37.69, 42.31)	(0.4711, 0.5288)	0.1198	0.4574

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APPENDIX A. PROOF OF PROPOSITION 2.1

By definition,

$$\begin{aligned}
H_{ij}(u, y, t; c) &= \mathbb{P}(U(t) \le y, J(t) = j | U(0) = u, J(0) = i) \\
&\quad - \mathbb{P}(U(t) \le y, \tau \le t, J(t) = j | U(0) = u, J(0) = i) \\
&= \mathbb{P}(S(t) \ge u + ct - y, J(t) = j | J(0) = i) \\
&\quad - \int_0^y \mathbb{P}(U(t) \in dz, \tau \le t, J(t) = j | U(0) = u, J(0) = i) \\
&= \overline{F}_{ij}(u + ct - y) - \int_0^y \mathbb{P}(U(t) \in dz, \tau \le t, J(t) = j | U(0) = u, J(0) = i).
\end{aligned} \tag{A.1}$$

By a one-to-one sample path mapping between $\{(U(s), J(s)), 0 \le s < t\}$ when $U(0) = u$ and $U(t) = z$ and $\{(\widehat{U}(s), \widehat{J}(s)), 0 \le s < t\}$ when $\widehat{U}(0) = u$ and $\widehat{U}(t) = z$, it is clear that

$$\begin{aligned}
&\mathbb{P}\left(U(t) \in dz, \tau \le t, J(t) = j \,|U(0) = u, J(0) = i\right) \\
&\quad = \mathbb{P}\left(\widehat{U}(t) \in dz, \widehat{\tau}^+_{z+u} \le t, \widehat{J}(t) = j \,\middle|\widehat{U}(0) = u, \widehat{J}(0) = i\right),
\end{aligned}$$

for $z > 0$ where $\widehat{\tau}^+_{z+u} = \inf\{0 \le s < t : \widehat{U}(s) = z + u\}$. By further conditioning on $\widehat{\tau}^+_{z+u}$ and $\widehat{J}(\widehat{\tau}^+_{z+u})$, one finds that

$$\begin{aligned}
&\mathbb{P}(U(t) \in dz, t \ge \tau, J(t) = j | U(0) = u, J(0) = i) \\
&= \left(\int_0^t \sum_{k=1}^n \mathbb{P}\left(\widehat{\tau}^+_{z+u} \in ds, \widehat{J}\left(\widehat{\tau}^+_{z+u}\right) = k \,\middle|\, \widehat{U}(0) = u, \widehat{J}(0) = j\right) \widehat{f}_{ki}(u + c(t-s), t-s) \right) dz \\
&= \left(\int_{\frac{z}{c} \wedge t}^t \sum_{k=1}^n \widehat{Z}_{jk}(ds, z; c) \widehat{f}_{ki}(u + c(t-s), t-s) \right) dz \qquad \text{(A.2)}
\end{aligned}$$

Substituting (A.2) into (A.1) easily leads to (15).

APPENDIX B. PROOF OF PROPOSITION 3.1

We silently assume throughout this proof that $U_1(0) = u_1$ and $U_2(0) = u_2$, unless otherwise stated. Given that $\tau_{or} = \tau_1$ if $\tau_1 \le T$ and $\tau_{or} = \tau_2$ if $\tau_1 > T$, we have

$$\begin{aligned}
&m_{\delta,ij}(u_1, u_2; \tau_{or}) \\
&= E[e^{-\delta\tau_1 - \xi_1 \Delta(\tau_1)} e^{-\eta U_1(\tau_1 -) - \gamma |U_1(\tau_1)|} \mathbb{I}\{\tau_1 \le T, J(\tau_1) = j\} | J(0) = i] \\
&\quad + E[e^{-\delta\tau_2 + \xi_2 \Delta(\tau_2)} e^{-\eta U_2(\tau_2 -) - \gamma |U_2(\tau_2)|} \mathbb{I}\{\tau_2 < \infty, J(\tau_2) = j\} \mathbb{I}\{\tau_1 > T\} | J(0) = i] \\
&= e^{-\xi_2(u_2 - u_1)} E[e^{-\delta_1\tau_1 - \eta U_1(\tau_1 -) - \gamma |U_1(\tau_1)|} \mathbb{I}\{\tau_1 < \infty, J(\tau_1) = j\} | J(0) = i] \\
&\quad - E[e^{-\delta\tau_1 - \xi_1 \Delta(\tau_1)} e^{-\eta U_1(\tau_1 -) - \gamma |U_1(\tau_1)|} \mathbb{I}\{\tau_1 < \infty, J(\tau_1) = j\} \mathbb{I}\{\tau_1 > T\} | J(0) = i] \\
&\quad + E[e^{-\delta\tau_2 + \xi_2 \Delta(\tau_2)} e^{-\eta U_2(\tau_2 -) - \gamma |U_2(\tau_2)|} \mathbb{I}\{\tau_2 < \infty, J(\tau_2) = j\} \mathbb{I}\{\tau_1 > T\} | J(0) = i], \qquad \text{(B.1)}
\end{aligned}$$

where $\delta_1 = \delta - \xi_2(c_1 - c_2) \ge 0$ and $\xi_i = \eta_i + \gamma_i$ $(i = 1, 2)$. Conditioning on the surplus level and the state at time T (with no ruin occurs before T), and utilizing the matrix renewal property, we have

$$\begin{aligned}
&E[e^{-\delta\tau_1 - \xi_1 \Delta(\tau_1)} e^{-\eta U_1(\tau_1 -) - \gamma |U_1(\tau_1)|} \mathbb{I}\{\tau_1 < \infty, J(\tau_1) = j\} \mathbb{I}\{\tau_1 > T\} \| J(0) = i] \\
&= \sum_{l=1}^n \int_0^\infty H_{il}(u_1, dy, T; c_1) \\
&\quad \times e^{-\delta T} E[e^{-\delta_1\tau_1} e^{-\eta U_1(\tau_1 -) - \gamma |U_1(\tau_1)|} \mathbb{I}\{\tau_1 < \infty, J(\tau_1) = j\} \| U_1(0) = y, J(0) = l] \\
&= e^{-\delta T} \sum_{l=1}^n \int_0^\infty H_{il}(u_1, dy, T; c_1) \times \phi_{\delta_1,\eta,\gamma,lj}(y; c_1). \qquad \text{(B.2)}
\end{aligned}$$

Recall that $\tau_1 > T$ implies $\tau_2 > T$ and thus $\tau_2 = \tau_2(T)$. Using virtually the same arguments, one finds

$$\begin{aligned}
&E[e^{-\delta\tau_2 + \xi_2 \Delta(\tau_2)} e^{-\eta U_2(\tau_2 -) - \gamma |U_2(\tau_2)|} \mathbb{I}\{\tau_2 < \infty, J(\tau_2) = j\} \mathbb{I}\{\tau_1 > T\} \| J(0) = i] \\
&= e^{-\delta T} \sum_{l=1}^n \int_0^\infty H_{il}(u_1, dy, T; c_1)\, \phi_{\delta_2,\eta,\gamma,lj}(y; c_2), \qquad \text{(B.3)}
\end{aligned}$$

where $\delta_2 = \delta + \xi_1(c_1 - c_2) \ge 0$. Substituting (B.2) and (B.3) into (B.1) leads to the matrix representation (22). Second, given that $\{\tau_1, \tau_2\} = \{\tau_{or}, \tau_{and}\}$, one easily obtain the following identity:

$$\mathbf{m}_\delta(u_1, u_2; \tau_{or}) + \mathbf{m}_\delta(u_1, u_2; \tau_{and}) = \mathbf{m}_\delta(u_1, u_2; \tau_1) + \mathbf{m}_\delta(u_1, u_2; \tau_2)$$
$$= e^{-\xi_2(u_2-u_1)}\mathbf{\Phi}_{\delta_1,\eta,\gamma}(u_1; c_1) + e^{\xi_1(u_2-u_1)}\mathbf{\Phi}_{\delta_2,\eta,\gamma}(u_2; c_2).$$

Equation (23) follows immediately from (22). Finally, from (19) for τ_{sim}, Eq. (24) can be validated using similar arguments.

APPENDIX C. PROOF OF PROPOSITION 3.2

For τ_{or}, we know that $\tau_{or} = \tau_1$ if $\tau_1 \le T$ and $\tau_{or} = \tau_2$ if $\tau_1 > T$. If $t \le T$, then

$$\psi_{or,ij}(u_1, u_2, t) = \mathbb{P}(\tau_{or} \le t, J(\tau_{or}) = j | U_1(0) = u_1, U_2(0) = u_2, J(0) = i)$$
$$= \mathbb{P}(\tau_1 \le t, J(\tau_1) = j | U_1(0) = u_1, U_2(0) = u_2, J(0) = i)$$
$$= \psi_{ij}(u_1, t; c_1).$$

If $t > T$, then

$$\psi_{or,ij}(u_1, u_2, t) = \mathbb{P}(\tau_1 \le t, \tau_1 \le T, J(\tau_1) = j | U_1(0) = u_1, U_2(0) = u_2, J(0) = i)$$
$$+ \mathbb{P}(\tau_2 \le t, \tau_1 > T, J(\tau_2) = j | U_1(0) = u_1, U_2(0) = u_2, J(0) = i)$$
$$= \psi_{ij}(u_1, T; c_1) + \mathbb{P}(\tau_2 \le t, \tau_1 > T, J(\tau_2) = j | U_1(0) = u_1, J(0) = i).$$

Following the same arguments as in Proposition 3.1, we can show that

$$\mathbb{P}(\tau_2 \le t, \tau_1 > T, J(\tau_2) = j | U_1(0) = u_1, J(0) = i) = \sum_{l=1}^{n} \int_0^\infty H_{il}(u_1, dy, T; c_1)\psi_{lj}(y, t - T; c_2).$$

This completes the proof of (25). Equation (26) can be obtained through the following identity:

$$\psi_{or,ij}(u_1, u_2, t) + \psi_{and,ij}(u_1, u_2, t) = \psi_{ij}(u_1, t; c_1) + \psi_{ij}(u_2, t; c_2).$$

Finally, Eq. (27) can be proved in a similar way to Eq. (25).