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Notions of multivariate dependence and their applications in optimal portfolio selections with dependent risks

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HIGHLIGHTS

- We propose the new dependence notions of LWSAI and WSAI.
- We present properties of the new dependence notions of LWSAI and WSAI.
- Dependent risks in portfolio section problems are modeled by the notions of dependence.
- Many existing studies on optimal portfolio selections are extended to more general dependent risks.

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ABSTRACT

In this paper, we propose the dependence notions of weakly stochastic arrangement increasing through left tail probability (LWSAI) and weakly stochastic arrangement increasing (WSAI) to model multivariate dependent risks. We derive properties and characterizations of these new notions and show that many existing dependence structures are the special cases of these notions of dependence. We apply the dependence notions of LWSAI and WSAI to the problem of optimal portfolio selections with dependent risks and generalize many existing studies.

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1. Introduction

The problem of optimal portfolio selections has been an interesting research topic in insurance and finance. In most of the existing studies on optimal portfolio selections, assets or risks in an investment/insurance portfolio are assumed to be independent or to have some special dependence structures such as exchangeable assets or comonotonic risks. However, even with the special dependence structures on the assets or risks, the solutions to the problem of optimal portfolio selections are usually not available if the joint distribution of the assets are unknown. Therefore, many studies investigated the properties of the solutions to the problem of optimal portfolio selections when assets in a portfolio have some dependence structures but their joint distribution is unknown. In particular, ordering optimal proportions or

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allocations has been a challenging problem due to the dependence of the risks and the unavailability of the joint distribution of the risks in a portfolio selection problem. Such researches in insurance and finance can be found in [6,12,11,7,3,2,4,13,19], and references therein. The optimal portfolio selection problems in these studies can be generally formulated as follows.

Let $X_1, \ldots, X_n$ be random variables, representing the stochastic return rates of n different assets in an investment portfolio of an investor. Let a_i be the investment weight on asset i, then $\sum_{i=1}^n a_i = 1$. Furthermore, we assume that $0 \le a_i \le 1$ for $i = 1, \ldots, n$ or short positions are not allowed. Thus, at the end of the investment term, the total return rate is $\sum_{i=1}^n a_i X_i$. The investor wants to choose a portfolio $(a_1, \ldots, a_n)$ so as to maximize the expected utility of his total return. Mathematically, we want to study the following problem:

$$\max_{(a_1,\ldots,a_n)\in\mathcal{A}_n} \mathbb{E}\left[u\left(\sum_{k=1}^n a_k X_k\right)\right], \tag{1.1}$$

where $\mathcal{A}_n$ is the collection of all the possible portfolios and is defined as

$$\mathcal{A}_n = \left\{(a_1, \ldots, a_n) \ge 0 : \sum_{k=1}^n a_k = 1\right\},$$

and u is a utility function.

Landsberger and Meilijson [12] investigated Problem (1.1) for two independent assets. Kijima and Ohnishi [11] generalized the studies of Landsberger and Meilijson [12] by introducing a dependence structure between two assets. Hennessy and Lapan [7] used the Archimedean copula to model the dependence between multiple assets and ordered the optimal allocations. Recently, Li and You [13] had generalized the studies of Hennessy and Lapan [7] by introducing a multivariate dependence structure defined through the arrangement increasing property of the joint density function of the assets. On the other hand, Hadar and Seo [6] discussed mutually independent assets ordered by the usual stochastic order. With certain utility functions u, they managed to order the optimal investment weights for Problem (1.1).

In practice, default may occur on an asset or bond investment. An interesting extension of Model (1.1) is to incorporate default risks. Assume that the investor faces the risk of default on each asset or bond. We use Bernoulli random variable I_k to indicate the default event of the asset or bond k. The return rate X_k is realized only if $I_k = 1$, otherwise the return rate of the asset k is 0 or default occurs on the asset k. Therefore, the total return rate with default risks is $\sum_{k=1}^n a_k X_k I_k$, and the optimal portfolio selection problem becomes

$$\max_{(a_1,\ldots,a_n)\in\mathcal{A}_n} \mathbb{E}\left[u\left(\sum_{k=1}^n a_k X_k I_k\right)\right]. \tag{1.2}$$

Model (1.2) was proposed and studied by Cheung and Yang [3]. They assumed that $(X_1, \ldots, X_n)$ are exchangeable and managed to order the optimal allocations with certain assumptions on the default indicators $I_1, \ldots, I_n$. Later, Chen and Hu [2] studied Problem (1.2) with independent return rates $X_1, \ldots, X_n$ and the certain utility functions u and generalized the results of Hadar and Seo [6].

Furthermore, Cheung and Yang [4] proposed a mixture risk model for the problem of optimal portfolio sections. In this model, we assume that there is a group of fundamental risks $\{X_j : j \in J\}$ in the financial market, where J is an arbitrary index set. These fundamental risks can be interpreted as stochastic return rates under different investment environments. Although the index of the fundamental risks can be uncountably infinite, we assume J to be a finite set $\{1, 2, \ldots, m\}$ for simplicity in this paper. We further assume that the return rate of any security in the market is a mixture of these fundamental risks and associate different securities with different mixing random variables $M_1, \ldots, M_n$. Mathematically, we denote the return rates of n assets or securities in an investment portfolio by

$$X_{M_i} = \sum_{j\in J} X_j \,\mathbb{I}\{M_i = j\} = \sum_{j=1}^m X_j \,\mathbb{I}\{M_i = j\}, \quad i = 1, \ldots, n,$$

where $\{M_i, i = 1, \ldots, n\}$ are random variables taking values in $J = \{1, 2, \ldots, m\}$. In this context, the optimal portfolio selection problem is formulated as

$$\max_{(a_1,\ldots,a_n)\in\mathcal{A}_n} \mathbb{E}\left[u\left(\sum_{k=1}^n a_k X_{M_k}\right)\right]. \tag{1.3}$$

To study Problem (1.3), Cheung and Yang [4] assumed that $X_1, \ldots, X_n$ are comonotonic with $X_1 \le_{st} \cdots \le_{st} X_n$. This mixture risk model with the same assumption was also adopted by Hu and Wang [8] to study some allocation problems in insurance.

We point out that, in most of the existing studies of Problems (1.1)–(1.3), the return rates of assets are assumed to be independent or comonotonic or exchangeable. Motivated by this observation, this paper proposes new dependence notions to model more general dependent risks and use these notions of dependence to unify and extend the existing studies on the optimal portfolio selection problems with dependent risks. The rest of the paper is organized as follows.

In Section 2, we present preliminaries about stochastic orders and define the dependence notions of LWSAI and WSAI. We also recall the dependence notions of stochastic arrangement increasing (SAI) and weakly stochastic arrangement increasing through right tail probability (RWSAI) defined by Cai and Wei [1] and cite some results on SAI and WSAI, which will be used in this paper. In Section 3, we derive properties of LWSAI and WSAI and give characterizations of LWSAI. In Section 4, we show how to construct LWSAI random vectors through certain copulas. By doing so, we demonstrate that the dependence structure studied in [7] is a special case of LWSAI. In Section 5, we use LWSAI and WSAI random vectors to model the return rates and restudy the optimal portfolio selection problems (1.1)–(1.3) with more general dependent risks. These studies generalize the results of Hennessy and Lapan [7], Chen and Hu [2], and Cheung and Yang [3,4]. In Section 6, we give concluding remarks.

2. Preliminaries and the dependence notions of LWSAI and WSAI

For the sake of convenience, we use the following notations throughout the paper. We denote an n-dimensional real-valued vector $(x_1, \ldots, x_n)$ by $\mathbf{x}$ and an n-dimensional random vector $(X_1, \ldots, X_n)$ by $\mathbf{X}$. We use $S(\mathbf{X})$ or $S(X_1, \ldots, X_n)$ to denote the support of random vector $\mathbf{X}$, which means that $\mathbb{P}\{\mathbf{X} \in S(\mathbf{X})\} = \mathbb{P}\{(X_1, \ldots, X_n) \in S(X_1, \ldots, X_n)\} = 1$. For any set $K = \{i_1, \ldots, i_k\} \subset \{1, \ldots, n\}$ where $1 \le i_1 < \cdots < i_k \le n$ and $k = 1, \ldots, n$, we denote $\mathbf{x}_K = (x_{i_1}, \ldots, x_{i_k})$ and $\mathbf{X}_K = (X_{i_1}, \ldots, X_{i_k})$. In particular, for any $1 \le i < j \le n$, if $K = \{i, j\}$, we write $\bar{K} = \overline{ij} = \{1, \ldots, n\} \setminus \{i, j\}$, $\mathbf{X}_K = \mathbf{X}_{ij}$, $\mathbf{X}_{\bar{K}} = \mathbf{X}_{\overline{ij}}$, $\mathbf{x}_K = \mathbf{x}_{ij}$, and $\mathbf{x}_{\bar{K}} = \mathbf{x}_{\overline{ij}}$.

Let $\pi = (\pi(1), \ldots, \pi(n))$ be any permutation of $\{1, \ldots, n\}$, we define $\pi(\mathbf{x}) = (x_{\pi(1)}, \ldots, x_{\pi(n)})$. For any $1 \le i \ne j \le n$, we denote the special permutation of transposition by $\pi_{ij} = (\pi_{ij}(1), \ldots, \pi_{ij}(n))$, where $\pi_{ij}(k) = k$ for $k \ne i, j$ and $\pi_{ij}(i) = j$, $\pi_{ij}(j) = i$.

We first recall definitions of comonotonicity and some stochastic orders. Readers are referred to Dhaene et al. [5] for a detailed discussion on comonotonicity, and [17] for a comprehensive study on various stochastic orders.

Definition 2.1. A random vector $(X_1, \ldots, X_n)$ is said to be *comonotonic*, if

$$\mathbb{P}\{X_1 \le x_1, \ldots, X_n \le x_n\} = \min\{\mathbb{P}\{X_1 \le x_1\}, \ldots, \mathbb{P}\{X_n \le x_n\}\},$$

for any $(x_1, \ldots, x_n) \in \mathbb{R}^n$. □

Definition 2.2. Let X and Y be two random variables with distribution functions $F_X(x) = \mathbb{P}\{X \le x\} = 1 - \bar{F}_X(x)$ and $F_Y(y) = \mathbb{P}\{Y \le y\} = 1 - \bar{F}_Y(y)$ and probability density functions (or probability mass function in discrete cases) $f_X(x)$ and $f_Y(y)$.

(i) We say that X is smaller than Y in *usual stochastic order*, denoted as $X \le_{st} Y$, if $\bar{F}_X(x) \le \bar{F}_Y(x)$ for all $x \in \mathbb{R}$.
(ii) We say that X is smaller than Y in *reversed hazard rate order*, denoted as $X \le_{rh} Y$, if $F_Y(x)/F_X(x)$ is increasing in $x \in \{x : F_X(x) > 0\}$.
(iii) We say that X is smaller than Y in *likelihood ratio order*, denoted as $X \le_{lr} Y$, if $f_Y(x)/f_X(x)$ is increasing in $x \in \{x : f_X(x) > 0\}$. □

These stochastic orders only involve comparison of marginal distributions. Shanthikumar and Yao [18] incorporated interdependence into comparisons of two random variables by introducing bivariate stochastic orders. Cai and Wei [1] generalized some of the bivariate stochastic orders to multivariate cases. In the following, we recall some of their notions.

Consider function $g : \mathbb{R}^n \to \mathbb{R}$. Denote $\Delta_{ij} g(x_1, \ldots, x_n) = g(x_1, \ldots, x_n) - g(\pi_{ij}(x_1, \ldots, x_n))$. For $1 \le i < j \le n$, define

$$\mathcal{G}^{ij}_{sai}(n) = \{g(x_1, \ldots, x_n) : \Delta_{ij} g(x_1, \ldots, x_n) \ge 0 \text{ for any } x_i \le x_j\}, \tag{2.1}$$

$$\mathcal{G}^{ij}_{lwsai}(n) = \{g(x_1, \ldots, x_n) : \Delta_{ij} g(x_1, \ldots, x_n) \text{ is decreasing in } x_i \le x_j\}, \tag{2.2}$$

$$\mathcal{G}^{ij}_{rwsai}(n) = \{g(x_1, \ldots, x_n) : \Delta_{ij} g(x_1, \ldots, x_n) \text{ is increasing in } x_j \ge x_i\}, \tag{2.3}$$

$$\mathcal{G}^{ij}_{wsai}(n) = \{g(x_1, \ldots, x_n) : \Delta_{ij} g(x_1, \ldots, x_n) \text{ is increasing in } x_j\}. \tag{2.4}$$

The class $\mathcal{G}^{ij}_{sai}(n)$ describes the arrangement increasing property of multivariate functions. Readers are referred to Marshall et al. [15] for more discussions about the arrangement increasing property. It is easy to verify that $\mathcal{G}^{ij}_{wsai}(n) \subset \mathcal{G}^{ij}_{lwsai}(n)$ $(\mathcal{G}^{ij}_{rwsai}(n)) \subset \mathcal{G}^{ij}_{sai}(n)$. The following definitions of SAI and RWSAI were proposed by Cai and Wei [1].

Definition 2.3. A random vector $\mathbf{X} = (X_1, \ldots, X_n)$ or its joint distribution is said to be *stochastic arrangement increasing* (SAI) if $\mathbb{E}[g(\mathbf{X})] \ge \mathbb{E}[g(\pi_{ij}(\mathbf{X}))]$ for any $1 \le i < j \le n$ and $g(x_1, \ldots, x_n) \in \mathcal{G}^{ij}_{sai}(n)$ such that the expectations exist.

A random vector $\mathbf{X} = (X_1, \ldots, X_n)$ or its joint distribution is said to be *weakly stochastic arrangement increasing through right tail probability* (RWSAI) if $\mathbb{E}[g(\mathbf{X})] \ge \mathbb{E}[g(\pi_{ij}(\mathbf{X}))]$ for any $1 \le i < j \le n$ and $g(x_1, \ldots, x_n) \in \mathcal{G}^{ij}_{rwsai}(n)$ such that the expectations exist. □

For a random vector with a joint density function, Cai and Wei [1] developed an equivalent characterization of SAI through the arrangement increasing property of the joint density function, as shown in Theorem 3.6 and Remark 3.7 of Cai and Wei [1]. We point out that, for a discrete random vector, we can derive a similar characterization through the joint probability mass function, as shown by the following proposition.

Proposition 2.4. *Let $(X_1, \ldots, X_n)$ be a discrete random vector with joint probability mass function $p(x_1, \ldots, x_n) = \mathbb{P}\{X_1 = x_1, \ldots, X_n = x_n\}$. The random vector $(X_1, \ldots, X_n)$ is SAI if and only if*

$$p(x_1, \ldots, x_n) \geq p(\pi_{ij}(x_1, \ldots, x_n)),$$

for any $1 \leq i < j \leq n$ and $(x_1, \ldots, x_n) \in S(X_1, \ldots, X_n)$ such that $x_i \leq x_j$.

Proof. We only prove the bivariate case. The proof for the multivariate case is similar to that for the bivariate case.

First, assume that (X_1, X_2) is SAI. For any $(x_1, x_2) \in S(X_1, X_2)$ such that $x_1 \leq x_2$, there exists $A \subset \{(x, y) \in \mathbb{R}^2 \,|\, x \leq y\}$ satisfying $S(X_1, X_2) \cap A = \{(x_1, x_2)\}$. Define $h(u, v) = \mathbb{I}\{(u, v) \in A\}$, it is easy to verify that $h(u, v) \in \mathcal{G}^{12}_{sai}(2)$. According to the definition of SAI, we have $E[h(X_1, X_2)] \geq \mathbb{E}[h(X_2, X_1)]$, which implies that $p(x_1, x_2) \geq p(x_2, x_1)$.

Now assume that $p(x_1, x_2) \geq p(x_2, x_1)$ for any $x_1 \leq x_2$. Consider any function $g \in \mathcal{G}^{12}_{sai}(2)$, i.e., $g(x_1, x_2) \geq g(x_2, x_1)$ for any $x_1 \leq x_2$. Note that

$$\begin{aligned}
\mathbb{E}[g(X_1, X_2)] &= \sum_{x_1 < x_2} \big(p(x_1, x_2)g(x_1, x_2) + p(x_2, x_1)g(x_2, x_1)\big) + \sum_{x_1 = x_2} p(x_1, x_2)g(x_1, x_2) \\
&\geq \sum_{x_1 < x_2} \big(p(x_1, x_2)g(x_2, x_1) + p(x_2, x_1)g(x_1, x_2)\big) + \sum_{x_1 = x_2} p(x_1, x_2)g(x_1, x_2) \\
&= \mathbb{E}[g(X_2, X_1)].
\end{aligned}$$

The inequality follows from the fact that $ac + bd \geq ad + bc$ for $a \geq b,\ c \geq d$. $\square$

Definition 2.5. A random vector $\mathbf{X} = (X_1, \ldots, X_n)$ or its joint distribution is said to be *weakly stochastic arrangement increasing through left tail probability* (LWSAI), if $\mathbb{E}\,[g(\mathbf{X})] \geq \mathbb{E}\,[g(\pi_{ij}(\mathbf{X}))]$ for any $1 \leq i < j \leq n$ and $g(x_1, \ldots, x_n) \in \mathcal{G}^{ij}_{lwsai}(n)$ such that the expectations exist.

A random vector $\mathbf{X} = (X_1, \ldots, X_n)$ or its joint distribution is said to be *weakly stochastic arrangement increasing* (WSAI), if $\mathbb{E}\,[g(\mathbf{X})] \geq \mathbb{E}\,[g(\pi_{ij}(\mathbf{X}))]$ for any $1 \leq i < j \leq n$ and $g(x_1, \ldots, x_n) \in \mathcal{G}^{ij}_{wsai}(n)$ such that the expectations exist. $\square$

As pointed out by Cai and Wei [1], the notion of RWSAI is a multivariate generalization of the joint hazard rate order, which was proposed by Shanthikumar and Yao [18]. Here, we point out that the notions of LWSAI and WSAI are respectively the multivariate generalizations of the joint reversed hazard rate order and the joint stochastic order proposed by Shanthikumar and Yao [18] as well. We also remark that the introduced concepts of LWSAI and WSAI are used to generalize the multivariate models used in the optimal portfolio selections. As illustrated in Sections 3 and 4, these concepts allow risks to be dependent in a more general structure than the existing multivariate models do. However, we point out that these concepts essentially describe the properties of multivariate distributions rather than study multivariate dependence itself since we do not study whether these concepts satisfy the desirable properties of a notion of dependence. For modeling dependence, we refer to the monographs of Joe [9,10].

At the end of this section, we use the multivariate normal random vectors and the exchangeable random vectors as examples to illustrate the notions of SAI, LWSAI, RWSAI, and WSAI. First, by noting that $\mathcal{G}^{ij}_{wsai}(n) \subset \mathcal{G}^{ij}_{lwsai}(n)$ $(\mathcal{G}^{ij}_{rwsai}(n)) \subset \mathcal{G}^{ij}_{sai}(n)$, we have the following implications:

$$\text{SAI} \Longrightarrow \text{LWSAI (RWSAI)} \Longrightarrow \text{WSAI}.$$

Furthermore, let $(X_1, \cdots, X_n)$ be any exchangeable random vector. It is easy to verify that the exchangeable random vector $(X_1, \ldots, X_n)$ is SAI and thus is LWSAI, RWSAI, and WSAI. Moreover, there are many nonexchangeable random vectors that are SAI, LWSAI, RWSAI, and WSAI. For instance, Section 4 presents how to construct nonexchangeable LWSAI (and thus WSAI) vectors. In addition, Theorem 2.15 of Shanthikumar and Yao [18] gives a sufficient condition for a bivariate normal random vector to be SAI, which shows that nonexchangeable random vectors can be SAI.

In the following, we first prove that the sufficient condition given by Shanthikumar and Yao [18] is also necessary and then derive a sufficient and necessary condition for a multivariate normal random vector to be SAI.

Lemma 2.6. *Assume that (X_1, X_2) is a bivariate normal random vector. Then (X_1, X_2) is SAI if and only if $Var(X_1) = Var(X_2)$ and $\mathbb{E}(X_1) \leq \mathbb{E}(X_2)$.*

Proof. Suppose that $Var(X_1) = Var(X_2)$ and $\mathbb{E}(X_1) \leq \mathbb{E}(X_2)$. Then, (X_1, X_2) is SAI by Theorem 2.15 (ii) of Shanthikumar and Yao [18].

Conversely, suppose that (X_1, X_2) is SAI. Let $\sqrt{Var(X_1)} = \sigma_1$, $\sqrt{Var(X_2)} = \sigma_2$, $\mathbb{E}(X_1) = \mu_1$, and $\mathbb{E}(X_2) = \mu_2$. According to Proposition 5.1 and relation (4.2) of Cai and Wei [1], we know that $X_1 \leq_{st} X_2$, which implies that $\mathbb{P}\{X_2 \leq t\} \leq \mathbb{P}\{X_1 \leq t\}$

or $\Phi(\frac{t-\mu_2}{\sigma_2}) \le \Phi(\frac{t-\mu_1}{\sigma_1})$ for all $t \in \mathbb{R}$, where $\Phi(x)$ is the standard normal distribution function. Noting that $\Phi(x)$ is strictly increasing, we have

$$\frac{t-\mu_2}{\sigma_2} \le \frac{t-\mu_1}{\sigma_1} \quad \text{for all } t \in \mathbb{R}, \tag{2.5}$$

which implies $\mu_1 \le \mu_2$ by setting $t = \mu_2$ in (2.5). Furthermore, (2.5) implies $\frac{1-\frac{\mu_2}{t}}{\sigma_2} \le \frac{1-\frac{\mu_1}{t}}{\sigma_1}$ for $t > 0$ and $\frac{1-\frac{\mu_2}{t}}{\sigma_2} \ge \frac{1-\frac{\mu_1}{t}}{\sigma_1}$ for $t < 0$. Thus, by letting $t \to \infty$ and $t \to -\infty$, we get $\sigma_1 \le \sigma_2$ and $\sigma_1 \ge \sigma_2$ and thus $\sigma_1 = \sigma_2$. $\square$

Note that if (X_1, X_2) is a bivariate normal random vector, then $(X_1+\alpha_1, X_2+\alpha_2)$ is a bivariate normal random vector for any constants α_1 and α_2. Furthermore, if (X_1, X_2) is an exchangeable bivariate random vector with $\mathbb{E}(X_1^2) < \infty$ and $\mathbb{E}(X_2^2) < \infty$, then $Var(X_1) = Var(X_2)$ and $\mathbb{E}(X_1) = \mathbb{E}(X_2)$. Thus, by Lemma 2.6, we immediately obtain the following corollary.

Corollary 2.7. *If (X_1, X_2) is an exchangeable bivariate normal random vector, then $(X_1 + \alpha_1, X_2 + \alpha_2)$ is SAI for any constants $\alpha_1 \le \alpha_2$.* $\square$

Proposition 2.8. *Assume that a random vector $(X_1, \dots, X_n)$ follows a multivariate normal distribution with mean $\boldsymbol{\mu} = (\mu_1, \dots, \mu_n)$. Denote $\mathrm{Var}[X_i] = \sigma_i^2$ for $i = 1, \dots, n$ and $\mathrm{Cov}[X_i, X_j] = c_{ij}$ for $1 \le i \ne j \le n$. Then $(X_1, \dots, X_n)$ is SAI if and only if the following three conditions hold:* (i) *$\mu_1 \le \cdots \le \mu_n$,* (ii) *$\sigma_1 = \cdots = \sigma_n$, and* (iii) *$c_{ij}$ are equal for all $1 \le i \ne j \le n$.*

Proof. Suppose that $(X_1, \dots, X_n)$ is SAI. According to Proposition 3.3(i) of Cai and Wei [1], we know that (X_i, X_j) and (X_i, X_j, X_k) are SAI for all $1 \le i < j < k \le n$. Therefore, we conclude that $\mu_1 \le \cdots \le \mu_n$ and $\sigma_1 = \cdots = \sigma_n$ from Lemma 2.6. Denote $\sigma = \sigma_1 = \cdots = \sigma_n$. Now we consider (X_1, X_2, X_3), which has a multivariate normal distribution as well. For any given $x_3 \in \mathbb{R}$, by the property of the multivariate normal distribution, see, for example, Joe [10], we know that $(X_1, X_2) \mid X_3 = x_3$ follows a bivariate normal distribution with the following mean

$$\hat{\boldsymbol{\mu}} = (\mu_1, \mu_2) + (x_3 - \mu_3) \times \frac{1}{\sigma^2} \times (c_{13}, c_{23}) = \left(\mu_1 + \frac{c_{13}}{\sigma^2}(x_3 - \mu_3),\ \mu_2 + \frac{c_{23}}{\sigma^2}(x_3 - \mu_3)\right).$$

On the other hand, according to Proposition 3.4 of Cai and Wei [1], we know that $(X_1, X_2)|X_3 = x_3$ is SAI for any $x_3 \in \mathbb{R}$. Therefore, from Lemma 2.6, we have

$$\mu_1 + \frac{c_{13}}{\sigma^2}(x_3 - \mu_3) \le \mu_2 + \frac{c_{23}}{\sigma^2}(x_3 - \mu_3), \quad \text{for all } x_3 \in \mathbb{R},$$

which implies that $\frac{\mu_1}{x_3} + \frac{c_{13}}{\sigma^2}(1 - \frac{\mu_3}{x_3}) \le \frac{\mu_2}{x_3} + \frac{c_{23}}{\sigma^2}(1 - \frac{\mu_3}{x_3})$ for $x_3 > 0$ and $\frac{\mu_1}{x_3} + \frac{c_{13}}{\sigma^2}(1 - \frac{\mu_3}{x_3}) \ge \frac{\mu_2}{x_3} + \frac{c_{23}}{\sigma^2}(1 - \frac{\mu_3}{x_3})$ for $x_3 < 0$. Thus, by letting $x_3 \to \infty$ and $x_3 \to -\infty$, we get $c_{13} \le c_{23}$ and $c_{13} \ge c_{23}$ and thus $c_{13} = c_{23}$. By the same arguments, we obtain that c_{ij} are equal for all $1 \le i \ne j \le n$.

Conversely, suppose that conditions (i), (ii), (iii) hold. Then $(X_1 - \mu_1, \dots, X_n - \mu_n)$ follows an exchangeable multivariate normal distribution. Hence,

$$\begin{aligned}(X_1 - \mu_1, X_2 - \mu_2) \mid (X_3 - \mu_3, \dots, X_n - \mu_n) &= (x_3 - \mu_3, \dots, x_n - \mu_n) \\ &= (X_1 - \mu_1, X_2 - \mu_2) \mid (X_3, \dots, X_n) = (x_3, \dots, x_n)\end{aligned}$$

follows an exchangeable bivariate normal distribution for any $(x_3, \dots, x_n) \in \mathbb{R}^{n-2}$. Thus, according to Corollary 2.7, we know that

$$\begin{aligned}(X_1, X_2) \mid (X_3, \dots, X_n) &= (x_3, \dots, x_n) \\ &= (X_1 - \mu_1, X_2 - \mu_2) \mid (X_3, \dots, X_n) = (x_3, \dots, x_n) + (\mu_1, \mu_2)\end{aligned}$$

is SAI. By the same arguments, we obtain that $(X_i, X_j) \mid \mathbf{X}_{\overline{ij}} = \mathbf{x}_{\overline{ij}}$ is SAI for any $1 \le i < j \le n$ and any $\mathbf{x}_{\overline{ij}} \in \mathbb{R}^{n-2}$. This implies that $(X_1, \dots, X_n)$ is SAI by Proposition 3.4 of Cai and Wei [1]. $\square$

3. Properties of LWSAI and WSAI

In this section, we present some properties of the dependence notions of LWSAI and WSAI.

Proposition 3.1. *A random vector $(X_1, \dots, X_n)$ is LWSAI if and only if $(-X_n, \dots, -X_1)$ is RWSAI.*

Proof. We first prove the "if" part. Assume $(X_1, \dots, X_n)$ is LWSAI. For any $1 \le i < j \le n$, consider any multivariate function $g(x_1, \dots, x_n) \in \mathcal{G}_{rwsai}^{ij}(n)$, we want to show that

$$\mathbb{E}[g(-X_n, \dots, -X_1)] \ge \mathbb{E}[g(\pi_{ij}(-X_n, \dots, -X_1))].$$

Define $h(x_1, \ldots, x_n) = g(-x_n, \ldots, -x_1)$, it is easy to verify that $h(x_1, \ldots, x_n) \in \mathcal{G}_{lwsai}^{n+1-j,\, n+1-i}(n)$. Therefore,

$$\begin{aligned}\mathbb{E}[g(-X_n, \ldots, -X_1)] &= \mathbb{E}[h(X_1, \ldots, X_n)] \\ &\geq \mathbb{E}[h(\pi_{n+1-j,\, n+1-i}(X_1, \ldots, X_n))] = \mathbb{E}[g(\pi_{ij}(-X_n, \ldots, -X_1))].\end{aligned}$$

The "only if" part can be similarly proved. □

Proposition 3.1 establishes a relation between the notions of LWSAI and RWSAI and thus it provides a shortcut to derive some properties of LWSAI based on the properties of RWSAI developed in Cai and Wei [1].

Proposition 3.2. *A random vector $\mathbf{X} = (X_1, \ldots, X_n)$ is LWSAI (WSAI) if and only if $(X_i, X_j) \,|\, \mathbf{X}_{\overline{ij}} = \mathbf{x}_{\overline{ij}}$ is LWSAI (WSAI) for any $1 \leq i < j \leq n$ and $\mathbf{x}_{\overline{ij}} \in S(\mathbf{X}_{\overline{ij}})$.*

Proof. We give the proof only for the notion of WSAI. The proof for LWSAI is analogous.

First, assume that $(X_1, \ldots, X_n)$ is WSAI. For any $1 \leq i < j \leq n$ and $g(x, y) \in \mathcal{G}_{wsai}^{12}(2)$, denote $h(\mathbf{x}) = g(x_i, x_j)\, \mathbb{I}\{\mathbf{x}_{\overline{ij}} \in A\}$ for any $A \in \sigma(\mathbf{X}_{\overline{ij}})$. It is easy to verify that $h(\mathbf{x}) \in \mathcal{G}_{wsai}^{ij}(n)$. Since $(X_1, \ldots, X_n)$ is WSAI, we have $\mathbb{E}[h(\mathbf{X})] \geq \mathbb{E}[h(\pi_{ij}(\mathbf{X}))]$, i.e., $\mathbb{E}[g(X_i, X_j)\, \mathbb{I}\{\mathbf{X}_{\overline{ij}} \in A\}] \geq \mathbb{E}[g(X_j, X_i)\, \mathbb{I}\{\mathbf{X}_{\overline{ij}} \in A\}]$ for any $A \in \sigma(\mathbf{X}_{\overline{ij}})$, or equivalently, $\mathbb{E}[\mathbb{E}[g(X_i, X_j) \mid \mathbf{X}_{\overline{ij}}]\, \mathbb{I}\{\mathbf{X}_{\overline{ij}} \in A\}] \geq \mathbb{E}[\mathbb{E}[g(X_j, X_i) \mid \mathbf{X}_{\overline{ij}}]\, \mathbb{I}\{\mathbf{X}_{\overline{ij}} \in A\}]$ for any $A \in \sigma(\mathbf{X}_{\overline{ij}})$. Recall Lemma 3.2 of Cai and Wei [1], which states that if $\mathbf{X} = (X_1, \ldots, X_n)$ is a random vector defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and $f(\mathbf{x}) = f(x_1, \ldots, x_n)$ is a multivariate function satisfying $\mathbb{E}[f(\mathbf{X})\, \mathbb{I}(A)] \leq 0$ for all $A \in \mathcal{F}$, then $f(\mathbf{X}) \leq_{a.s.} 0$. Hence, according to Lemma 3.2 of Cai and Wei [1], we have $\mathbb{E}[g(X_i, X_j) \,|\, \mathbf{X}_{\overline{ij}}] \geq_{a.s.} \mathbb{E}[g(X_j, X_i) \,|\, \mathbf{X}_{\overline{ij}}]$, which implies that $(X_i, X_j) \,|\, \mathbf{X}_{\overline{ij}} = \mathbf{x}_{\overline{ij}}$ is WSAI.

Conversely, assume that $(X_i, X_j) \,|\, \mathbf{X}_{\overline{ij}} = \mathbf{x}_{\overline{ij}}$ is WSAI for any $1 \leq i < j \leq n$, consider any $1 \leq i < j \leq n$ and any function $g \in \mathcal{G}_{wsai}^{ij}(n)$, we need to show $\mathbb{E}\,[g(\mathbf{X})] \geq \mathbb{E}\,[g(\pi_{ij}(\mathbf{X}))]$. Note that for any fixed $\mathbf{x}_{\overline{ij}} \in S(\mathbf{X}_{\overline{ij}})$, $g(x_1, \ldots, x_n) \in \mathcal{G}_{wsai}^{12}(2)$ as a bivariate function of (x_i, x_j). Since $(X_i, X_j) \,|\, \mathbf{X}_{\overline{ij}} = \mathbf{x}_{\overline{ij}}$ is WSAI, we have $\mathbb{E}\,[g(\mathbf{X}) \,|\, \mathbf{X}_{\overline{ij}} = \mathbf{x}_{\overline{ij}}] \geq \mathbb{E}\,[g(\pi_{ij}(\mathbf{X})) \,|\, \mathbf{X}_{\overline{ij}} = \mathbf{x}_{\overline{ij}}]$, which implies $\mathbb{E}\,[g(\mathbf{X})] \geq \mathbb{E}\,[g(\pi_{ij}(\mathbf{X}))]$. □

LWSAI and WSAI random vectors can be constructed through special dependence structures, such as independence and comonotonicity. As a matter of fact, following Proposition 5.5 of Cai and Wei [1], a comonotonic random vector $(X_1, \ldots, X_n)$ with $X_1 \leq_{st} \cdots \leq_{st} X_n$ is SAI and thus is LWSAI and WSAI. The following Proposition 3.3 shows how to construct WSAI random vectors with independence.

Proposition 3.3. *If $X_1, \ldots, X_n$ are mutually independent and $X_1 \leq_{st} \cdots \leq_{st} X_n$, then $(X_1, \ldots, X_n)$ is WSAI.*

Proof. It follows from Theorem 4.3 of Shanthikumar and Yao [18] that, if $X \leq_{st} Y$ and X is independent of Y, then (X, Y) is WSAI. Therefore, we have $(X_i, X_j) \,|\, \mathbf{X}_{\overline{ij}} = \mathbf{x}_{\overline{ij}}$ is WSAI for any $\mathbf{x}_{\overline{ij}} \in S(\mathbf{X}_{\overline{ij}})$, which implies that $(X_1, \ldots, X_n)$ is WSAI by Proposition 3.2. □

In the following, we develop some equivalent characterizations for the notion of LWSAI.

Theorem 3.4. *Let (X, Y) be a bivariate random vector. The following statements are equivalent.*

(i) *(X, Y) is LWSAI;*
(ii) *$\mathbb{P}\{X \leq x,\ y < Y \leq y + t\} \geq \mathbb{P}\{Y \leq x,\ y < X \leq y + t\}$ for any $x \leq y$ and $t > 0$;*
(iii) *$\mathbb{E}[h(X)\, \mathbb{I}\{X \leq x,\ y < Y \leq y + t\}] \geq \mathbb{E}[h(Y)\, \mathbb{I}\{Y \leq x,\ y < X \leq y + t\}]$ for any $x \leq y$, $t > 0$, and nonnegative decreasing function $h(x)$.*

Proof. (i) ⇒ (ii). This implication follows immediately from the fact that the indicator function $\mathbb{I}\{(u, v) \in (-\infty, x] \times (y, y + t]\}$ belongs to the class $\mathcal{G}_{lwsai}^{12}(2)$ for any $x \leq y$ and $t > 0$.

(ii) ⇒ (iii). It is easy to show by limiting arguments that (ii) implies

$$\mathbb{P}\{X < x,\ y < Y \leq y + t\} \geq \mathbb{P}\{Y < x,\ y < X \leq y + t\},$$

for any $x \leq y$ and $t > 0$. Therefore, we conclude that

$$\mathbb{P}\{X \in I,\ y < Y \leq y + t\} \geq \mathbb{P}\{Y \in I,\ y < X \leq y + t\}, \tag{3.1}$$

for any $t > 0$ and interval I such that $\inf I = -\infty$ and $\sup I \leq y$.

Recalling that $\mathbb{E}[Z] = \int_0^\infty \mathbb{P}\{Z > z\} dz$ for any nonnegative random variable Z, we have

$$\begin{aligned}&\mathbb{E}[h(X)\, \mathbb{I}\{X \leq x,\ y < Y \leq y + t\}] = \int_0^\infty \mathbb{P}\big\{h(X)\, \mathbb{I}\{X \leq x,\ y < Y \leq y + t\} > z\big\} dz \\ &= \int_0^\infty \mathbb{P}\{h(X) > z,\ X \leq x,\ y < Y \leq y + t\} dz = \int_0^\infty \mathbb{P}\{X \in I_z,\ y < Y \leq y + t\} dz \\ &\geq \int_0^\infty \mathbb{P}\{Y \in I_z,\ y < X \leq y + t\} dz = \mathbb{E}[h(Y)\, \mathbb{I}\{Y \leq x, y < X \leq y + t\}],\end{aligned}$$

where $I_z = h^{-1}((z,\infty)) \cap (-\infty, x]$ and the inequality holds because I_z is either an empty set or an interval satisfying the condition for (3.1) to hold.

(iii) $\Rightarrow$ (i). Consider any $g(x,y) \in \mathcal{G}^{12}_{lwsai}(2)$, we want to show that $\mathbb{E}[g(X,Y)] \geq \mathbb{E}[g(X,Y)]$, or $\mathbb{E}[\Delta_{12}g(X,Y)] \geq 0$. Noting that $\Delta_{12}g(x,y) = -\Delta_{12}g(y,x)$ and $\Delta_{12}g(x,y) = 0$ if $x = y$, we have

$$\begin{aligned}\mathbb{E}[\Delta_{12}g(X,Y)] &= \mathbb{E}[\Delta_{12}g(X,Y)\,\mathbb{I}\{X<Y\}] + \mathbb{E}[\Delta_{12}g(X,Y)\,\mathbb{I}\{X>Y\}] \\ &= \mathbb{E}[\Delta_{12}g(X,Y)\,\mathbb{I}\{X<Y\}] - \mathbb{E}[\Delta_{12}g(Y,X)\,\mathbb{I}\{Y<X\}].\end{aligned} \tag{3.2}$$

For a fixed positive integer n, define

$$h_{n,i}(x) = \inf_{s\in[\,i2^{-n},\,(i+1)2^{-n})} \Delta_{12}g(x,s) \times \mathbb{I}\left\{x \leq \frac{i}{2^n}\right\}, \quad -n2^n \leq i < n2^n - 1.$$

The infimum always exists since $\Delta_{12}g(x,y) \geq 0$ for any $x \leq y$. Recalling that $\Delta_{12}g(x,y)$ is decreasing in $x \in (-\infty, y]$, we conclude that $h_{n,i}(x)$ is decreasing in x. Therefore, according to (iii), we have

$$\begin{aligned}&\mathbb{E}\left[h_{n,i}(X)\,\mathbb{I}\left\{\frac{i}{2^n} < Y \leq \frac{i+1}{2^n}\right\}\right] = \mathbb{E}\left[h_{n,i}(X)\,\mathbb{I}\left\{X \leq \frac{i}{2^n},\ \frac{i}{2^n} < Y \leq \frac{i+1}{2^n}\right\}\right] \\ &\quad\geq \mathbb{E}\left[h_{n,i}(Y)\,\mathbb{I}\left\{Y \leq \frac{i}{2^n},\ \frac{i}{2^n} < X \leq \frac{i+1}{2^n}\right\}\right] = \mathbb{E}\left[h_{n,i}(Y)\,\mathbb{I}\left\{\frac{i}{2^n} < X \leq \frac{i+1}{2^n}\right\}\right].\end{aligned}$$

Furthermore, define

$$H_n(x,y) = \sum_{i=-n2^n}^{n2^n-1} h_{n,i}(x) \times \mathbb{I}\left\{\frac{i}{2^n} < y \leq \frac{i+1}{2^n}\right\}.$$

Then,

$$\begin{aligned}\mathbb{E}[H_n(X,Y)] &= \sum_{i=-n2^n}^{n2^n-1} \mathbb{E}\left[h_{n,i}(X)\,\mathbb{I}\left\{\frac{i}{2^n} < Y \leq \frac{i+1}{2^n}\right\}\right] \\ &\geq \sum_{i=-n2^n}^{n2^n-1} \mathbb{E}\left[h_{n,i}(Y)\,\mathbb{I}\left\{\frac{i}{2^n} < X \leq \frac{i+1}{2^n}\right\}\right] = \mathbb{E}[H_n(Y,X)].\end{aligned}$$

On the other hand, it is easy to verify that $\{H_n(x,y),\ n = 1,2,\ldots\}$ is an increasing sequence and $\lim_{n\to\infty} H_n(x,y) = \Delta_{12}g(x,y) \times \mathbb{I}\{x<y\}$. According to the monotone convergence theorem, we have

$$\begin{aligned}\mathbb{E}[\Delta_{12}g(X,Y)\,\mathbb{I}\{X<Y\}] &= \lim_{n\to\infty} \mathbb{E}[H_n(X,Y)] \\ &\geq \lim_{n\to\infty} \mathbb{E}[H_n(Y,X)] = \mathbb{E}[\Delta_{12}g(Y,X)\,\mathbb{I}\{Y<X\}],\end{aligned}$$

which implies that $\mathbb{E}[\Delta_{12}g(X,Y)] \geq 0$ from (3.2). □

Proposition 3.5. *A random vector $\mathbf{X} = (X_1,\ldots,X_n)$ is LWSAI if and only if*

$$\mathbb{P}\{X_i \leq x_i,\ x_j < X_j \leq x_j + t,\ \mathbf{X}_{\overline{ij}} \in A_{\overline{ij}}\} \geq \mathbb{P}\{X_j \leq x_i,\ x_j < X_i \leq x_j + t,\ \mathbf{X}_{\overline{ij}} \in A_{\overline{ij}}\}, \tag{3.3}$$

for any $1 \leq i < j \leq n$, $x_i \leq x_j$, $t > 0$, and $A_{\overline{ij}} \in \sigma(\mathbf{X}_{\overline{ij}})$.

Proof. The "only if" part is obvious by noting that $\mathbb{I}\{y_i \leq x_i,\ x_j < y_j \leq x_j + t,\ \mathbf{y}_{\overline{ij}} \in A_{\overline{ij}}\} \in \mathcal{G}^{ij}_{lwsai}(n)$.

For the "if" part, we first rewrite (3.3) as

$$\mathbb{E}[\mathbb{E}[\mathbb{I}\{X_i \leq x_i, x_j < X_j \leq x_j + t\} \mid \mathbf{X}_{\overline{ij}}]\,\mathbb{I}\{\mathbf{X}_{\overline{ij}} \in A_{\overline{ij}}\}] \geq \mathbb{E}[\mathbb{E}[\mathbb{I}\{X_j \leq x_i,\ x_j < X_i \leq x_j + t\} \mid \mathbf{X}_{\overline{ij}}]\,\mathbb{I}\{\mathbf{X}_{\overline{ij}} \in A_{\overline{ij}}\}].$$

As in the proof of Proposition 3.2, according to Lemma 3.2 of Cai and Wei [1], we have

$$\mathbb{E}[\mathbb{I}\{X_i \leq x_i,\ x_j < X_j \leq x_j + t\} \mid \mathbf{X}_{\overline{ij}}] \geq_{a.s.} \mathbb{E}[\mathbb{I}\{X_j \leq x_i,\ x_j < X_i \leq x_j + t\} \mid \mathbf{X}_{\overline{ij}}],$$

or

$$\mathbb{P}\{X_i \leq x_i,\ x_j < X_j \leq x_j + t \mid \mathbf{X}_{\overline{ij}} = \mathbf{x}_{\overline{ij}}\} \geq \mathbb{P}\{X_j \leq x_i,\ x_j < X_i \leq x_j + t \mid \mathbf{X}_{\overline{ij}} = \mathbf{x}_{\overline{ij}}\},$$

for any $1 \leq i < j \leq n$, $x_i \leq x_j$, and $\mathbf{x}_{\overline{ij}} \in S(\mathbf{X}_{\overline{ij}})$.

From Theorem 3.4 (ii), we know that the (conditional) distribution of $(X_i, X_j) \mid \mathbf{X}_{\overline{ij}} = \mathbf{x}_{\overline{ij}}$ is LWSAI for any $1 \leq i < j \leq n$ and $\mathbf{x}_{\overline{ij}} \in \sigma(\mathbf{X}_{\overline{ij}})$, which implies that $(X_1,\ldots,X_n)$ is LWSAI by Proposition 3.2. □

Based on Proposition 3.1, we can easily derive an analogue to Proposition 3.5 for the notion of RWSAI. The proof is straightforward and is thus omitted.

Proposition 3.6. *A random vector* $\mathbf{X} = (X_1, \dots, X_n)$ *is RWSAI if and only if*

$$\mathbb{P}\{x_i - t < X_i \le x_i,\ X_j > x_j,\ \mathbf{X}_{\overline{ij}} \in A_{\overline{ij}}\} \ge \mathbb{P}\{x_i - t < X_j \le x_i,\ X_i > x_j,\ \mathbf{X}_{\overline{ij}} \in A_{\overline{ij}}\},$$

for any $1 \le i < j \le n$, $x_i \le x_j$, $t > 0$, *and* $A_{\overline{ij}} \in \sigma(\mathbf{X}_{\overline{ij}})$. □

Propositions 3.5 and 3.6 provide an easy way to verify the notions of LWSAI and RWSAI, especially in the absence of joint density functions. The application of Proposition 3.5 will be given in next section.

On the other hand, if the joint density function exists, we have the following characterization of the notion of LWSAI.

Proposition 3.7. *Assume a random vector* $(X_1, \dots, X_n)$ *has a joint density function. Then* $(X_1, \dots, X_n)$ *is LWSAI if and only if*

$$\frac{\partial}{\partial x_j}\mathbb{P}\{X_i \le x_i,\ X_j \le x_j \,|\, \mathbf{X}_{\overline{ij}} = \mathbf{x}_{\overline{ij}}\} \ge \frac{\partial}{\partial x_j}\mathbb{P}\{X_i \le x_j,\ X_j \le x_i \,|\, \mathbf{X}_{\overline{ij}} = \mathbf{x}_{\overline{ij}}\}, \tag{3.4}$$

for any $1 \le i < j \le n$, $x_i \le x_j$, *and* $\mathbf{x}_{\overline{ij}} \in S(\mathbf{X}_{\overline{ij}})$.

Proof. We first give the proof for the case that $n = 2$. Assume random vector (X, Y) has a joint density function, we want to show that (X, Y) is LWSAI if and only if

$$\frac{\partial}{\partial y}\mathbb{P}\{X \le x,\ Y \le y\} \ge \frac{\partial}{\partial y}\mathbb{P}\{X \le y,\ Y \le x\}, \quad \forall x \le y. \tag{3.5}$$

By Proposition 3.1 of this section and Theorem 3.14 of Cai and Wei [1], we see that (X, Y) is LWSAI $\Longleftrightarrow$ $(-Y, -X)$ is RWSAI $\Longleftrightarrow$

$$\frac{\partial}{\partial u}\mathbb{P}\{-Y > u,\ -X > v\} \le \frac{\partial}{\partial u}\mathbb{P}\{-Y > v,\ -X > u\}, \quad \forall u \le v. \tag{3.6}$$

Note that X and Y have the joint density function, (3.6) is equivalent to

$$\frac{\partial}{\partial u}\mathbb{P}\{Y \le -u,\ X \le -v\} \le \frac{\partial}{\partial u}\mathbb{P}\{Y \le -v,\ X \le -u\}, \quad \forall u \le v,$$

which is equivalent to (3.5) by letting $-u = y$ and $-v = x$.

The proof for the case that $n \ge 3$ follows from Proposition 3.2. □

You and Li [19] proposed the notion of LTPD for random vectors with joint density functions. Proposition 3.7 shows that the notion of LWSAI is reduced to LTPD if the joint density function of a random vector exists.

4. Construction of LWSAI random vectors

In this section, we show how to construct LWSAI random vectors through Archimedean copulas and certain marginal distributions. We first recall the definition of an Archimedean copula. Let $\Psi : (0, 1] \to [0, \infty)$ be continuous, strictly decreasing and satisfy: (i) $\Psi(1) = 0$, $\lim_{x \downarrow 0} \Psi(x) = \infty$, and (ii) $\Lambda(x)$ is completely monotonic, namely $(-1)^k \Lambda^{(k)}(x) = (-1)^k \frac{d^k}{dx^k}\Lambda(x) \ge 0$ for all $k = 0, 1, \dots$, where $\Lambda(x) = \Psi^{-1}(x)$. Define

$$C(u_1, \dots, u_n) = \Lambda\left(\sum_{k=1}^{n} \Psi(u_k)\right), \quad u_1, \dots, u_n \in [0, 1]. \tag{4.1}$$

Then $C(u_1, \dots, u_n)$ is an Archimedean copula.

Assume that the random vector $(X_1, \dots, X_n)$ has joint distribution function $F(x_1, \dots, x_n)$ and marginal distribution functions $F_k(x)$, $k = 1, \dots, n$. Then $(X_1, \dots, X_n)$ is said to be linked by an Archimedean copula given by (4.1) if

$$F(x_1, \dots, x_n) = \Lambda\left(\sum_{k=1}^{n} \Psi(F_k(x_k))\right). \tag{4.2}$$

Proposition 4.1. *Assume a random vector* $(X_1, \dots, X_n)$ *is linked by an Archimedean copula given by* (4.1) *with positive joint density functions. If* $X_1 \le_{rh} \cdots \le_{rh} X_n$, *then* $(X_1, \dots, X_n)$ *is LWSAI.*

Proof. By combining Theorem 5.7 of Cai and Wei [1] and Proposition 3.1, we know that Proposition 4.1 holds if $x\Psi'(x)$ is increasing in $x \in (0, 1)$. (We point that the condition "$x\Psi'(x)$ is increasing in $x \in [0, 1]$" in Theorem 5.7 of Cai and Wei [1] can be reduced to "$x\Psi'(x)$ is increasing in $x \in (0, 1)$".) In the following, we shall show that $x\Psi'(x)$ is increasing in $x \in (0, 1)$ under the assumption of the Archimedean copula defined by (4.1).

Note that $\Psi : (0, 1] \to [0, \infty)$ is continuous, strictly decreasing and satisfies $\Psi(1) = 0$ and $\lim_{x\downarrow 0} \Psi(x) = \infty$. Thus, $\Lambda(t) = \Psi^{-1}(t) \neq 0$ and $\Lambda'(t) = \frac{d}{dt}\Psi^{-1}(t) = (\Psi'(\Psi^{-1}(t)))^{-1} \neq 0$ for any $t \in (0, \infty)$. Furthermore, note that the generator Λ of the Archimedean copula is completely monotonic. Hence, by Theorem 2.14 of Müller and Scarsini [16], we know that $\Lambda(t)$ is log-convex in $t \in (0, \infty)$, namely, $\log(\Lambda(t))$ is convex in $t \in (0, \infty)$, which implies that $(\log(\Lambda(t)))' = \frac{\Lambda'(t)}{\Lambda(t)}$ is increasing in $t \in (0, \infty)$ and thus $\frac{\Lambda(t)}{\Lambda'(t)}$ is decreasing in $t \in (0, \infty)$. Therefore, $\frac{\Lambda(\Psi(x))}{\Lambda'(\Psi(x))} = x\Psi'(x)$ is increasing in $x \in (0, 1)$ since $\Psi(x)$ is decreasing in $x \in (0, 1)$ and $0 < \Psi(x) < \infty$ for $x \in (0, 1)$. $\square$

With the assumption of the existence of density functions, Hennessy and Lapan [7] introduced the following dependence structure. They defined

$$s_k(x) = \Psi'(F_k(x))f_k(x), \quad k = 1, \dots, n, \tag{4.3}$$

where $f_k(x)$ is the probability density function of X_k, and modeled dependence of $(X_1, \dots, X_n)$ by assuming that $s_i(x) \geq s_j(x)$ for any $1 \leq i < j \leq n$. In the following, we show that this dependence structure is a special case of the LWSAI notion.

Proposition 4.2. *Assume a random vector $(X_1, \dots, X_n)$ is linked by an Archimedean copula given by (4.1) with marginal density functions. If $s_1(x) \geq \cdots \geq s_n(x)$, then $(X_1, \dots, X_n)$ is LWSAI.*

Proof. For any $1 \leq i < j \leq n$ and $x_i \leq x_j$, we have

$$\Psi(F_j(x_j)) - \Psi(F_j(x_i)) = \int_{x_i}^{x_j} s_j(t)\mathrm{d}t,$$

$$\Psi(F_i(x_j)) - \Psi(F_i(x_i)) = \int_{x_i}^{x_j} s_i(t)\mathrm{d}t,$$

then

$$\Psi(F_j(x_j)) - \Psi(F_j(x_i)) \leq \Psi(F_i(x_j)) - \Psi(F_i(x_i)),$$

or

$$\Psi(F_i(x_i)) + \Psi(F_j(x_j)) \leq \Psi(F_i(x_j)) + \Psi(F_j(x_i)),$$

which implies that

$$\Psi(F_i(x_i)) + \Psi(F_j(x_j)) + \sum_{k\neq i,j} \Psi(F_k(x_k)) \leq \Psi(F_i(x_j)) + \Psi(F_j(x_i)) + \sum_{k\neq i,j} \Psi(F_k(x_k)).$$

Recall that Λ is completely comonotonic, which means that $(-1)^{n-1}\Lambda^{(n-1)}(x)$ is nonnegative and decreasing. Therefore, by the similar arguments for (5.3) of Cai and Wei [1], it is not hard to see that

$$\begin{aligned}
&\frac{\partial}{\partial x_j}\mathbb{P}\{X_i \leq x_i, X_j \leq x_j \mid \mathbf{X}_{\overline{ij}} = \mathbf{x}_{\overline{ij}}\} \\
&= \Lambda^{(n-1)}\left(\sum_{k=1} \Psi(F_k(x_k))\right)\Psi'(F_j(x_j))f_j(x_j)\prod_{k\neq i,j}\Psi'(F_k(x_k))f_k(x_k) \\
&= (-1)^{n-1}\Lambda^{(n-1)}\left(\sum_{k=1} \Psi(F_k(x_k))\right)(-s_j(x_j))\prod_{k\neq i,j}(-s_k(x_k)) \\
&\geq (-1)^{n-1}\Lambda^{(n-1)}\left(\Psi(F_i(x_j)) + \Psi(F_j(x_i)) + \sum_{k\neq i,j}\Psi(F_k(x_k))\right)(-s_i(x_j))\prod_{k\neq i,j}(-s_k(x_k)) \\
&= \frac{\partial}{\partial x_j}\mathbb{P}\{X_i \leq x_j, X_j \leq x_i \mid \mathbf{X}_{\overline{ij}} = \mathbf{x}_{\overline{ij}}\}.
\end{aligned}$$

According to Proposition 3.7, we conclude that $(X_1, \dots, X_n)$ is LWSAI. $\square$

We point out that the results of Proposition 4.2 have been also obtained in the discussion (i) right after Theorem 3.5 of Li and You [14].

5. Applications in optimal portfolio selections with dependent risks

In this section, we use the dependence notions of LWSAI, SAI, WSAI to model dependent return rates and restudy the optimal portfolio selection problems (1.1)–(1.3). We also discuss a default risk model with threshold indicators. The results of this section generalize the many existing results.

5.1. Optimal portfolio selections with dependent risks

Lemma 5.1. *Let $\mathbf{a} = (a_1, \ldots, a_n)$ and $\mathbf{x} = (x_1, \ldots, x_n)$ be real-valued vectors and $\mathbf{a} \geq 0$. Define $h(\mathbf{x}) = u(\mathbf{a} \cdot \mathbf{x})$, where $\mathbf{a} \cdot \mathbf{x} = a_1x_1 + \cdots + a_nx_n$. Then, for any $1 \leq i < j \leq n$ and $a_i \leq a_j$, we have*

(i) *$h(\mathbf{x}) \in \mathcal{G}^{ij}_{sai}(n)$ if $u(z)$ is increasing;*
(ii) *$h(\mathbf{x}) \in \mathcal{G}^{ij}_{lwsai}(n)$ if $u(z)$ is increasing and concave;*
(iii) *$h(\mathbf{x}) \in \mathcal{G}^{ij}_{wsai}(n)$ for $\mathbf{x} \geq 0$ if $u(z)$ is increasing and concave, and $zu'(z)$ is increasing.*

Proof. (i) Note that $a_i \leq a_j$ and $x_i \leq x_j$ imply $a_ix_i + a_jx_j \geq a_ix_j + a_jx_i$ and thus $\mathbf{a} \cdot \mathbf{x} \geq \mathbf{a} \cdot \pi_{ij}(\mathbf{x})$, and therefore $u(\mathbf{a} \cdot \mathbf{x}) \geq u(\mathbf{a} \cdot \pi_{ij}(\mathbf{x}))$ since u is increasing.

(ii) Noting that u is increasing and concave, the right derivative $u'^{+}(x)$ always exists and $u'^{+}(x)$ is nonnegative and decreasing. Then $0 \leq u'^{+}(\mathbf{a} \cdot \mathbf{x}) \leq u'^{+}(\mathbf{a} \cdot \pi_{ij}(\mathbf{x}))$ for any $a_i \leq a_j$ and $x_i \leq x_j$. Therefore,

$$\frac{\partial^{+}}{\partial x_i} \Delta_{ij} h(\mathbf{x}) = a_i u'^{+}(\mathbf{a} \cdot \mathbf{x}) - a_j u'^{+}(\mathbf{a} \cdot \pi_{ij}(\mathbf{x})) \leq 0,$$

which implies that $h(\mathbf{x}) \in \mathcal{G}^{ij}_{lwsai}(n)$.

(iii) Note that $\frac{\partial}{\partial x_j} \Delta_{ij} h(\mathbf{x}) = a_j u'(\mathbf{a} \cdot \mathbf{x}) - a_i u'(\mathbf{a} \cdot \pi_{ij}(\mathbf{x}))$.

If $x_i \geq x_j$, we have $\mathbf{a} \cdot \mathbf{x} \leq \mathbf{a} \cdot \pi_{ij}(\mathbf{x})$, and thus $u'(\mathbf{a} \cdot \mathbf{x}) \geq u'(\mathbf{a} \cdot \pi_{ij}(\mathbf{x})) \geq 0$, which implies that $\frac{\partial}{\partial x_j} \Delta_{ij} h(\mathbf{x}) \geq 0$ since $a_j \geq a_i \geq 0$ and $u'^{+} \geq 0$ is decreasing.

If $x_j > x_i$, we have $\mathbf{a} \cdot \mathbf{x} \geq \mathbf{a} \cdot \pi_{ij}(\mathbf{x})$ and thus $u'(\mathbf{a} \cdot \mathbf{x}) \leq u'(\mathbf{a} \cdot \pi_{ij}(\mathbf{x}))$. Denote $c = \sum_{k \neq i,j} a_k x_k$, then $c \geq 0$ since $a_k \geq 0$ and $x_k \geq 0$ for all $k = 1, \ldots, n$. Therefore, we have

$$(a_ix_i + c)u'(\mathbf{a} \cdot \mathbf{x}) \leq (a_jx_i + c)u'(\mathbf{a} \cdot \pi_{ij}(\mathbf{x})). \tag{5.1}$$

On the other hand, recalling that $zu'(z)$ is increasing, we have

$$(\mathbf{a} \cdot \mathbf{x})\, u'(\mathbf{a} \cdot \mathbf{x}) \geq (\mathbf{a} \cdot \pi_{ij}(\mathbf{x}))\, u'(\mathbf{a} \cdot \pi_{ij}(\mathbf{x})). \tag{5.2}$$

Subtracting (5.1) from (5.2), we get $a_jx_ju'(\mathbf{a} \cdot \mathbf{x}) \geq a_ix_ju'(\mathbf{a} \cdot \pi_{ij}(\mathbf{x}))$, which implies that $\frac{\partial}{\partial x_j} \Delta_{ij} h(\mathbf{x}) \geq 0$ if $x_j > 0$. Thus, we conclude that $h(\mathbf{x}) \in \mathcal{G}^{ij}_{wsai}(n)$. □

Theorem 5.2. *The optimal solution $(a_1^*, \ldots, a_n^*)$ to Problem (1.1) should satisfy $a_1^* \leq \cdots \leq a_n^*$ if any of the following conditions holds:*

(i) *$(X_1, \ldots, X_n)$ is SAI and $u(z)$ is increasing;*
(ii) *$(X_1, \ldots, X_n)$ is LWSAI and $u(z)$ is increasing and concave;*
(iii) *$(X_1, \ldots, X_n)$ is nonnegative and WSAI, $u(z)$ is increasing and concave, and $zu'(z)$ is increasing.*

Proof. It suffices to show that

$$\mathbb{E}\left[u\left(\sum_{k=1}^{n} a_k X_k\right)\right] \geq \mathbb{E}\left[u\left(a_iX_j + a_jX_i + \sum_{k \neq i,j} a_k X_k\right)\right],$$

for any $a_i \leq a_j$ under each condition, which directly follows from Lemma 5.1 and the definitions of SAI, LWSAI and WSAI, respectively. □

Hennessy and Lapan [7] studied Problem (1.1) with the dependence structure specified by the assumption of Proposition 4.2. Hadar and Seo [6] studied Problem (1.1) with the assumption that $X_1, \ldots, X_n$ are independent and $X_1 \leq_{st} \cdots \leq_{st} X_n$. We point out that their studies are generalized by Theorem 5.2 (ii) and (iii), since the dependence structures used by Hennessy and Lapan [7] and Hadar and Seo [6] are LWSAI and WSAI, respectively. Theorem 5.2 (i) coincides with Theorem 5 of Li and You [13], but is proved by a different approach.

5.2. Default risk model with independent default indicators

In this subsection, we consider the default risk model and study Problem (1.2). In [3], the return rate random vector $(X_1, \ldots, X_n)$ is assumed to be exchangeable. In the following, we shall relax the assumption of exchangeability to the LWSAI dependence.

In addition, Cheung and Yang [3] made some assumptions on the default indicator random vector $\mathbf{I} = (I_1, \ldots, I_n)$. Specifically, they assumed that

$$\mathbb{P}\{I_i = 0,\ I_j = 1,\ \mathbf{I}_{\overline{ij}} = \mathbf{i}_{\overline{ij}}\} \ge \mathbb{P}\{I_i = 1,\ I_j = 0,\ \mathbf{I}_{\overline{ij}} = \mathbf{i}_{\overline{ij}}\},$$

for any $1 \le i < j \le n$ and $\mathbf{i}_{\overline{ij}} \in \{0, 1\}^{n-2}$. We point out that this assumption is equivalent to the conditions that $(I_1, \ldots, I_n)$ is LWSAI as shown by the following lemma.

Lemma 5.3. *A multivariate Bernoulli random vector* $\mathbf{I}$ *is LWSAI if and only if*

$$\mathbb{P}\{I_i = 0,\ I_j = 1,\ \mathbf{I}_{\overline{ij}} = \mathbf{i}_{\overline{ij}}\} \ge \mathbb{P}\{I_i = 1,\ I_j = 0,\ \mathbf{I}_{\overline{ij}} = \mathbf{i}_{\overline{ij}}\}, \tag{5.3}$$

for any $1 \le i < j \le n$ *and* $\mathbf{i}_{\overline{ij}} \in \{0, 1\}^{n-2}$.

Proof. Without loss of generality, we give the proof only for the case that $i = 1, j = 2$. Also, we denote the joint probability mass function of $\mathbf{I}$ by $p(\mathbf{i}) = p(i_1, \ldots, i_n) = \mathbb{P}\{\mathbf{I} = \mathbf{i}\}$, where $\mathbf{i} \in \{0, 1\}^n$. In particular, we denote $p(i_1, i_2, \mathbf{i}_{\overline{12}}) = \mathbb{P}\{I_1 = i_1,\ I_2 = i_2,\ \mathbf{I}_{\overline{12}} = \mathbf{i}_{\overline{12}}\}$.

"$\Longleftarrow$" Let $g(x_1, \ldots, x_n)$ be defined on $\{0, 1\}^n$. For any $g(x_1, \ldots, x_n) \in \mathcal{G}^{12}_{lwsai}(n)$, we have $g(0, 1, \mathbf{i}_{\overline{12}}) \ge g(1, 0, \mathbf{i}_{\overline{12}})$ for any $\mathbf{i}_{\overline{12}} \in \{0, 1\}^{n-2}$. Therefore,

$$\mathbb{E}[\Delta_{12}\, g(I_1, \ldots, I_n)] = \sum_{\mathbf{i}_{\overline{12}} \in \{0,1\}^{n-2}} \big(p(0, 1, \mathbf{i}_{\overline{12}}) - p(1, 0, \mathbf{i}_{\overline{12}})\big) \times \big(g(0, 1, \mathbf{i}_{\overline{12}}) - g(1, 0, \mathbf{i}_{\overline{12}})\big) \ge 0.$$

"$\Longrightarrow$" For any $\mathbf{i}_{\overline{12}} \in \{0, 1\}^{n-2}$, define $g(x_1, \ldots, x_n) = \mathbb{I}\{x_1 < 1,\ x_2 = 1\}\, \mathbb{I}\{\mathbf{x}_{\overline{12}} = \mathbf{i}_{\overline{12}}\}$. It is easy to verify that $g(x_1, \ldots, x_n) \in \mathcal{G}^{12}_{lwsai}(n)$. Since $(I_1, \ldots, I_n)$ is LWSAI, we have $\mathbb{E}[g(I_1, I_2, \ldots, I_n)] \ge \mathbb{E}[g(I_2, I_1, \ldots, I_n)]$, which implies that $p(0, 1, \mathbf{i}_{\overline{12}}) \ge p(1, 0, \mathbf{i}_{\overline{12}})$. □

It follows from Proposition 2.4 and Lemma 5.3 that, for a multivariate Bernoulli random vector, LWSAI is equivalent to SAI.

Proposition 5.4. *Let a random vector* $(X_1, \ldots, X_n)$ *be nonnegative and independent of the multivariate Bernoulli random vector* $(I_1, \ldots, I_n)$*. If* $(X_1, \ldots, X_n)$ *is WSAI and* $(I_1, \ldots, I_n)$ *is LWSAI, then* $(X_1 I_1, \ldots, X_n I_n)$ *is WSAI.*

Proof. It suffices to show that

$$\mathbb{E}[g(X_1 I_1, \ldots, X_n I_n)] \ge \mathbb{E}[g(\pi_{ij}(X_1 I_1, \ldots, X_n I_n))], \tag{5.4}$$

for any $1 \le i < j \le n$ and $g \in \mathcal{G}^{ij}_{wsai}$. Without loss of generality, we assume $i = 1, j = 2$.

Noting that

$$\begin{aligned}\mathbb{E}[g(X_1 I_1, \ldots, X_n I_n)] = \sum_{\mathbf{i}_{\overline{12}} \in \{0,1\}^{n-2}} \Big\{ & p(0, 0, \mathbf{i}_{\overline{12}})\, \mathbb{E}[g(0, 0, X_3 i_3, \ldots, X_n i_n)] + p(0, 1, \mathbf{i}_{\overline{12}})\, \mathbb{E}[g(0, X_2, X_3 i_3, \ldots, X_n i_n)] \\ & + p(1, 0, \mathbf{i}_{\overline{12}})\, \mathbb{E}[g(X_1, 0, X_3 i_3, \ldots, X_n i_n)] + p(1, 1, \mathbf{i}_{\overline{12}})\, \mathbb{E}[g(X_1, X_2, X_3 i_3, \ldots, X_n i_n)] \Big\}.\end{aligned} \tag{5.5}$$

Similarly,

$$\begin{aligned}\mathbb{E}[g(X_2 I_2, X_1 I_1, \ldots, X_n I_n)] = \sum_{\mathbf{i}_{\overline{12}} \in \{0,1\}^{n-2}} \Big\{ & p(0, 0, \mathbf{i}_{\overline{12}})\, \mathbb{E}[g(0, 0, X_3 i_3, \ldots, X_n i_n)] \\ & + p(0, 1, \mathbf{i}_{\overline{12}})\, \mathbb{E}[g(X_2, 0, X_3 i_3, \ldots, X_n i_n)] \\ & + p(1, 0, \mathbf{i}_{\overline{12}})\, \mathbb{E}[g(0, X_1, X_3 i_3, \ldots, X_n i_n)] \\ & + p(1, 1, \mathbf{i}_{\overline{12}})\, \mathbb{E}[g(X_2, X_1, X_3 i_3, \ldots, X_n i_n)] \Big\}.\end{aligned} \tag{5.6}$$

Recalling that $(X_1, \ldots, X_n)$ is WSAI and $g(x_1, x_2, x_3 i_3, \ldots, x_n i_n) \in \mathcal{G}^{12}_{wsai}(n)$ for any $\mathbf{i}_{\overline{12}} \in \{0, 1\}^{n-2}$, we have

$$\mathbb{E}[g(X_1, X_2, X_3 i_3, \ldots, X_n i_n)] \ge \mathbb{E}[g(X_2, X_1, X_3 i_3, \ldots, X_n i_n)], \tag{5.7}$$

for any $\mathbf{i}_{\overline{12}} \in \{0, 1\}^{n-2}$.

Denote $l(x_1,\dots,x_n) = g(0,x_2,x_3i_3,\dots,x_ni_n) + g(x_1,0,x_3i_3,\dots,x_ni_n)$. It is easy to verify that $\Delta_{12}l(x_1,\dots,x_n)$ is increasing in x_2, i.e., $l \in \mathcal{G}^{12}_{wsai}(n)$. Recalling that $(X_1,\dots,X_n)$ is WSAI, we have $\mathbb{E}[l(X_1,X_2,\dots,X_n)] \ge \mathbb{E}[l(X_2,X_1,\dots,X_n)]$, which implies

$$\begin{aligned}&\mathbb{E}[g(0,X_2,X_3i_3,\dots,X_ni_n)] - \mathbb{E}[g(X_2,0,X_3i_3,\dots,X_ni_n)]\\ &\quad\ge \mathbb{E}[g(0,X_1,X_3i_3,\dots,X_ni_n)] - \mathbb{E}[g(X_1,0,X_3i_3,\dots,X_ni_n)].\end{aligned} \tag{5.8}$$

Note that both sides of (5.8) are nonnegative since $X_1, X_2 \ge 0$ and $g \in \mathcal{G}^{12}_{wsai}(n) \subset \mathcal{G}^{12}_{sai}(n)$. Recalling that $p(0,1,\mathbf{i}_{\overline{12}}) \ge p(1,0,\mathbf{i}_{\overline{12}}) \ge 0$ since $(I_1,\dots,I_n)$ is LWSAI, we have

$$\begin{aligned}&p(0,1,\mathbf{i}_{\overline{12}})\big(\mathbb{E}[g(0,X_2,X_3i_3,\dots,X_ni_n)] - \mathbb{E}[g(X_2,0,X_3i_3,\dots,X_ni_n)]\big)\\ &\quad\ge p(1,0,\mathbf{i}_{\overline{12}})\big(\mathbb{E}[g(0,X_1,X_3i_3,\dots,X_ni_n)] - \mathbb{E}[g(X_1,0,X_3i_3,\dots,X_ni_n)]\big),\end{aligned}$$

or

$$\begin{aligned}&p(0,1,\mathbf{i}_{\overline{12}})\,\mathbb{E}[g(0,X_2,X_3i_3,\dots,X_ni_n)] + p(1,0,\mathbf{i}_{\overline{12}})\,\mathbb{E}[g(X_1,0,X_3i_3,\dots,X_ni_n)]\\ &\quad\ge p(0,1,\mathbf{i}_{\overline{12}})\,\mathbb{E}[g(X_2,0,X_3i_3,\dots,X_ni_n)] + p(1,0,\mathbf{i}_{\overline{12}})\,\mathbb{E}[g(0,X_1,X_3i_3,\dots,X_ni_n)].\end{aligned} \tag{5.9}$$

A combination of (5.7) and (5.9) implies that the right side of (5.5) is no less than that of (5.6), which means that (5.4) holds. □

Combining Theorem 5.2(iii) and Proposition 5.4, we immediately get the following result.

Corollary 5.5. *Assume that a random vector $(X_1,\dots,X_n)$ is nonnegative and WSAI, and the multivariate Bernoulli random vector $(I_1,\dots,I_n)$ is LWSAI and independent of $(X_1,\dots,X_n)$. The optimal solution $(a_1^*,\dots,a_n^*)$ to Problem (1.2) should satisfy $a_1^* \le \cdots \le a_n^*$ for any increasing concave utility function $u(z)$ such that $zu'(z)$ is increasing.* □

Corollary 5.5 generalizes Theorem 3.1(1) of Chen and Hu [2]. Furthermore, many utility functions satisfy the condition that $zu'(z)$ is increasing. For example, the power utility function $u(z) = z^\theta$ with $0 < \theta \le 1$, the log utility function $u(z) = \log z$, the exponential utility function $u(z) = 1 - e^{-\gamma z}$, $\gamma > 0$ with $z \le \frac{1}{\gamma}$.

Proposition 5.6. *Let a random vector $(X_1,\dots,X_n)$ be independent of the multivariate Bernoulli random vector $(I_1,\dots,I_n)$. If $(X_1,\dots,X_n)$ is exchangeable and $(I_1,\dots,I_n)$ is LWSAI, then $(X_1I_1,\dots,X_nI_n)$ is LWSAI.*

Proof. It suffices to show that

$$\mathbb{E}[g(X_1I_1,X_2I_2,\dots,X_nI_n)] \ge \mathbb{E}[g(X_2I_2,X_1I_1,\dots,X_nI_n)], \tag{5.10}$$

for any $g \in \mathcal{G}^{12}_{lwsai}(n)$.

The proof is the same as the proof of Proposition 5.4 except that the condition $g \in \mathcal{G}^{12}_{wsai}(n)$ is now replaced by $g \in \mathcal{G}^{12}_{lwsai}(n)$. Note that the exchangeability of $(X_1,\dots,X_n)$ implies that $(X_1,\dots,X_n)$ is LWSAI and also implies that (5.8) holds with equality. □

Proposition 5.6 together with Theorem 5.2 (ii) implies Corollary 2 of Cheung and Yang [3].

5.3. Default risk model with threshold default indicators

In this subsection, we consider the default risk model, but with a different type of default indicators. We assume that each return rate is not realized until the rate reaches a certain threshold, otherwise the actual return rate is 0. Specifically, denote the actual return rates as X_k^{act}, $k = 1,\dots,n$, thus $X_k^{act} = X_k\,\mathbb{I}\{X_k > l_k\}$, $k = 1,\dots,n$, where $l_1,\dots,l_n$ are predetermined thresholds. Then the investor's objective is to maximize the expected utility of the total return rate, i.e., to study the following optimal selection problem:

$$\max_{(a_1,\dots,a_n)\in\mathcal{A}_n} \mathbb{E}\left[u\left(\sum_{k=1}^n a_kX_k\,\mathbb{I}\{X_k > l_k\}\right)\right], \tag{5.11}$$

for some utility function u.

Proposition 5.7. *Let $l_1,\dots,l_n$ be real numbers such that $l_1 \ge \cdots \ge l_n$. If a random vector $(X_1,\dots,X_n)$ is nonnegative and LWSAI, then $(X_1\,\mathbb{I}\{X_1 > l_1\},\dots,X_n\,\mathbb{I}\{X_n > l_n\})$ is LWSAI.*

Proof. Denote $\mathbf{Y} = (Y_1, \ldots, Y_n) = (X_1 \mathbb{I}\{X_1 > l_1\}, \ldots, X_n \mathbb{I}\{X_n > l_n\})$. Without loss of generality, it suffices to show that

$$\mathbb{P}\{Y_1 \le x_1,\ x_2 < Y_2 \le x_2 + t,\ \mathbf{Y}_{\overline{12}} \in B_{\overline{12}}\} \ge \mathbb{P}\{Y_2 \le x_1,\ x_2 < Y_1 \le x_2 + t,\ \mathbf{Y}_{\overline{12}} \in B_{\overline{12}}\}, \tag{5.12}$$

for any $x_1 \le x_2$, $t > 0$, and $B_{\overline{12}} \in \sigma(\mathbf{Y}_{\overline{12}})$. We only consider the case that $x_1 \ge 0$. Otherwise, if $x_1 < 0$, both sides of (5.12) are equal to 0.

When $x_1 \ge 0$, we see that $Y_1 \le x_1$ is equivalent to $X_1 \le x_1 \vee l_1$, where $x_1 \vee l_1 = \max\{x_1, l_1\}$. Then (5.12) is equivalent to

$$\begin{aligned} &\mathbb{P}\{X_1 \in (-\infty,\ x_1 \vee l_1],\ X_2 \in (x_2, x_2 + t] \cap (l_2, \infty),\ \mathbf{X}_{\overline{12}} \in A_{\overline{12}}\} \\ &\quad \ge \mathbb{P}\{X_2 \in (-\infty,\ x_1 \vee l_2],\ X_1 \in (x_2, x_2 + t] \cap (l_1, \infty),\ \mathbf{X}_{\overline{12}} \in A_{\overline{12}}\}, \end{aligned} \tag{5.13}$$

where $A_{\overline{12}} \in \sigma(\mathbf{X}_{\overline{12}})$ is such that $\mathbf{X}_{\overline{12}} \in A_{\overline{12}} \Longleftrightarrow \mathbf{Y}_{\overline{12}} \in B_{\overline{12}}$.

On the other hand, (5.13) follows from

$$\begin{aligned} &\mathbb{P}\{X_1 \in (-\infty,\ x_1 \vee l_1],\ X_2 \in (x_2, x_2 + t] \cap (l_2, \infty),\ \mathbf{X}_{\overline{12}} \in A_{\overline{12}}\} \\ &\quad \ge \mathbb{P}\{X_1 \in (-\infty,\ x_1 \vee l_2],\ X_2 \in (x_2, x_2 + t] \cap (l_1, \infty),\ \mathbf{X}_{\overline{12}} \in A_{\overline{12}}\} \\ &\quad \ge \mathbb{P}\{X_2 \in (-\infty,\ x_1 \vee l_2],\ X_1 \in (x_2, x_2 + t] \cap (l_1, \infty),\ \mathbf{X}_{\overline{12}} \in A_{\overline{12}}\}, \end{aligned}$$

where the first inequality holds because $x_1 \vee l_1 \ge x_1 \vee l_2$ and $(x_2, x_2+t] \cap (l_2, \infty) \supset (x_2, x_2+t] \cap (l_1, \infty)$ for $l_1 > l_2$, and the second inequality is due to Proposition 3.5 and the fact that $\sup\{(-\infty,\ x_1 \vee l_2]\} = x_1 \vee l_2 \le x_2 \vee l_1 = \inf\{(x_2, x_2+t] \cap (l_1, \infty)\}$ for $x_1 \le x_2$ and $l_1 > l_2$. In the case of $(x_2, x_2 + t] \cap (l_1, \infty) = \emptyset$, the second inequality holds with equality. □

Combining Proposition 5.7 and Theorem 5.2, we draw the following conclusion on the optimal solution to Problem (5.11).

Proposition 5.8. *Assume $l_1 \ge \cdots \ge l_n$. If the random vector $(X_1, \ldots, X_n)$ in (5.11) is nonnegative and LWSAI, then the optimal solution $(a_1^*, \ldots, a_n^*)$ to Problem (5.11) should satisfy $a_1^* \le \cdots \le a_n^*$ for any increasing concave utility function u.* □

Proposition 5.8 generalizes Theorem 1 of Cheung and Yang [3].

5.4. Mixture risk model with fundamental risks

In this subsection, we consider the mixture risk model and study Problem (1.3). Cheung and Yang [4] first studied Problem (1.3), for $J = \{1, \ldots, m\}$, their assumptions on the dependence structures of risks are reduced to the following:

(a) $(X_1, \ldots, X_m)$ is independent of $(M_1, \ldots, M_n)$.
(b) $X_1, \ldots, X_m$ are comonotonic with $X_1 \le_{st} \cdots \le_{st} X_m$.
(c) $M_1, \ldots, M_n$ are mutually independent with $M_1 \le_{lr} \cdots \le_{lr} M_n$.

In the following, we relax the assumption (c) and restudy Problem (1.3).

Lemma 5.9. *Let f be a nondecreasing univariate function. If $(X_1, \ldots, X_n)$ is SAI/LWSAI/RWSAI/WSAI, then $(f(X_1), \ldots, f(X_n))$ is SAI/LWSAI/RWSAI/WSAI.*

Proof. The proof is straightforward by noting that

$$g(x_1, \ldots, x_n) \in \mathcal{G}_{sai}^{ij}(n)\ (\mathcal{G}_{lwsai}^{ij}(n),\ \mathcal{G}_{rwsai}^{ij}(n),\ \mathcal{G}_{wsai}^{ij}(n))$$

implies that $g(f(x_1), \ldots, f(x_n)) \in \mathcal{G}_{sai}^{ij}(n)\ (\mathcal{G}_{lwsai}^{ij}(n),\ \mathcal{G}_{rwsai}^{ij}(n),\ \mathcal{G}_{wsai}^{ij}(n))$ for any nondecreasing f and $1 \le i < j \le n$. □

Theorem 5.10. *Let $(X_1, \ldots, X_m)$ and $(M_1, \ldots, M_n)$ be two independent random vectors with M_i taking values in $\{1, \ldots, m\}$ for $i = 1, \ldots, n$. Assume that $(X_1, \ldots, X_n)$ is comonotonic and $X_1 \le_{st} \cdots \le_{st} X_m$. If $(M_1, \ldots, M_n)$ is SAI/LWSAI/RWSAI/WSAI, then $(X_{M_1}, \ldots, X_{M_n})$ is also SAI/LWSAI/RWSAI/WSAI.*

Proof. We only provide the proof for the LWSAI case. The proofs for the other cases are similar.

Consider any real-valued vector $(x_1, \ldots, x_m)$ such that $x_1 \le \cdots \le x_m$. Let f be a function defined on $\{1, \ldots, m\}$ with $f(j) = x_j$ for any $j \in \{1, 2, \ldots, m\}$. Then f is nondecreasing. Assume that $(M_1, \ldots, M_n)$ is LWSAI, following Lemma 5.9, we know that $(x_{M_1}, \ldots, x_{M_n})$ is LWSAI. Recalling Lemma 5.3 of Cai and Wei [1], we have $X_1 \le_{a.s.} \cdots \le_{a.s.} X_m$. Therefore, for any $1 \le i < j \le n$ and $g \in \mathcal{G}_{lwsai}^{ij}(n)$, we have

$$\begin{aligned} \mathbb{E}[g(X_{M_1}, \ldots, X_{M_n})] &= \mathbb{E}[\mathbb{E}[g(X_{M_1}, \ldots, X_{M_n}) \mid (X_1, \ldots, X_m)]] \\ &\ge \mathbb{E}[\mathbb{E}[g(\pi_{ij}((X_{M_1}, \ldots, X_{M_n}))) \mid (X_1, \ldots, X_m)]] = \mathbb{E}[g(\pi_{ij}((X_{M_1}, \ldots, X_{M_n})))], \end{aligned}$$

which implies that $(X_{M_1}, \ldots, X_{M_n})$ is LWSAI. □

From Theorems 5.2 and 5.10, we immediately get the following result.

Proposition 5.11. *Let $(X_1, \ldots, X_m)$ be comonotonic and $X_1 \le_{st} \cdots \le_{st} X_m$. Let $(M_1, \ldots, M_n)$ be independent of $(X_1, \ldots, X_m)$. The optimal solution $(a_1^*, \ldots, a_n^*)$ to Problem (1.3) satisfies $a_1^* \le \cdots \le a_n^*$ if any of the following conditions holds:*

(i) *$(M_1, \ldots, M_n)$ is SAI and $u(z)$ is increasing;*
(ii) *$(M_1, \ldots, M_n)$ is LWSAI and $u(z)$ is increasing and concave;*
(iii) *$(M_1, \ldots, M_n)$ is WSAI, $X_i \ge 0$ for all $i = 1, \ldots, m$, $u(z)$ is increasing and concave, and $zu'(z)$ is increasing.* □

It follows from Propositions 5.4 and 5.5 of Cai and Wei [1] that assumption (c) implies that $(M_1, \ldots, M_n)$ is SAI. Therefore, we conclude that Theorem 1 of Cheung and Yang [4] is a special case of Proposition 5.11(i) for the case of $J = \{1, \ldots, m\}$.

Moreover, Hu and Wang [8] applied the mixture risk model, together with assumptions (a), (b), and (c), to study the optimal allocation of deductibles for insurance companies. We point out that, with the help of Theorem 5.10, their results can be also generalized as well.

6. Concluding remarks

In this paper, we propose the new dependence notions of LWSAI and WSAI. These dependence notions of LWSAI and WSAI are characterized by the joint distribution function of a random vector and are the complementary to the dependence notions RWSAI and SAI, which are proposed by Cai and Wei [1], in the sense that the dependence notions of Cai and Wei [1] are characterized by the joint survival function of a random vector. We apply the notions of LWSAI and WSAI to study the optimal portfolio selections with more general dependent risks and generalize many existing studies in this field including those of Hadar and Seo [6], Hennessy and Lapan [7], Cheung and Yang [3], Chen and Hu [2], and Cheung and Yang [4]. Furthermore, we point out that the notions of LWSAI and WSAI allow potential applications in other fields such as scheduling, insurance, and so on. We will present more applications of these notions of dependence in our future research.

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