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Joint stochastic orders of high degrees and their applications in portfolio selections

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ABSTRACT

In this paper, we propose two new classes of joint stochastic orders, namely joint (reversed) hazard order of degree n and joint n-increasing convex/concave order, and establish their theoretical properties. These new orders substantially generalize the existing class of joint stochastic orders, and incorporate them in one general framework. We also explore the applications of these orders in portfolio selections and unify similar studies on this problem.

1. Introduction

Stochastic orders are well-established tools to compare random variables. The standard literature in this area includes Müller and Stoyan (2002) and Shaked and Shanthikumar (2007). However, traditional stochastic orders compare only marginal distributions and thus do not concern dependence structures of random variables in comparison. To address this limitation, Shanthikumar and Yao (1991) were among the first to incorporate dependence structures into stochastic comparisons. Specifically, they proposed a few concepts of joint stochastic orders: $\leq_{lr:j}$, $\leq_{hr:j}$, $\leq_{rh:j}$ and $\leq_{st:j}$, which are dependent versions of $\leq_{lr}$, $\leq_{hr}$, $\leq_{rh}$ and $\leq_{st}$, respectively. The joint stochastic orders show wide applications in different areas. Shanthikumar and Yao (1991) themselves gave applications in operations research immediately after proposing these concepts. Later on, Kijima and Ohnishi (1996) employed the joint likelihood ratio order $\leq_{lr:j}$ to study portfolio selections and thereby motivated a sequence of similar studies in finance.

Note that the above joint stochastic orders do not possess transitivity. For example, $X \leq_{lr:j} Y$ and $Y \leq_{lr:j} Z$ do not necessarily imply $X \leq_{lr:j} Z$. The lack of transitivity makes it difficult to extend the stochastic comparison to multiple random variables. Motivated by this limitation, Cai and Wei (2014, 2015) proposed the notions of SAI, RWSAI and LWSAI, which are essentially multivariate generalizations of $\leq_{lr:j}$, $\leq_{hr:j}$, and $\leq_{rh:j}$. They established different characterizations of these notions and constructed typical examples of these notions. Recent applications of these notions in finance and insurance can be found in Pan et al. (2015) and You and Li (2016). Applications in reliability theory are discussed in Li and Li (2016).

All the above joint stochastic orders, including the multivariate generalizations, imply the usual stochastic order $\leq_{st}$ between marginal distributions. We point out that the usual stochastic order is a very strong order and thus limits the applications of these joint stochastic orders in modeling random quantities in real-life. In this paper, we propose to generalize the existing joint stochastic orders to more flexible orders, which on one hand still incorporate dependence structures, and one the other hand relax the requirement on marginal distributions. In doing so, the new orders are more flexible in modeling reality.

The rest of the paper is organized as follows. Section 2 recalls definitions of some classical stochastic orders and introduces some useful notations. Section 3 proposes new orders $\leq_{n\text{-}hr:j}$ and $\leq_{n\text{-}rh:j}$, which are essentially high degree generalizations of $\leq_{hr:j}$ and $\leq_{rh:j}$. Section 3 also establishes theoretical properties of the new orders. Section 4 defines another sequence of new stochastic orders: $\leq_{n\text{-}icx:j}$ and $\leq_{n\text{-}icv:j}$, and thus incorporates dependence structures into the classical high degree stochastic dominance. In this section, examples with typical dependence structures have been constructed. Section 5 discusses applications of new stochastic orders in finance. Section 6 gives concluding remarks.

2. Preliminaries

We first recall from Shaked and Shanthikumar (2007) definitions of four commonly used stochastic orders.

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Definition 2.1. Let X and Y be two random variables with distribution functions F_X and F_Y. Denote their survival functions by $\bar{F}_X$ and $\bar{F}_Y$, respectively.

(i) X is said to be smaller than Y in *usual stochastic order*, denoted as $X \le_{st} Y$, if $\bar{F}_X(x) \le \bar{F}_Y(x)$ for all $x \in \mathbb{R}$.

(ii) X is said to be smaller than Y in *hazard rate order*, denoted as $X \le_{hr} Y$, if $\bar{F}_Y(x)/\bar{F}_X(x)$ is increasing in $x \in \{x : \bar{F}_X(x) > 0\}$.

(iii) X is said to be smaller than Y in *reverse hazard rate order*, denoted as $X \le_{rh} Y$, if $F_Y(x)/F_X(x)$ is increasing in $x \in \{x : F_X(x) > 0\}$.

(iv) Assume X and Y have density functions f_X and f_Y. X is said to be smaller than Y in *likelihood ratio order*, denoted as $X \le_{lr} Y$, if $f_Y(x)/f_X(x)$ is increasing in $x \in \{x : f_X(x) > 0\}$.

It is well known that these orders have the following implications: $\le_{lr} \Rightarrow \le_{hr}(\le_{rh}) \Rightarrow \le_{st}$. In addition, the latter three orders can be characterized by the following classes of bivariate functions, respectively:

$$\mathcal{G}_{lr} = \{g : \Delta g(x, y) \ge 0 \text{ for all } y \ge x\},$$
$$\mathcal{G}_{hr} = \{g : \Delta g(x, y) \text{ is increasing in } y \in [x, \infty)\},$$
$$\mathcal{G}_{rh} = \{g : \Delta g(x, y) \text{ is decreasing in } y \in (-\infty, x]\},$$

where $\Delta g(x, y) = g(x, y) - g(y, x)$.

Theorem 2.2. *$X \le_{lr} Y$ (respectively $X \le_{hr}$, $X \le_{rh}$) if and only if $\mathbb{E}[g(X', Y')] \ge \mathbb{E}[g(Y', X')]$ for any $g \in \mathcal{G}_{lr}$ (respectively $g \in \mathcal{G}_{hr}$, $g \in \mathcal{G}_{rh}$), where $X' \overset{d}{=} X$, $Y' \overset{d}{=} Y$, and X' is independent of Y'.*

The proof of Theorem 2.2 can be found in the classical literature, for example, Shaked and Shanthikumar (2007) and Shanthikumar and Yao (1991) among others.

Note that the above orders are defined through only marginal distributions and thus do not involve dependence structure. Based on the bivariate characterization in Theorem 2.2, Shanthikumar and Yao (1991) proposed the concepts of joint likelihood ratio orders, joint hazard rate order and joint reversed hazard rate order, which incorporate dependence structure into comparison of random variables. These orders are defined as follows.

Definition 2.3. Random variable X is said to be smaller than Y in *joint likelihood ratio order* (respectively, *joint hazard rate order*, *joint reversed hazard rate order*), denoted as $X \le_{lr:j} Y$ (respectively, $X \le_{hr:j} Y$, $X \le_{rh:j} Y$), if $\mathbb{E}[g(X, Y)] \ge \mathbb{E}[g(Y, X)]$ for any $g \in \mathcal{G}_{lr}$ (respectively $g \in \mathcal{G}_{hr}$, $g \in \mathcal{G}_{rh}$).

For advanced properties regarding these joint stochastic orders, readers are referred to Shanthikumar and Yao (1991) and Righter and Shanthikumar (1992). Cai and Wei (2014, 2015) have generalized these orders to multivariate random variables and discussed their applications in finance and insurance. For more applications, see Li and You (2015), and You and Li (2016).

For positive integer n, define

$$\mathcal{U}_{n\text{-}icx} = \left\{u : \frac{d^k}{dz^k}u(z) \text{ is increasing in } z \text{ for all } k = 0, 1, \ldots, n-1\right\},$$
$$\mathcal{U}_{n\text{-}icv} = \left\{u : (-1)^{k-1}\frac{d^k}{dz^k}u(z) \text{ is decreasing in } z \text{ for all } k = 0, 1, \ldots, n-1\right\}.$$

Definition 2.4. Random variable X is said to be less than random variable Y in the *n-increasing convex (concave) order*, denoted as $X \le_{n\text{-}icx} Y$ ($X \le_{n\text{-}icv} Y$), if $\mathbb{E}[u(X)] \le \mathbb{E}[u(Y)]$ for any function $u \in \mathcal{U}_{n\text{-}icx}$ ($u \in \mathcal{U}_{n\text{-}icv}$) such that the expectations exist.

There are different versions of definitions for $\le_{n\text{-}icx}$ and $\le_{n\text{-}icv}$. Definition 2.4 is taken from Müller and Stoyan (2002, Definition 1.6.2). Readers are referred to Shaked and Shanthikumar (2007) for a comprehensive review on different variations of this definition. We point that function classes $\mathcal{U}_{n\text{-}icx}$ and $\mathcal{U}_{n\text{-}icv}$ can be defined more generally by removing differentiability assumptions, see Müller and Stoyan (2002) or Shaked and Wong (1995). In this paper, we keep differentiability assumptions to avoid technical discussions. However, to slightly enlarge the classes of $\mathcal{U}_{n\text{-}icx}$ and $\mathcal{U}_{n\text{-}icv}$, we allow the notation $\frac{d}{dz}u(z)$ to represent the left derivative when two sided derivatives do not coincide. Under this convention, $\mathcal{U}_{2-icx}$ denotes the collection of all increasing and convex functions.

We remark that $\le_{1\text{-}icx}$ or $\le_{1\text{-}icv}$ is the usual stochastic order $\le_{st}$, and $\le_{2\text{-}icx}$ ($\le_{2\text{-}icv}$) is the regular increasing convex (concave) order. Clearly, $X \le_{n_1\text{-}icx} Y$ implies $X \le_{n_2\text{-}icx} Y$ for any $n_1 \le n_2$.

At the end of the section, we present some general assumptions and notations that will be used throughout the paper. For a random vector (X, Y), unless otherwise specified, we assume the joint density function, denoted as $f_{XY}(x, y)$, exists and is continuous. Its marginal density functions are denoted as $f_X(x)$ and $f_Y(x)$, respectively. Consider a bivariate function $g(x, y)$. Whenever its cross partial derivatives exist, we assume they are commutable, that is, $\frac{\partial^{i+j}}{\partial x^i \partial y^j}g(x, y) = \frac{\partial^{i+j}}{\partial y^j \partial x^i}g(x, y)$. For notional convenience, denote $g^{(i,j)}(x, y) = \frac{\partial^{i+j}}{\partial x^i \partial y^j}g(x, y)$. The notation $\Delta g(x, y)$ has been used to denote $g(x, y) - g(y, x)$. Clearly, $\Delta g(x, x) = 0$ and $\Delta g(x, y) = -\Delta g(y, x)$. Furthermore, denote $\Delta^{(i,j)}g(x, y) = \frac{\partial^{i+j}}{\partial x^i \partial y^j}\Delta g(x, y)$.

It is important to note the difference between $\Delta^{(i,j)}g(x, y)$ and $\Delta g^{(i,j)}(x, y)$. The former operation takes difference first and then takes derivatives, while the latter one operates in an opposite order. For example, $\Delta^{(0,1)}g(x, y) = g^{(0,1)}(x, y) - g^{(1,0)}(y, x)$, while $\Delta g^{(0,1)}(x, y) = g^{(0,1)}(x, y) - g^{(0,1)}(y, x)$. This paper involves the use of only the former notation.

3. Generalization of $\le_{lr:j}$, $\le_{hr:j}$, $\le_{rh:j}$ to high degrees

In this section, we shall generalize the joint stochastic orders $\le_{lr:j}$, $\le_{hr:j}$, $\le_{rh:j}$ to higher degrees. In doing so, we define two classes of bivariate functions. For a positive integer n, define

$$\mathcal{G}_{n\text{-}rh} = \{g : \Delta g(x, y) \in \mathcal{U}_{n\text{-}icv} \text{ as a function of } y \text{ on } (-\infty, x]\}, \quad (3.1)$$
$$\mathcal{G}_{n\text{-}hr} = \{g : \Delta g(x, y) \in \mathcal{U}_{n\text{-}icx} \text{ as a function of } y \text{ on } [x, \infty)\}. \quad (3.2)$$

Clearly, $\mathcal{G}_{1\text{-}rh} = \mathcal{G}_{rh}$ and $\mathcal{G}_{1\text{-}hr} = \mathcal{G}_{hr}$. Furthermore, with the convention $\mathcal{U}_{0\text{-}icx} = \mathcal{U}_{0\text{-}icv} = \{u(x), u(x) \ge 0\}$, both $\mathcal{G}_{0\text{-}hr}$ and $\mathcal{G}_{0\text{-}rh}$ reduce to $\mathcal{G}_{lr}$. A trivial example belonging to the class $\mathcal{G}_{n\text{-}rh}$ ($\mathcal{G}_{n\text{-}hr}$) is $g(x, y) = u(y)$ with $u \in \mathcal{U}_{n\text{-}icv}$ ($\mathcal{U}_{n\text{-}icx}$). As illustrated in Section 5, $g(x, y) = u(ax + by)$ with $a \le b$ and certain assumptions on u also belongs to $\mathcal{G}_{n\text{-}rh}$.

Definition 3.1. Random variable X is said to be smaller than Y in the *joint reversed hazard rate order of degree n (joint hazard rate order of degree n)*, denoted as $X \le_{n\text{-}rh:j} Y$ ($X \le_{n\text{-}hr:j} Y$), if $\mathbb{E}[g(X, Y)] \ge \mathbb{E}[g(Y, X)]$ for all $g \in \mathcal{G}_{n\text{-}rh}$ ($g \in \mathcal{G}_{n\text{-}hr}$) such that the expectations exist.

We note that $\le_{0\text{-}rh:j}$, $\le_{1\text{-}rh:j}$ and $\le_{1\text{-}hr:j}$ correspond to $\le_{lr:j}$, $\le_{rh:j}$ and $\le_{hr:j}$ (see Section 2), respectively. It is easy to see that, $\le_{n_1\text{-}rh:j}$ ($\le_{n_1\text{-}hr:j}$) implies $\le_{n_2\text{-}rh:j}$ ($\le_{n_2\text{-}hr:j}$) for any $n_1 \le n_2$. In this section, we shall focus on the new orders (with $n \ge 2$) and establish their properties.

First of all, there is a close relationship between $\le_{n\text{-}rh:j}$ and $\le_{n\text{-}hr:j}$.

Proposition 3.2. *$X \le_{n\text{-}rh:j} Y$ if and only if $-Y \le_{n\text{-}hr:j} -X$.*

Proof. For any bivariate function $g(x,y)$, denote $h(x,y) = -g(-x,-y)$. Then $\Delta h(x,y) = -\Delta g(-x,-y) = \Delta g(-y,-x)$. It is easy to verify that $h \in \mathcal{U}_{n\text{-}rh}$ if and only if $g \in \mathcal{U}_{n\text{-}hr}$. Then the proof is completed by applying the definitions of $\le_{n\text{-}rh:j}$ and $\le_{n\text{-}hr:j}$. □

Proposition 3.2 indicates that $\le_{n\text{-}rh:j}$ and $\le_{n\text{-}hr:j}$ are a pair of dual notions. Therefore, they share many similar properties. In the following, we shall focus on developing properties of the order $\le_{n\text{-}rh:j}$ and briefly mention their analogues for $\le_{n\text{-}hr:j}$.

In order to characterize the orders $\le_{n\text{-}rh:j}$ and $\le_{n\text{-}hr:j}$ from a different perspective, we introduce two sequences of functions. For a random vector (X,Y) and a positive integer n, assume $\mathbb{E}[|X|^{n-1}] < \infty$ and $\mathbb{E}[|Y|^{n-1}] < \infty$. Define

$$M_{XY}^{[0]}(x,y) = f_{XY}(x,y),$$
$$M_{XY}^{[n]}(x,y) = \frac{1}{(n-1)!}\int_{-\infty}^{x}(x-s)^{n-1}f_{XY}(s,y)ds, \tag{3.3}$$
$$N_{XY}^{[0]}(x,y) = f_{XY}(x,y),$$
$$N_{XY}^{[n]}(x,y) = \frac{1}{(n-1)!}\int_{y}^{\infty}(t-y)^{n-1}f_{XY}(x,t)dt. \tag{3.4}$$

It is easy to verify that for $k = 1,2,\ldots,n-1$,

$$M_{XY}^{[k+1]}(x,y) = \int_{-\infty}^{x} M_{XY}^{[k]}(s,y)ds \quad \text{and}$$
$$N_{XY}^{[k+1]}(x,y) = \int_{y}^{\infty} N_{XY}^{[k]}(x,t)dt.$$

For $n \ge 2$, these functions admit the following representations in terms of expectations.

$$M_{XY}^{[n]}(x,y) = \frac{1}{(n-1)!}f_Y(y)\mathbb{E}[(x-X)_+^{n-1}|Y=y],$$
$$N_{XY}^{[n]}(x,y) = \frac{1}{(n-1)!}f_X(x)\mathbb{E}[(Y-y)_+^{n-1}|X=x],$$

where $f_X(x)$ and $f_Y(y)$ denote the pdf's of X and Y, respectively, and $x_+ = \max\{x,0\}$.

Theorem 3.3. *Assume $\mathbb{E}[|X|^{n-1}] < \infty$ and $\mathbb{E}[|Y|^{n-1}] < \infty$. $X \le_{n\text{-}rh:j} Y$ if and only if $M_{XY}^{[k]}(x,x) \ge M_{YX}^{[k]}(x,x)$ for any x and $k = 1,2,\ldots,n-1$, and $M_{XY}^{[n]}(x,y) \ge M_{YX}^{[n]}(x,y)$ for any $x \le y$.*

Proof. The "if" part. Noting that $\Delta g(x,x) = 0$ and $\Delta g(x,y) = -\Delta g(y,x)$, one obtains that $\Delta g(x,y) = \Delta g(x,y)\mathbb{I}\{y \le x\} - \Delta g(y,x)\mathbb{I}\{x \le y\}$. It suffices to show that $\mathbb{E}[\Delta g(X,Y)] \ge 0$, or $\mathbb{E}[\Delta g(X,Y)\mathbb{I}\{Y \le X\}] \ge \mathbb{E}[\Delta g(Y,X)\mathbb{I}\{X \le Y\}]$ for any $g \in \mathcal{G}_{n\text{-}rh}$.

For simplicity, assume $\Delta^{(0,k)}g(x,y)$, $k = 1,2,\ldots,n$ all exist. Then $g \in \mathcal{G}_{n\text{-}rh}$ implies $(-1)^{k-1}\Delta^{(0,k)}g(x,y) \ge 0$ for all $x \ge y$ and $k = 1,2,\ldots,n-1$. Using integration by parts, one gets

$$\begin{aligned}&\int_{-\infty}^{\infty}\int_{-\infty}^{x}\Delta^{(0,k)}g(x,y)M_{YX}^{[k]}(y,x)dydx\\ &\quad = -\int_{-\infty}^{\infty}\Delta^{(0,k)}g(x,x)M_{YX}^{[k+1]}(x,x)dx\\ &\qquad + \int_{-\infty}^{\infty}\int_{-\infty}^{x}\Delta^{(0,k+1)}g(x,y)M_{YX}^{[k+1]}(y,x)dydx.\end{aligned}$$

Applying this formula recursively, one obtains

$$\begin{aligned}&\mathbb{E}[\Delta g(X,Y)\mathbb{I}\{Y \le X\}]\\ &\quad = \int_{-\infty}^{\infty}\int_{-\infty}^{x}\Delta g(x,y)f_{XY}(x,y)dydx\\ &\quad = \int_{-\infty}^{\infty}\int_{-\infty}^{x}\Delta^{(0,0)}g(x,y)M_{YX}^{[0,0]}(y,x)dydx\\ &\quad = \sum_{k=0}^{n-1}\int_{-\infty}^{\infty}(-1)^k\Delta^{(0,k)}g(x,x)M_{YX}^{[k+1]}(x,x)dx\\ &\qquad + \int_{-\infty}^{\infty}\int_{-\infty}^{x}(-1)^n\Delta^{(0,n)}g(x,y)M_{YX}^{[n]}(y,x)dydx.\end{aligned}$$

Interchanging the position of X and Y, one gets

$$\begin{aligned}&\mathbb{E}[\Delta g(Y,X)\mathbb{I}\{X \le Y\}]\\ &\quad = \sum_{k=0}^{n-1}\int_{-\infty}^{\infty}(-1)^k\Delta^{(0,k)}g(x,x)M_{XY}^{[k+1]}(x,x)dx\\ &\qquad + \int_{-\infty}^{\infty}\int_{-\infty}^{x}(-1)^n\Delta^{(0,n)}g(x,y)M_{XY}^{[n]}(y,x)dydx.\end{aligned}$$

Noting $\Delta^{(0,0)}g(x,x) = 0$, the inequality $\mathbb{E}[\Delta g(X,Y)\mathbb{I}\{Y \le X\}] \ge \mathbb{E}[\Delta g(Y,X)\mathbb{I}\{X \le Y\}]$ follows from the assumptions and the fact that $(-1)^{k-1}\Delta^{(0,k)}g(x,y) \ge 0$ for any $x \ge y$ and $k = 1,2,\ldots,n$.

The "only if" part. Define a bivariate function $g_\epsilon(s,t) = (x-s)_+^{n-1}\mathbb{I}\{t \in [y, y+\epsilon)\}$ with $\epsilon > 0$. Since $\mathbb{E}[|X|^{n-1}] < \infty$, then $\mathbb{E}[|g_\epsilon(X,Y)|] \le \mathbb{E}[|x-X|^{n-1}] < \infty$. Thus, $\mathbb{E}[g_\epsilon(X,Y)] = \int_y^{y+\epsilon}\int_{-\infty}^{x}(x-s)^{n-1}f_{XY}(s,t)dsdt$. Therefore, $\lim_{\epsilon\downarrow 0}\frac{\mathbb{E}[g_\epsilon(X,Y)]}{\epsilon} = \int_{-\infty}^{x}(x-s)^{n-1}f_{XY}(s,y)ds = M_{XY}^{[n]}(x,y)$. Similarly, $\mathbb{E}[|Y|^{n-1}] < \infty$ implies $\lim_{\epsilon\downarrow 0}\frac{\mathbb{E}[g_\epsilon(Y,X)]}{\epsilon} = M_{YX}^{[n]}(x,y)$. On the other hand, it is easy to verify that $g_\epsilon \in \mathcal{G}_{n\text{-}rh}$. Therefore, $\mathbb{E}[g_\epsilon(X,Y)] \ge \mathbb{E}[g_\epsilon(Y,X)]$ and thus $M_{XY}^{[n]}(x,y) \ge M_{YX}^{[n]}(x,y)$ for any $x \le y$.

For any $k = 1,2,\ldots,n-1$, define $h_\epsilon(s,t) = (t-s)_+^{k-1}\mathbb{I}\{t \in [x, x+\epsilon)\}$ with $\epsilon > 0$. Noting that $\mathbb{E}[h_\epsilon(X,Y)] = \int_x^{x+\epsilon}\int_{-\infty}^{t}(t-s)^{n-1}f_{XY}(s,t)dsdt$, one obtains that $\lim_{\epsilon\downarrow 0}\frac{\mathbb{E}[h_\epsilon(X,Y)]}{\epsilon} = M_{XY}^{[k]}(x,x)$. Similarly, $\lim_{\epsilon\downarrow 0}\frac{\mathbb{E}[h_\epsilon(Y,X)]}{\epsilon} = M_{YX}^{[k]}(x,x)$. For any $s \ge t$, it is easy to see that $h_\epsilon(s,t) = 0$, and thus $\Delta h_\epsilon(s,t) = -(s-t)^{k-1}\mathbb{I}\{s \in [x, x+\epsilon)\}$ belongs to $\mathcal{U}_{n\text{-}icv}$ as a function of t. This implies $h_\epsilon \in \mathcal{G}_{n\text{-}rh}$. Therefore, $\mathbb{E}[h_\epsilon(X,Y)] \ge \mathbb{E}[h_\epsilon(Y,X)]$ and thus $M_{XY}^{[k]}(x,x) \ge M_{YX}^{[k]}(x,x)$ for any x. □

When $n = 0$, the order $\le_{0\text{-}rh:j}$ reduces to $\le_{lr:j}$. When $n = 1$, the order $\le_{1\text{-}rh:j}$ reduces to $\le_{rh:j}$. Therefore, the characterization provided by Theorem 3.3 covers Theorems 2.12(a) in Shanthikumar and Yao (1991), and Proposition 3.7 (bivariate version) in Cai and Wei (2015) as special cases.

For a random pair (X,Y) following certain order, applications usually call for the comparison of $\mathbb{E}[g_1(X,Y)]$ and $\mathbb{E}[g_2(X,Y)]$ for two different functions g_1, g_2. The following two theorems provides a tool for such comparison.

Theorem 3.4. *$X \le_{n\text{-}rh:j} Y$ if and only if $\mathbb{E}[g_2(X,Y)] \ge \mathbb{E}[g_1(X,Y)]$ for any bivariate functions g_1, g_2 such that*

(i) *$g_2(x,y) - g_1(x,y) \in \mathcal{U}_{n\text{-}icv}$ as a function of y on $(-\infty, x]$,*
(ii) *$g_2(x,y) + g_2(y,x) \ge g_1(x,y) + g_1(y,x)$ for any x and y, with equality at $y = x$.*

Proof. The "if" part. For any $g \in \mathcal{G}_{n\text{-}rh}$, define $g_2(x,y) = g(x,y)$ and $g_1(x,y) = g(y,x)$. It is easy to verify that g_1, g_2 satisfy Conditions (i) and (ii). Therefore, $\mathbb{E}[g_2(X,Y)] \ge \mathbb{E}[g_1(X,Y)]$, which implies that $\mathbb{E}[g(X,Y)] \ge \mathbb{E}[g(Y,X)]$.

The "only if" part. For any g_1, g_2 satisfying Conditions (i) and (ii), define $h(x,y) = (g_2(x,y) - g_1(x,y)) \times \mathbb{I}\{x \ge y\}$. Noting that $g_1(x,x) = g_2(x,x)$ from (ii), one concludes that $\Delta h(x,y) = g_2(x,y) - g_1(x,y)$ for any $x \geq y$ and thus belongs to $\mathcal{G}_{n\text{-}rh}$. It follows from $X \leq_{n\text{-}rh:j} Y$ that $\mathbb{E}[h(X,Y)] \geq \mathbb{E}[h(Y,X)]$, i.e.,

$$\mathbb{E}[(g_2(X,Y) - g_1(X,Y))\mathbb{I}\{X \geq Y\}] \geq \mathbb{E}[(g_2(Y,X) - g_1(Y,X))\mathbb{I}\{Y \geq X\}]. \tag{3.5}$$

Recalling Condition (ii), one obtains $g_2(Y,X) - g_1(Y,X) \geq_{a.s.} g_1(X,Y) - g_2(X,Y)$, and thus $(g_2(Y,X) - g_1(Y,X))\mathbb{I}\{Y \geq X\} \geq_{a.s.} (g_1(X,Y) - g_2(X,Y))\mathbb{I}\{Y \geq X\}$, which implies

$$\mathbb{E}[(g_2(Y,X) - g_1(Y,X))\mathbb{I}\{Y \geq X\}] \geq \mathbb{E}[(g_1(X,Y) - g_2(X,Y))\mathbb{I}\{Y \geq X\}]. \tag{3.6}$$

From (3.5), (3.6), and the equality in Condition (ii), one gets $\mathbb{E}[g_2(X,Y)] \geq \mathbb{E}[g_1(X,Y)]$. □

Theorem 3.4 covers Theorem 1(i) of Righter and Shanthikumar (1992) and Theorem 6.1 of Cai and Wei (2014) as special cases, and generalize those results to joint stochastic orders of high degrees. We remark that the equality in Condition (ii) is not necessary when $n = 0$ or $n = 1$ (corresponding to the orders $\leq_{lr:j}$ and $\leq_{rh:j}$), since the continuity of $\Delta g(x,y)$ is not required in these cases. An applications of Theorem 3.4 is illustrated in Section 5.

It is worth mentioning that, by using Proposition 3.2, it is easy to develop analogues of Theorems 3.3 and 3.4 for the order $\leq_{n\text{-}hr:j}$. Those results would be useful in solving stochastic scheduling problems and reliability analysis.

4. Incorporate dependence into $\leq_{n\text{-}icx}$ and $\leq_{n\text{-}icv}$

In this section, we propose some new orders to incorporate dependence structure into the nth degree increasing convex/concave orders. Define

$$\mathcal{G}_{icx} = \left\{g(x,y) \,\middle|\, \Delta g(x,y) \text{ is increasing and convex in } y\right\}$$
$$\mathcal{G}_{icv} = \left\{g(x,y) \,\middle|\, \Delta g(x,y) \text{ is increasing and concave in } y\right\}.$$

Shanthikumar and Yao (1991) provide a bivariate characterization of the increasing convex order by using $\mathcal{G}_{icx}$ without proof. Below we cite their result and present a proof.

Theorem 4.1. *$X \leq_{icx} Y$ if and only if there exist $X' \overset{d}{=} X$ and $Y' \overset{d}{=} Y$ such that X' is independent of Y' and $\mathbb{E}[g(X',Y')] \geq \mathbb{E}[g(Y',X')]$ for all $g \in \mathcal{G}_{icx}$.*

Proof. For the "if" part, consider any increasing convex function u. Define $g(x,y) = u(y) - u(x)$. It is easy to verify that $g \in \mathcal{G}_{icx}$. Therefore, $\mathbb{E}[u(Y') - u(X')] \geq \mathbb{E}[u(X') - u(Y')]$, which implies that $\mathbb{E}[u(Y')] \geq \mathbb{E}[u(X')]$ and thus $\mathbb{E}[u(Y)] \geq \mathbb{E}[u(X)]$.

For the "only if" part, assume X is independent of Y for notational convenience. Let random variable Z be such that $Z \overset{d}{=} X$ and Z is independent of X and Y. For any $g \in \mathcal{G}_{icx}$, it holds that $\mathbb{E}[\Delta g(X,Y)] \geq \mathbb{E}[\Delta g(X,Z)]$, since $Y \geq_{icx} Z$ and $\Delta g(x,y)$ is increasing and convex in y.

Recall that X and Z are independent and have the same distribution, and thus are exchangeable. It follows that $\mathbb{E}[g(X,Z)] = \mathbb{E}[g(Z,X)]$, i.e., $\mathbb{E}[\Delta g(X,Z)] = 0$. Therefore, it holds that $\mathbb{E}[\Delta g(X,Y)] \geq 0$, or $\mathbb{E}[g(X,Y)] \geq \mathbb{E}[g(Y,X)]$. □

Based on the bivariate characterization, Shanthikumar and Yao (1991) proposed the concept of joint increasing convex order $\leq_{icx:j}$. In this section, we shall propose a sequence of joint n-increasing convex/concave order and develop their theoretical properties.

For a positive integer n, define

$$\mathcal{G}_{n\text{-}icx} = \{g(x,y) \mid \Delta g(x,y) \in \mathcal{U}_{n\text{-}icx} \text{ as a function of } y \text{ for any } x\},$$
$$\mathcal{G}_{n\text{-}icv} = \{g(x,y) \mid \Delta g(x,y) \in \mathcal{U}_{n\text{-}icv} \text{ as a function of } y \text{ for any } x\}.$$

Clearly, $\mathcal{G}_{n_2\text{-}icx} \subset \mathcal{G}_{n_1\text{-}icx}$ and $\mathcal{G}_{n_2\text{-}icv} \subset \mathcal{G}_{n_1\text{-}icv}$ for any $n_1 \leq n_2$.

Definition 4.2. Random variable X is said to be less than random variable Y in the *joint n-increasing convex (concave) order*, denoted as $X \leq_{n\text{-}icx:j} Y$ ($X \leq_{n\text{-}icv:j} Y$), if $\mathbb{E}[g(X,Y)] \geq \mathbb{E}[g(Y,X)]$ for any $g \in \mathcal{G}_{n\text{-}icx}$ ($g \in \mathcal{G}_{n\text{-}icv}$) provided the expectations exist.

When $n = 1$, $\leq_{1\text{-}icx:j}$ and $\leq_{1\text{-}icv:j}$ both reduce to $\leq_{st:j}$ (Shanthikumar and Yao, 1991), which is also the bivariate version of WSAI (Cai and Wei, 2015). When $n = 2$, $\leq_{2\text{-}icx:j}$ and $\leq_{2\text{-}icv:j}$ are simply denoted as $\leq_{icx:j}$ and $\leq_{icv:j}$. Note that $\leq_{icx:j}$ is the one proposed by Shanthikumar and Yao (1991). It is easy to see that $X \leq_{n_1\text{-}icx:j} Y$ implies $X \leq_{n_2\text{-}icx:j} Y$ and $X \leq_{n_1\text{-}icv:j} Y$ implies $X \leq_{n_2\text{-}icv:j} Y$ for any $n_1 \leq n_2$.

Similarly as Proposition 3.2, there is a close relationship between $\leq_{n\text{-}icx:j}$ and $\leq_{n\text{-}icv:j}$.

Proposition 4.3. *$X \leq_{n\text{-}icx:j} Y$ if and only if $-Y \leq_{n\text{-}icv:j} -X$.*

In the rest of this section, we shall develop more properties of the orders $\leq_{n\text{-}icx:j}$ and $\leq_{n\text{-}icv:j}$. Since they are dual orders, we will state properties of only $\leq_{n\text{-}icx:j}$. It should be kept in mind that analogues of the other order $\leq_{n\text{-}icv:j}$ can be easily obtained by applying Proposition 4.3.

Proposition 4.4. *The following statements are true:*

(i) *If $\leq_{n\text{-}hr:j} Y$, then $X \leq_{n\text{-}icx:j} Y$;*
(ii) *If $X \leq_{n\text{-}icx:j} Y$, then $X \leq_{n\text{-}icx} Y$.*

Proof. The first statement follows from the fact that $\mathcal{G}_{n\text{-}icx} \subset \mathcal{G}_{n\text{-}hr}$. The second statement can be verified by taking $g(x,y) = u(y)$ with $u \in \mathcal{U}_{n\text{-}icx}$. □

Proposition 4.5. *Assume X and Y are independent. $X \leq_{n\text{-}icx:j} Y$ if and only if $X \leq_{n\text{-}icx} Y$.*

Proof. The statement for $\leq_{2-icx:j}$, i.e., $\leq_{icx:j}$, directly follows from the proof of Theorem 4.1. The statements for other cases can be proved similarly. □

Lemma 4.6. *Let X, Y, Z be comonotonic and $Z \geq 0$. If $X \leq_{icx} Y$, then $\mathbb{E}[XZ] \leq \mathbb{E}[YZ]$.*

Proof. There exists X' such that $X \leq_{st} X' \leq_{cx} Y$. Furthermore, X' can be constructed in such a way that X, X', Y are comonotonic. Following from Lemma 3.12.13 of Müller and Stoyan (2002), $\mathbb{E}[X'Z] \leq \mathbb{E}[YZ]$.

According to Lemma 5.3 of Cai and Wei (2014), X and X' comonotonic together with $X \leq_{st} X'$ implies $X \leq_{as} X'$. Since $Z \geq 0$, then $\mathbb{E}[XZ] \leq \mathbb{E}[X'Z]$ and thus $\mathbb{E}[XZ] \leq \mathbb{E}[YZ]$. □

The following theorem provides a sufficient conditions that lead to the order $\leq_{icx:j}$.

Theorem 4.7. *If $\mathbb{E}[Y|X=x]$ is increasing in x and $\mathbb{E}[Y|X] \geq_{icx} X$, then $X \leq_{icx:j} Y$.*

Proof. Consider any function $g \in \mathcal{G}_{icx}$. It suffices to show that $\mathbb{E}[\Delta g(X,Y)] \geq 0$.

Since $\Delta g(x,y) = g(x,y) - g(y,x)$ is increasing and convex in y, it follows that $\Delta g(X,Y) \geq \Delta^{(0,1)} g(X,X)(Y-X) = f(X)(Y-X)$, where $f(x) = \Delta^{(0,1)} g(x,x)$. Note that $f(x) \geq 0$ and $f'(x) = \Delta^{(0,2)} g(x,x) \geq 0$.

Denote $h(x) = \mathbb{E}[Y|X=x]$. Since h and f are both increasing function, then $f(X), X, h(X)$ are comonotonic. Noting that $X \leq_{icx} h(X)$ and $f(X) \geq 0$, it follows from Lemma 4.6 that $\mathbb{E}[f(X)X] \leq \mathbb{E}[f(X)h(X)] = \mathbb{E}[f(X)Y]$, which implies $\mathbb{E}[\Delta g(X,Y)] \geq 0$. □

Proposition 4.8. *Assume X and Y are comonotonic. $X \leq_{icx:j} Y$ if and only if $X \leq_{icx} Y$.*

Proof. The proof is similar to that of Theorem 4.7, and thus is omitted. □

The following proposition presents a closure property of the joint orders, which allows to construct new random vectors satisfying certain order $\leq_{n\text{-}icx:j}$ from known ones.

Proposition 4.9. *The order $\leq_{n\text{-}icx:j}$ has the following properties:*

(i) *Let $h \in \mathcal{U}_{n\text{-}icx}$. $X \leq_{n\text{-}icx:j} Y$ implies $h(X) \leq_{n\text{-}icx:j} h(Y)$*
(ii) *Let $Y \leq_{a.s.} Z$. $X \leq_{n\text{-}icx:j} Y$ implies $X \leq_{n\text{-}icx:j} Z$.*

Proof. (i). It is straightforward by verifying that $g(h(x), h(y)) \in \mathcal{G}_{n\text{-}icx}$ for any $g \in \mathcal{G}_{n\text{-}icx}$ and $h \in \mathcal{U}_{n\text{-}icx}$.

(ii) For any $g \in \mathcal{G}_{n\text{-}icx}$, $\Delta g(x, y)$ is increasing in y. Therefore, $\Delta g(X, Z) \geq_{a.s.} \Delta g(X, Y)$, which implies $\mathbb{E}[\Delta g(X, Z)] \geq \mathbb{E}[\Delta g(X, Y)] \geq 0$ for any $g \in \mathcal{G}_{n\text{-}icx}$. □

Corollary 4.10. *If $X \leq_{n\text{-}icx:j} Y$, then*

(i) *$\mu_1 + X \leq_{n\text{-}icx:j} \mu_2 + Y$ for any $\mu_1 \leq \mu_2$,*
(ii) *$aX \leq_{n\text{-}icx:j} aY$ for any $a > 0$.*

Proof. It directly follows from Proposition 4.9. □

Propositions 4.5 and 4.8 show how to construct joint increasing convex and concave orders under two special dependence structures: independence and comonotonicity. The following proposition shows the construction of those orders for bivariate normal random vectors.

Proposition 4.11. *Let $(X, Y) \sim BVN(\mu_1, \mu_2; \sigma_1, \sigma_2, \rho)$ with $\rho \geq 0$ and $n \geq 2$. $X \leq_{n\text{-}icx:j} Y$ if and only if $\mu_1 \leq \mu_2, \sigma_1 \leq \sigma_2$.*

Proof. The "only if" part. According to Proposition 4.4, $X \leq_{n\text{-}icx:j} Y$ implies $X \leq_{n\text{-}icx} Y$, that is, $\mathbb{E}[u(X)] \leq \mathbb{E}[u(Y)]$ for any $u \in \mathcal{U}_{n\text{-}icx}$. Setting $u(z) = z$ immediately yields $\mathbb{E}[X] \leq \mathbb{E}[Y]$, i.e., $\mu_1 \leq \mu_2$. Setting $u(z) = e^{sz}$ yield $\mathbb{E}[e^{sX}] \leq \mathbb{E}[e^{sY}]$, i.e., $e^{\mu_1 s + \frac{1}{2}\sigma_1^2 s^2} \leq e^{\mu_2 s + \frac{1}{2}\sigma_2^2 s^2}$ for any $s > 0$, which implies that $\sigma_1 \leq \sigma_2$.

The "if" part. If suffices to show $X \leq_{icx:j} Y$, since $X \leq_{icx:j} Y$ implies $X \leq_{n\text{-}icx:j} Y$ for all $n \geq 3$, according to the remark below Definition 4.2. We shall focus on the proof for the special case $\mu_1 = \mu_2 = 0$ and $\sigma_1 = 1 \leq \sigma_2$. The general case immediately follows from Corollary 4.10.

Denote $Y' = Y/\sigma_2$, then (X, Y') is exchangeable and thus $\mathbb{E}[g(X, Y')] = \mathbb{E}[g(Y', X)]$, i.e., $\mathbb{E}[\Delta g(X, Y')] = 0$. Recalling that $\Delta g(x, y)$ is increasing and convex in y, it follows that

$$\begin{aligned}\Delta g(X, \sigma_2 Y') - \Delta g(X, Y') &\geq (\sigma_2 Y' - Y')\Delta^{(0,1)} g(X, Y') \\ &= (\sigma_2 - 1) Y' \Delta^{(0,1)} g(X, Y'). \end{aligned} \tag{4.1}$$

Since $(X, Y') \sim BVN(0, 0; 1, 1, \rho)$ with $\rho \geq 0$, there exist $U, V, W \overset{i.i.d.}{\sim} N(0, 1)$ such that $X = \sqrt{1-\rho}\,U + \sqrt{\rho}\,W$ and $Y' = \sqrt{1-\rho}\,V + \sqrt{\rho}\,W$. Denote $h(u, v, w) = \Delta^{(0,1)} g(\sqrt{1-\rho}\,u + \sqrt{\rho}\,w, \sqrt{1-\rho}\,v + \sqrt{\rho}\,w)$. Note that $\Delta^{(0,1)} g(X, Y') = h(U, V, W)$. Recalling that $g \in \mathcal{G}_{icx}$, i.e., $\Delta^{(0,1)} g(x, y)$ is nonnegative and increasing in y, one observes that $h(u, v, w)$ is increasing in v. Thus, $\sqrt{1-\rho}V$ and $h(U, V, W)$ are positively correlated since U, W are independent of V. Therefore,

$$\begin{aligned}\mathbb{E}[\sqrt{1-\rho}\,V\,\Delta^{(0,1)} g(X, Y')] &= \mathbb{E}[\sqrt{1-\rho}\,V\,h(U, V, W)] \\ &\geq \mathbb{E}[\sqrt{1-\rho}\,V] \times \mathbb{E}[h(U, V, W)] = 0. \end{aligned} \tag{4.2}$$

On the other hand,

$$\begin{aligned}\frac{\partial}{\partial w} h(u, v, w) &= \Delta^{(0,2)} g(\sqrt{1-\rho}\,u + \sqrt{\rho}\,w, \sqrt{1-\rho}\,v + \sqrt{\rho}\,w) \\ &\quad + \Delta^{(1,1)} g(\sqrt{1-\rho}\,u + \sqrt{\rho}\,w, \sqrt{1-\rho}\,v + \sqrt{\rho}\,w).\end{aligned}$$

Note that $\Delta^{(0,2)} g(x, y) \geq 0$ for any $g \in \mathcal{G}_{icx}$ and $\Delta g^{(1,1)}(x, y) = -\Delta g^{(1,1)}(y, x)$. Since U and V are exchangeable, then $\mathbb{E}[\Delta^{(1,1)} g(\sqrt{1-\rho}\,U + \sqrt{\rho}\,w, \sqrt{1-\rho}\,V + \sqrt{\rho}\,w)] = 0$. Therefore, $\mathbb{E}[\frac{\partial}{\partial w} h(U, V, w)] \geq 0$ and thus $\mathbb{E}[h(U, V, w)]$ is increasing in w. Similar as (4.2), one obtains that

$$\begin{aligned}\mathbb{E}[\sqrt{\rho}\,W\,\Delta^{(0,1)} g(X, Y')] &= \mathbb{E}[\sqrt{\rho}\,W\,h(U, V, W)] \\ &= \mathbb{E}[\sqrt{\rho}\,W\,\mathbb{E}[h(U, V, W)|(U, V)]] \geq 0,\end{aligned}$$

which, together with (4.1) and (4.2), implies $\mathbb{E}[\Delta g(X, Y)] \geq 0$ and thus $X \leq_{icx:j} Y$. □

Remark 4.12. For a bivariate normal random vector with nonnegative correlation coefficient, the orders $\leq_{n\text{-}icx:j}$, $n = 2, 3, \ldots$ are all equivalent, and they are further equivalent to $\leq_{icx}$.

To close this section, we establish equivalent characterizations for by using two different bivariate functions. This result is analogous to Theorem 3.4.

Theorem 4.13. *$X \leq_{icv:j} Y$ if and only if $\mathbb{E}[g_2(X, Y)] \geq \mathbb{E}[g_1(X, Y)]$ for any bivariate functions g_1, g_2 such that*

(i) *$g_2(x, y) - g_1(x, y) \in \mathcal{U}_{icv}$ as a function of y on $(-\infty, x]$,*
(ii) *$g_1(x, y) - g_2(x, y) \in \mathcal{U}_{icv}$ as a function of x on $[y, \infty)$,*
(iii) *$g_2(x, y) + g_2(y, x) \geq g_1(x, y) + g_1(y, x)$ for any x and y, with equality at $y = x$.*

Proof. The "if" part. For any $g \in \mathcal{G}_{icv}$, $\Delta g(x, y) = g(x, y) - g(y, x) \in \mathcal{U}_{icv}$ for any fixed x. Define $g_2(x, y) = g(x, y)$ and $g_1(x, y) = g(y, x)$. Then it is easy to verify that g_1, g_2 satisfy Conditions (i) and (iii). Noting that $g_1(x, y) - g_2(x, y) = -\Delta g(x, y) = \Delta g(y, x) \in \mathcal{U}_{icv}$ as a function of x for any y, Condition (ii) is also verified. Therefore, $\mathbb{E}[g_2(X, Y)] \geq \mathbb{E}[g_1(X, Y)]$, which implies that $\mathbb{E}[g(X, Y)] \geq \mathbb{E}[g(Y, X)]$.

The "only if" part. For any g_1, g_2 satisfying Conditions (i), (ii) and (iii), define $h(x, y) = (g_2(x, y) - g_1(x, y)) \times \mathbb{I}\{x \geq y\}$. Then $\Delta h(x, y) = g_2(x, y) - g_1(x, y)$ if $y \leq x$, and $\Delta h(x, y) = g_1(y, x) - g_2(y, x)$ if $y \geq x$. The equality in Condition (iii) yields $g_2(x, x) = g_1(x, x)$, which implies that $\Delta h(x, y)$ is continuous at $y = x$. Furthermore, differentiating $g_2(x, x) = g_1(x, x)$ yields $g_2^{(0,1)}(x, x) + g_2^{(1,0)}(x, x) = g_1^{(0,1)}(x, x) + g_1^{(1,0)}(x, x)$, and implies that the two-sided derivatives of Δh with respect to y coincide at $y = x$. Combining Conditions (i) and (ii), one concludes that $\Delta h(x, y) \in \mathcal{U}_{icv}$ for any fixed x, i.e., $h \in \mathcal{G}_{icv}$. It follows from $X \leq_{icv:j} Y$ that $\mathbb{E}[h(X, Y)] \geq \mathbb{E}[h(Y, X)]$. The rest proof is the same as that of Theorem 3.4. □

We remark that, for high degree orders, i.e., $\leq_{n\text{-}icv:j}$ and $\leq_{n\text{-}icx:j}$ with $n \geq 3$, more assumptions on g_1, g_2 are needed to make the function Δh (constructed in Theorem 4.13) smooth enough to fall into $\mathcal{U}_{n\text{-}icv}$.

5. Applications in portfolio selections

A typical application of the above proposed orders is to study the optimal portfolio selections. The classical problem of optimal portfolio selections concerning two risky assets is formulated as follows.

$$\max_{a,\,b} \mathbb{E}[u(aX + bY)]. \tag{5.1}$$

In (5.1), X, Y denote the stochastic return rates of two risky assets, which are usually assumed to be nonnegative. Real numbers a and b are investment weights on two assets and thus satisfy: $a + b = 1$ and $a, b \geq 0$. The function u belongs to certain class of utility functions. Typical choices of utility function classes are $\mathcal{U}_{1-icv}$ and $\mathcal{U}_{2-icv}$. In those cases, the optimization problem is interpreted as:

maximizing the expected utility of total return in the sense of the first or second stochastic dominance.

In the early literature, Problem (5.1) has been studied mainly under special dependence structures, namely, independence and comonotonicity. See Hadar and Seo (1988) and Landsberger and Meilijson (1990) among others. Later on, dependence structures has been introduced to the model through different joint stochastic orders, see for example, Kijima and Ohnishi (1996) and Hennessy and Lapan (2002). Recent studies can be found in Cai and Wei (2015), Li and You (2015), You and Li (2016), and Li and Li (2016), where the joint reversed hazard order and its multivariate generalizations are used. In a different stream, Cheung and Yang (2004) and Chen and Hu (2008) study the problem with default risks.

In the following, we shall study this problem by using the orders $\le_{n\text{-}rh:j}$ and $\le_{n\text{-}icv:j}$ and thus unify the existing studies in a general framework.

5.1. Applications of $\le_{n\text{-}rh:j}$

Theorem 5.1. *Let (a_1, b_1) and (a_2, b_2) be two real vectors satisfying $a_1 \ge b_1 \ge 0$, $a_1 + b_1 = a_2 + b_2$, and $b_1 \le \min\{a_2, b_2\}$. If $X \le_{n\text{-}rh:j} Y$, then $a_2X + b_2Y \ge_{(n+1)\text{-}icv} a_1X + b_1Y$.*

Proof. It suffices to show that $\mathbb{E}[u(a_2X + b_2Y)] \ge \mathbb{E}[u(a_1X + b_1Y)]$ for any $u \in \mathcal{U}_{(n+1)\text{-}icv}$. Denote $g_1(x, y) = u(a_1x + b_1y)$ and $g_2(x, y) = u(a_2x + b_2y)$. We shall show that g_1, g_2 satisfy Conditions (i) and (ii) in Theorem 3.4 and thus complete the proof.

Taking partial derivative of $g_2 - g_1$, one gets

$$\frac{\partial^k}{\partial y^k}(g_2(x, y) - g_1(x, y)) = b_2^k u^{(k)}(a_2x + b_2y) - b_1^k u^{(k)}(a_1x + b_1y).$$

Since $(a_2, b_2) \le_m (a_1, b_1)$, then $a_1 + b_1 = a_2 + b_2$, and thus $(a_2x + b_2y) - (a_1x + b_1y) = -(a_1 - a_2)(x - y) \le 0$ for any $x \ge y$. Recalling that $(-1)^{k-1}u^{(k)}(z)$ is nonnegative and decreasing, one obtains $(-1)^{(k-1)}u^{(k)}(a_2x + b_2y) \ge (-1)^{(k-1)}u^{(k)}(a_1x + b_1y) \ge 0$. Since $b_2 \ge b_1 \ge 0$, then $(-1)^{(k-1)} \frac{\partial^k}{\partial y^k}(g_2(x, y) - g_1(x, y)) \ge 0$ for any $x \ge y$ and $k = 1, 2, \ldots, n$. This verifies Condition (i).

In order to verify Condition (ii), assume $x \le y$ without loss of generality. Note that

$$\begin{aligned} &g_2(x, y) + g_2(y, x) - g_1(x, y) - g_1(y, x) \\ &= u(a_2x + b_2y) + u(a_2y + b_2x) - u(a_1x + b_1y) - u(a_1y + b_1x) \\ &= \int_{a_1x+b_1y}^{a_2x+b_2y} u'(z)dz - \int_{a_2y+b_2x}^{a_1y+b_1x} u'(z)dz. \end{aligned}$$

Since $(a_2x + b_2y) - (a_1x + b_1y) = (a_1y + b_1x) - (a_2y + b_2x) \ge 0$ and $(a_2x + b_2y) \le (a_1y + b_1x)$, then the inequality $\int_{a_1x+b_1y}^{a_2x+b_2y} u'(z)dz - \int_{a_2y+b_2x}^{a_1y+b_1x} u'(z)dz \ge 0$ follows from the fact that $u \in \mathcal{U}_{(n+1)\text{-}icv}$ is increasing and concave. This verifies $g_2(x, y) + g_2(y, x) \ge g_1(x, y) + g_1(y, x)$. The equality at $y = x$ follows from the fact that $a_2 + b_2 = a_1 + b_1$. □

Theorem 1 of Hua and Cheung (2008) and Theorem 2 of You and Li (2016) derive similar inequalities by using (multivariate version of) $\le_{lr:j}$ and $\le_{rh:j}$, respectively. Those results in bivariate case are covered by Theorem 5.1 as a special case by setting $n = 2$.

The implication of Theorem 5.1 is, a portfolio with more investment on the asset with smaller return rate can be always improved by transferring a portion of investment to the asset with higher return rate. In some circumstances of practice, the investment on certain asset is restricted. For example, an insurance company is to construct a portfolio consisting of a low risk bond and a stock with higher return, with the investment weight on stock market restricted (by the regulator) to below certain level $q \le \frac{1}{2}$. Under this constraint, Theorem 5.1 suggests that the insurance company should invest on the stock as much as allowed. In other words, $(1 - q, q)$ is the best portfolio, since the vectors (a, b) and $(1 - q, q)$ satisfy the conditions of Theorem 5.1 for any $a + b = 1$ and $0 \le b \le q$.

Corollary 5.2. *If $X \le_{n\text{-}rh:j} Y$, then $aX + bY \ge_{(n+1)\text{-}icv} aY + bX$ for any $0 \le a \le b$.*

Corollary 5.2 is a direct corollary of Theorem 5.1. The intuition is very clear. One should invest more on the asset with larger return rate and less on the one with smaller return rate to maximize the expected utility. This intuition agrees with many existing studies, see for example, Hadar and Seo (1988), Hennessy and Lapan (2002), and Cai and Wei (2015). Compared to those results, the words "smaller" and "larger" here are in a more general sense.

5.2. Applications of $\le_{n\text{-}icv:j}$

In this subsection, we use the order $\le_{n\text{-}icv:j}$ to study the portfolio selections. Since $\le_{n\text{-}icv:j}$ is an order weaker than $\le_{n\text{-}rh:j}$, we anticipate more assumptions on the utility function if we want to reach the same conclusion as in Section 5.1. We first introduce the concept of absolute risk aversion and use it to describe utility functions.

Let u be a utility function in $\mathcal{U}_{(n+1)\text{-}icv}$. Define

$$a_u^{(k)}(z) = -\frac{u^{(k+1)}(z)}{u^{(k)}(z)}, \qquad k = 1, 2, \ldots, n.$$

$a_u^{(k)}(z)$ is called the kth degree index of the absolute risk aversion of u. This concept is widely used in economics to describe risk preference. In particular, $a_u^{(1)}$ is the Arrow index of absolute risk aversion, $a_u^{(2)}$ is the Kimball index of absolute risk prudence, and $a_u^{(3)}$ is the index of absolute temperance. Readers are referred to Denuit and Eeckhoudt (2010) for a literature review on these indices.

Theorem 5.3. *Let (a_1, b_1) and (a_2, b_2) be two real vectors satisfying $a_1 \ge b_1 \ge 0$, $a_1 + b_1 = a_2 + b_2$, and $b_1 \le \min\{a_2, b_2\}$. If $X \ge 0$, $Y \ge 0$, and $X \le_{icv:j} Y$, then $\mathbb{E}[u(a_2X + b_2Y)] \ge \mathbb{E}[u(a_1Y + b_1X)]$ for any $u \in \mathcal{U}_{3-icv}$ such that $(-1)^{k-1}z^k u^{(k)}(z)$ is increasing in z for $k = 1, 2$.*

Proof. Denote $g_2(x, y) = u(a_2x + b_2y)$ and $g_1(x, y) = u(a_1x + b_1y)$. It suffices to verify that g_1 and g_2 satisfy Conditions (i), (ii) and (iii) in Theorem 4.13.

Conditions (i) and (iii) can be verified similarly as in Theorem 5.1.

In order to verify Condition (ii), one needs to show

$$\begin{aligned} &(-1)^{k-1}a_1^k u^{(k)}(a_1x + b_1y) \\ &\ge (-1)^{k-1}a_2^k u^{(k)}(a_2x + b_2y), \qquad k = 1, 2, \end{aligned} \tag{5.2}$$

for any $x \ge y$.

For any $x \ge y$, $a_1x + b_1y \ge a_2x + b_2y$, and thus

$$\begin{aligned} &(-1)^{(k-1)}u^{(k)}(a_1x + b_1y) \\ &\le (-1)^{(k-1)}u^{(k)}(a_2x + b_2y) \qquad k = 1, 2. \end{aligned} \tag{5.3}$$

The increasing property of $(-1)^{k-1}z^k u^{(k)}(z)$ implies

$$\begin{aligned} &(-1)^{(k-1)}(a_1x + b_1y)^k u^{(k)}(a_1x + b_1y) \\ &\ge (-1)^{(k-1)}(a_2x + b_2y)^k u^{(k)}(a_2x + b_2y) \; k = 1, 2. \end{aligned} \tag{5.4}$$

Noting that $0 \le b_1 \le b_2$ and $0 \le a_1b_1 \le a_2b_2$ algebraic manipulations of (5.3) and (5.4) yield (5.2). □

Theorem 5.3 makes it possible for the comparison between the portfolios (a_2, b_2) and (a_1, b_1) with $a_1 + b_1 = a_2 + b_2$. When it comes to the comparison between portfolios (a, b) and (b, a), the utility function can be more general.

Theorem 5.4. *Assume $X, Y \geq 0$. If $X \leq_{n\text{-}icv:j} Y$, then $\mathbb{E}[u(aX+bY)] \geq \mathbb{E}[u(aY+bX)]$ for any $0 \leq a \leq b$ and $u \in \mathcal{U}_{(n+1)\text{-}icv}$ satisfying the following conditions:*

(i) $a_u^{(k)}(z) \leq \frac{k}{z}$ *for* $k = 1, 2, \ldots, n$;
(ii) $a_u^{(k)}(z)$ *is decreasing in* z *for* $k = 1, 2, \ldots, n$.

Proof. It suffices to verify $u(ax+by) \in \mathcal{G}_{n\text{-}rh}$, or

$$\begin{aligned}&(-1)^{k-1} b^k u^{(k)}(ax+by)\\&\quad \geq (-1)^{k-1} a^k u^{(k)}(ay+bx), \qquad k = 1, 2, \ldots, n, \end{aligned} \tag{5.5}$$

for any x and y. Noting that $u \in \mathcal{U}_{(n+1)\text{-}icv}$ implies $(-1)^{(k-1)} u^{(k)}(z)$ is nonnegative and decreasing in z, inequalities in (5.5) immediately follow for all $x > y$ since $ax + by \leq ay + bx$ when $x > y$.

Now consider $x \leq y$. For any integer $k \geq 3$, denote $h(z) = \log(-1)^{(k-1)} u^{(k)}(z)$. Noting that $h'(z) = -a_u^{(k)}(z)$ and $a_u^{(k)}(z)$, one concludes that $h(z)$ is decreasing and convex. It follows from $ay \leq by$ and $ax \geq bx$ that $h(by) - h(ax+by) \leq h(ay) - h(ay+bx)$, or

$$\frac{(-1)^{(k-1)} u^{(k)}(by)}{(-1)^{(k-1)} u^{(k)}(ax+by)} \leq \frac{(-1)^{(k-1)} u^{(k)}(ay)}{(-1)^{(k-1)} u^{(k)}(ay+bx)}.$$

On the other hand, Condition (i) implies $(-1)^{(k-1)} z^k u^{(k)}(z)$ is increasing in z, and thus $(-1)^{k-1}(by)^k u^{(k)}(by) \geq (-1)^{k-1}(ay)^k u^{(k)}(ay)$ for all $y \geq 0$. Taking ratio of these two inequalities yields the required inequalities in (5.5). □

We point out that the assumptions of decreasing property of $a_u^{(1)}$ and $a_u^{(2)}$ in Condition (ii) can be dropped. Without these assumptions, equality (5.5) with $k = 1, 2$ still holds since it is implied by (5.2). By setting $n = 1$, Theorem 5.4 covers Theorem 5 of Li and You (2015) and Theorem 5.2(ii) of Cai and Wei (2015).

We remark that Condition (i) in Theorem 5.4 is equivalent to "$(-1)^{(k-1)} z^k u^{(k)}(z)$ is increasing for $k = 1, 2, \ldots, n$". The condition "$zu'(z)$ increasing in z" has been commonly used in the literature. When u is concave, Kijima and Ohnishi (1996) prove this condition equivalent to "$zu'(z+b)$ increasing in z for any $b \geq 0$" (see their Lemma A.2). In addition, they employ the index of relative risk aversion, defined to be $-\frac{zu''(z)}{u'(z)}$, to explain the intuition of this condition. Below we interpret Conditions (i) and (ii) of Theorem 5.4 from a different perspective.

Define $v(z) = \log z$. Clearly, v belongs to $\mathcal{U}_{n\text{-}icv}$ for any n and thus can be viewed as a utility function. Noting that $a_v^{(k)}(z) = \frac{k}{z}$, Condition (i) requires the first n degree indices of absolute risk aversion of u to be dominated by those of v, respectively. Recall that $a_u^{(1)} \leq a_v^{(1)}$ is interpreted as "v is more risk-averse than u" (Arrow, 1971; Pratt, 1964), and $a_u^{(2)} \leq a_v^{(2)}$ is interpreted as "v is more prudent than u" (Kimball, 1990). Following these interpretations, we generally interpret Condition (i) as "u is less conservative than v, the logarithm utility function, with respect to indices of absolute risk aversion of all degrees up to n".

For a utility function u, if $a_u^{(1)}(z)$ is decreasing in z, u is said to exhibit decreasing absolute risk aversion, or DARA. DARA property is frequently used in the economic literature, see for example Vickson (1974). In this sense, Condition (ii) is an extension of the DARA property to indices of absolute risk aversion of all degrees up to n. Caballé and Pomansky (1996) show that, if $u \in \mathcal{U}_{n\text{-}icv}$ for all n (referred to as mixed risk aversion), the u satisfies Condition (ii).

An example of utility function satisfying Conditions (i) and (ii) is z^γ with $0 < \gamma < 1$.

6. Concluding remarks

In this paper, we propose two new classes of joint stochastic orders: $\leq_{n\text{-}icx:j}(\leq_{n\text{-}icv:j})$ and $\leq_{n\text{-}hr:j}(\leq_{n\text{-}rh:j})$. These two classes substantially generalize the existing joint orders (such as $\leq_{st:j}$, $\leq_{icx:j}$, $\leq_{lr:j}$, $\leq_{hr:j}$, $\leq_{rh:j}$) and put them into a unified framework. Apart from their theoretical interests, these orders also provide new tools in studying optimization problems in the field of finance.

For the orders $\leq_{n\text{-}rh:j}$ and $\leq_{n\text{-}hr:j}$, we establish equivalent characterizations from different perspectives and thus lay down a solid theoretical foundation. The distributional characterizations (Theorem 3.3) enhance our understanding on the new orders. The general functional characterization (Theorem 3.4), on the other hand, demonstrates its power in applications. For the orders $\leq_{n\text{-}rh:j}$ and $\leq_{n\text{-}hr:j}$, we construct some examples with typical dependence structures. Furthermore, we develop general functional characterization (Theorem 4.13), and also establish closure properties (Proposition 4.9).

In this paper, all the orders are defined for two random variables. We point out that these notions and their properties can extend to multiple random variables, by using a similar conditioning treatment as in Cai and Wei (2014, 2015).

At the end, we would like to mention two notions: CLOAI and WCLOAI. Definitions and applications of the two notions can be found in Li and Li (2016). We point out that these two notions are closely related to the order $\leq_{n\text{-}rh:j}$ proposed in this paper. Indeed, the structure of their definitions (bivariate version) shares commonalities with that of the distributional characterization (Theorem 3.3) for $\leq_{n\text{-}rh:j}$. In addition, when it comes to applications to portfolio selections, the orders $\leq_{2\text{-}rh:j}$, and $\leq_{3\text{-}rh:j}$ yield the same outcomes as CLOAI and WCLOAI, respectively (see Corollary 5.2 in this paper and Theorems 3.2 and 3.3 of Li and Li, 2016). On the other hand, we remark that the notions of $\leq_{2\text{-}rh:j}$ and $\leq_{3\text{-}rh:j}$ are different from CLOAI and WCLOAI. There is no necessary implication between them. In an upcoming paper, we shall propose a more general framework to unify these notions.

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References

Arrow, K.J., 1971. Essays in the Theory of Risk-Bearing. Markham Publishing, Chicago.
Caballé, J., Pomansky, A., 1996. Mixed risk aversion. J. Econom. Theory 71 (2), 485–513.
Cai, J., Wei, W., 2014. Some new notions of dependence with applications in optimal allocation problems. Insurance Math. Econom. 55 (2), 200–209.
Cai, J., Wei, W., 2015. Notions of multivariate dependence and their applications in optimal portfolio selections with dependent risks. J. Multivariate Anal. 138, 156–169.
Chen, Z., Hu, T., 2008. Asset proportions in optimal portfolios with dependent default risks. Insurance Math. Econom. 43 (2), 223–226.
Cheung, K.C., Yang, H., 2004. Ordering optimal proportions in the asset allocation problem with dependent default risks. Insurance Math. Econom. 35 (3), 595–609.
Denuit, M.M., Eeckhoudt, L., 2010. A general index of absolute risk attitude. Manage. Sci. 56 (4), 712–715.
Hadar, J., Seo, T.K., 1988. Asset proportions in optimal portfolios. Rev. Econom. Stud. 55 (3), 459–468.
Hennessy, D.A., Lapan, H.E., 2002. The use of archimedean copulas to model portfolio allocations. Math. Finance 12 (2), 143–154.
Hua, L., Cheung, K.C., 2008. Stochastic orders of scalar products with applications. Insurance Math. Econom. 42 (3), 865–872.
Kijima, M., Ohnishi, M., 1996. Portfolio selection problems via the bivariate characterization of stochastic dominance relations. Math. Finance 6 (3), 237–277.
Kimball, M.S., 1990. Precautionary savings in the small and in the large. Econometrica 58 (1), 53–73.
Landsberger, M., Meilijson, I., 1990. Demand for risky financial assets: a portfolio analysis. J. Econom. Theory 50 (1), 204–213.
Li, X., Li, C., 2016. On allocations to portfolios of assets with statistically dependent potential risk returns. Insurance Math. Econom. 68, 178–186.

Li, X., You, Y., 2015. Permutation monotone functions of random vector with applications in financial and actuarial risk management. Adv. Appl. Probab. 47 (01), 270–291.
Müller, A., Stoyan, D., 2002. Comparison Methods for Stochastic Models and Risks, Vol. 389. Wiley.
Pan, X., Yuan, M., Kochar, S.C., 2015. Stochastic comparisons of weighted sums of arrangement increasing random variables. Statist. Probab. Lett. 102, 42–50.
Pratt, J.W., 1964. Risk aversion in the small and in the large. Econometrica 32 (1–2), 122–136.
Righter, R., Shanthikumar, J.G., 1992. Extension of the bivariate characterization for stochastic orders. Adv. Appl. Probab. 24 (02), 506–508.
Shaked, M., Shanthikumar, J.G., 2007. Stochastic Orders. Springer, New York.
Shaked, M., Wong, T., 1995. Preservation of stochastic orderings under random mapping by point processes. Probab. Engrg. Inform. Sci. 9 (04), 563–580.
Shanthikumar, J.G., Yao, D., 1991. Bivariate characterization of some stochastic order. Adv. Appl. Probab. 93 (3), 642–659.
Vickson, R.G., 1974. Stochastic dominance tests for decreasing absolute risk aversion I. Discrete random variables. Manage. Sci. 21, 1438–1446.
You, Y., Li, X., 2016. Ordering scalar products with applications in financial engineering and actuarial science. J. Appl. Probab. 53 (1), 47–56.