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Ordering Gini indexes of multivariate elliptical risks

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A B S T R A C T

Gini index is a well-known tool in economics that is often used for measuring income inequality. In insurance, the index and its modifications have been used to compare the riskiness of portfolios, to order reinsurance contracts, and to summarize insurance scores (relativities). In this paper, we establish several stochastic orders between the Gini indexes of multivariate elliptical risks with the same marginals but different dependence structures. This work is motivated by the applied studies of Brazauskas et al. (2007) and Samanthi et al. (2015), who employed the Gini index to compare the riskiness of insurance portfolios. Based on extensive Monte Carlo simulations, these authors have found that the power function of the associated hypothesis test increases as portfolios become more positively correlated. The comparison of the Gini indexes (of empirically estimated risk measures) presented in this paper provides a theoretical explanation to this statistical phenomenon. Moreover, it enriches the studies of the problem of central concentration of elliptical distributions and generalizes the pd-1 order proposed by Shaked and Tong (1985).

1. Introduction

Over a hundred years ago, Corrado Gini introduced an index to measure concentration or inequality of incomes (see Gini, 1936, for English translation of the original article). It later became known as the Gini index and has been extensively studied in many fields such as economics, insurance, finance, and statistics. At the intersection of insurance and statistics, for example, the index has been used for comparing distributions of risks and prices (see Frees et al., 2011). The comparisons are usually based on insurance scores relative to price, also known as "relativities", that point to areas of potential discrepancies between risk and price distributions. After ordering both risks and prices based on relativities, one arrives at an ordered Lorenz curve that can be summarized using a Gini index. Interestingly, the Lorenz curve and Gini index defined via relativities can cope with adverse selection, help measure potential profit, and serve as useful tools in predictive modeling (for more information, see Frees et al., 2014). Moreover, Lorenz curve and Lorenz order, the concepts closely related to Gini index, have been employed by Denuit and Vermandele (1999) to order reinsurance contracts.

Other statistical applications in insurance have emphasized the fact that the Gini index is an *L*-statistic, theoretical properties of which are well-established and thus can be employed to construct statistical inferential tools. For instance, Jones and Zitikis (2005), Jones et al. (2006), and Brazauskas et al. (2007) have designed several hypothesis tests to compare the riskiness of insurance portfolios by using the Gini index. Samanthi et al. (submitted for publication) have conducted an extensive simulation study by incorporating various types of dependence between portfolios and found that the power function of the associated hypothesis test increases as portfolios become more positively correlated. The comparison of the Gini indexes (of empirically estimated risk measures) presented in this paper provides a theoretical explanation to this statistical phenomenon.

As described by Samanthi et al. (submitted for publication), the power function of the hypothesis test under consideration is a probability event involving the Gini index $\frac{1}{n^2}\sum_{1\le i,j\le n}|X_i - X_j|$, where the random variables $X_1, \ldots, X_n$ represent empirical risk measures estimated from observations on n insurance portfolios. It is also known that the random vector $\mathbf{X} = (X_1, \ldots, X_n)$ follows an asymptotically multivariate normal distribution. For more details about the design of the hypothesis test, the reader may be referred to Samanthi et al. (submitted for publication) and Brazauskas et al. (2007). In order to explain the monotonicity of the test power function, with respect to the strength of dependence, we propose the following conjecture.

Conjecture 1.1. *Let $(X_1, \ldots, X_n)$ follow a multivariate normal distribution with mean $\mathbf{0}$ and covariance matrix $\boldsymbol{\Sigma}$, i.e., $(X_1, \ldots, X_n) \sim$*

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$MVN(\mathbf{0}, \mathbf{\Sigma})$. Then its Gini index $\frac{1}{n^2}\sum_{1\le i,j\le n}|X_i - X_j|$ decreases in the sense of usual stochastic order (see Section 2 for definition) as the covariance matrix $\mathbf{\Sigma}$ increases componentwise with diagonal elements remaining unchanged.

Basically, Conjecture 1.1 aims to order Gini indexes of multivariate normal risks with same marginals but different strength of dependence. Proving Conjecture 1.1 is a challenging task. This paper partially completes this task and generalizes the conclusion to elliptical distributions, yet still leaves some open problems.

Besides its usefulness in actuarial applications, the comparison of Gini indexes of multivariate elliptical risks shows its own independent interest. Intuitively, Conjecture 1.1 suggests that the probability $\mathbb{P}\left\{\frac{1}{n^2}\sum_{1\le i,j\le n}|X_i - X_j| \le t\right\}$ increases as $\mathbf{\Sigma}$ increases for any $t \ge 0$. In this sense, the study of Conjecture 1.1 falls into the scope of the problem of central concentration of elliptical distributions, which is formulated as follows: how the probability

$$\mathbb{P}_{\Sigma}(C) = \mathbb{P}\{(X_1, \ldots, X_n) \in C\} \tag{1.1}$$

changes according to the change of $\mathbf{\Sigma}$, where $(X_1, \ldots, X_n)$ follows an elliptical distribution with mean $\mathbf{0}$ and dispersion matrix $\mathbf{\Sigma}$?

This problem was first studied by Slepain (1962), which states that if $(X_1, \ldots, X_n)$ follows a multivariate normal distribution with mean $\mathbf{0}$ and covariance matrix $\mathbf{\Sigma}$, then $\mathbb{P}_{\Sigma}(C)$ increases as $\mathbf{\Sigma}$ increases componentwise with diagonal elements remaining unchanged for any lower orthant set C. Later literature has generalized the study to elliptical distributions while regions of different shapes have been considered, such as upper orthant sets, rectangles, and convex and centrally symmetric regions. Interested readers are referred to Das Gupta et al. (1972), Joe (1990), Eaton and Perlman (1991), and Anderson (1996). All these studies imposed certain assumptions on the structure of the covariance matrix. The results derived in this paper enriches the studies on this problem in the sense that it broadens the choice of the set C.

In addition, comparison of Gini index has another fold of meaning. The methodologies can be used to generalize the pd-1 order. The pd-1 order was proposed by Shaked and Tong (1985) and used to compare the strength of dependence of exchangeable random vectors. Chang (1992) extended this concept from the perspective of stochastic majorization and explored applications in operations research. Readers are referred to Chapter 9 of Shaked and Shanthikumar (2007) for a conclusive summary of the studies on the pd-1 order. In the existing literature, the application of pd-1 order is very restrictive since the comparison applies only to exchangeable random vectors. In this paper, we manage to generalize the pd-1 order to non-exchangeable random vectors.

The rest of the paper is organized as follows. Section 2 introduces some basics about stochastic orders, elliptical distributions, and comonotonicity. Section 3 compares Gini indexes of multivariate elliptical risks in the sense of a relatively weaker order: the increasing convex order. Section 4 imposes certain assumptions on the structure of covariance matrices and establishes the usual stochastic orders between Gini indexes. In Section 5, we discuss the pd-1 order and its generalization by using similar techniques before. Section 6 provides concluding remarks of the paper.

2. Preliminaries

Throughout the paper, we use bold letters to denote vectors or matrices. For example, $\mathbf{x} = (x_1, \ldots, x_n)$ is a row vector and $\mathbf{\Sigma} = (\sigma_{ij})_{n\times n}$ is an $n \times n$ matrix. In particular, the symbol $\mathbf{0}$ denotes the row vector with all entries equal to 0, and $\mathbf{1}_{n\times n}$ denotes the $n\times n$ matrix with all entries equal to 1. The inequality between vectors or matrices denotes componentwise inequalities. For example, $(x_1, \ldots, x_n) \le (y_1, \ldots, y_n)$ implies that $x_i \le y_i$ for all $i = 1, \ldots, n$.

Consider a random vector $\mathbf{X} = (X_1, \ldots, X_n)$. Its Gini index is defined to be $\frac{1}{n^2}\sum_{1\le i,j\le n}^{n}|X_i - X_j|$. Gini index measures how dispersive the components of the random vector are. For example, if all the components are identical, then the Gini index is 0, which indicates a perfect concentration. For notational convenience, denote

$$G(\mathbf{X}) = \sum_{1\le i,j\le n}^{n}|X_i - X_j|. \tag{2.1}$$

$G(\mathbf{X})$ is the scaled Gini index and is the random variable we shall study throughout the paper. It is easy to see that $G(\mathbf{X})$ can be rewritten in terms of order statistics as follows.

$$G(\mathbf{X}) = \sum_{i=1}^{n}(4i - 2n - 2)X_{(i)}, \tag{2.2}$$

where $X_{(i)}$ denotes the ith largest component of $\{X_1, \ldots, X_n\}$.

In order to compare Gini indexes, we recall definitions of some stochastic orders.

Definition 2.1. Let X and Y be two random variables.

X is said to be smaller than Y in *usual stochastic order*, denoted as $X \le_{st} Y$, if $\mathbb{P}\{X > t\} \le \mathbb{P}\{Y > t\}$ for all $t \in \mathbb{R}$.

X is said to be smaller than Y in *increasing convex order*, denoted as $X \le_{icx} Y$, if $\mathbb{E}[u(X)] \le \mathbb{E}[u(Y)]$ for any increasing convex function u such that the expectations exist.

The above definitions are taken from Shaked and Shanthikumar (2007), which also provide the following characterization for the usual stochastic order.

Proposition 2.2. *Let X, Y be two random variables. $X \le_{st} Y$ if and only if $\mathbb{E}[u(X)] \le \mathbb{E}[u(Y)]$ for any increasing function u such that the expectations exist.*

Furthermore, in order to compare random vectors, the concept of supermodular order is needed. There is rich literature on the subject of supermodular order, see, for example, Marshall et al. (2010), Müller and Stoyan (2002), and Shaked and Shanthikumar (2007). We cite the definition of supermodular function and supermodular order from Shaked and Shanthikumar (2007).

Definition 2.3. A function $f : \mathbb{R}^n \to \mathbb{R}$ is said to be *supermodular* if for any $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ it holds that

$$f(\mathbf{x}) + f(\mathbf{y}) \le f(\mathbf{x} \wedge \mathbf{y}) + f(\mathbf{x} \vee \mathbf{y}),$$

where the operators $\wedge$ and $\vee$ denote coordinatewise minimum and maximum respectively, i.e.,

$$(x_1, \ldots, x_n) \wedge (y_1, \ldots, y_n) = (\min(x_1, y_1), \ldots, \min(x_n, y_n)),$$
$$(x_1, \ldots, x_n) \vee (y_1, \ldots, y_n) = (\max(x_1, y_1), \ldots, \max(x_n, y_n)).$$

Random vector $\mathbf{X}$ is said to be smaller than random vector $\mathbf{Y}$ in the *supermodular order*, denoted as $\mathbf{X} \le_{sm} \mathbf{Y}$, if $\mathbb{E}[f(\mathbf{X})] \le \mathbb{E}[f(\mathbf{Y})]$ for any supermodular function f such that the expectations exist.

It is easy to verify that, if $\mathbf{X} \le_{sm} \mathbf{Y}$ and $\mathbf{X} \ge_{sm} \mathbf{Y}$, then $\mathbf{X} \overset{d}{=} \mathbf{Y}$, where $\overset{d}{=}$ denotes "equal in distribution". According to Kemperman (1977, Assertion (i)), we have the following result.

Proposition 2.4. *A function $f : \mathbb{R}^n \to \mathbb{R}$ is supermodular if and only if $f(x_1, \ldots, x_n)$ is supermodular as a function of (x_i, x_j) for any other fixed x_k, $k \ne i, j$ for any $1 \le i < j \le n$.*

To investigate the effect of dependence on the test statistic of the hypothesis test proposed by Brazauskas et al. (2007), it suffices to study multivariate normal distribution since that is the asymptotic distribution of empirical risk measures. However, because of its independent interest, we want to extend the study to elliptical distributions. We first state some basics about elliptical distributions. The following definition and characterization of elliptical distribution are taken from McNeil et al. (2005).

Definition 2.5. An n-dimensional random vector $\mathbf{X}$ has an *elliptical distribution* if its characteristic function has the following form:

$$\mathbb{E}[e^{i\mathbf{t}'\mathbf{X}}] = e^{i\mathbf{t}'\boldsymbol{\mu}}\psi(\mathbf{t}'\boldsymbol{\Sigma}\mathbf{t}),$$

where $\boldsymbol{\mu} \in \mathbb{R}^n$, $\boldsymbol{\Sigma} \in \mathbb{R}^{n\times n}$ is a positive semidefinite matrix.

In this case we denote $\mathbf{X} \sim EC_n(\boldsymbol{\mu}, \boldsymbol{\Sigma}, \psi)$. ψ is referred to as the characteristic generator of the elliptical distribution. $\boldsymbol{\mu}$ is referred to as location vector and is equal to the mean of $\mathbf{X}$ if exists, and $\boldsymbol{\Sigma}$ is referred to as dispersion matrix.

McNeil et al. (2005) point out that, generally, characteristic generators may be used only in certain dimensions. In this paper, we shall focus on a special class of generators and the elliptical distributions induced by this class. Specifically, we consider all the generators that can be used in any arbitrary dimension and denote this class by Ψ_∞.

The elliptical distribution family induced by Ψ_∞ includes many important distributions, such as multivariate normal distribution and multivariate t distribution. For more discussion about this family, readers are referred to Chapter 3 of McNeil et al. (2005). Furthermore, a useful property about this family is that it has a stochastic representation in terms of multivariate normal distribution, as shown by Proposition 2.6. Proposition 2.6 is essentially a combination of Theorem 3.25 and Definition 3.26 of McNeil et al. (2005), and the proof is thus omitted.

Proposition 2.6. *Random vector* $\mathbf{X} \sim EC_n(\boldsymbol{\mu}, \boldsymbol{\Sigma}, \psi)$ *with* $\psi \in \Psi_\infty$ *if and only if there exist random vector* $\mathbf{Z}$ *and random variable* R *such that*

$$\mathbf{X} \stackrel{d}{=} \boldsymbol{\mu} + R\mathbf{Z}, \tag{2.3}$$

where $\mathbf{Z} \sim MVN(\mathbf{0}, \boldsymbol{\Sigma})$ *and* $R \geq 0$ *is a random variable independent of* $\mathbf{Z}$.

Proposition 2.6 presents an important relationship between the multivariate normal and elliptical distributions. With this representation, many properties of the multivariate normal distribution can be easily generalized to elliptical distribution. In later sections, we shall see some examples.

This paper concerns dependence structure. Comonotonicity is a perfect positive dependence and has important applications in actuarial science and finance. Dhaene et al. (2002) conduct a comprehensive study on the concept of comonotonicity and its applications. Below we cite their definition and several equivalent characterizations of comonotonicity.

Definition 2.7. A set $A \subset \mathbb{R}^n$ is said to be *comonotonic*, if for any $\mathbf{x}, \mathbf{y} \in A$, either $\mathbf{x} \leq \mathbf{y}$ or $\mathbf{y} \leq \mathbf{x}$ holds.

Intuitively, a set is comonotonic if and only if it is totally ordered.

Definition 2.8. For a random vector, its *support* is defined by

$$supp(\mathbf{X}) = \{\mathbf{x} \in \mathbb{R}^n : \mathbb{P}\{\mathbf{X} \in B(\mathbf{x}, r)\} > 0,\ \text{for any } r > 0\},$$

where $B(\mathbf{x}, r)$ denote the ball centered at $\mathbf{x}$ with radius r.

Definition 2.9. A random vector $\mathbf{X}$ is *comonotonic* if its support is comonotonic.

Dhaene et al. (2002) also develop several well-known characterizations of comonotonicity. For example, the following statements (a), (b), and (c) are equivalent. (a) $\mathbf{X} = (X_1, \dots, X_n)$ is comonotonic, (b) there exist a random variable Z and increasing functions $f_1, \dots, f_n$ such that $(X_1, \dots, X_n) \stackrel{d}{=} (f_1(Z), \dots, f_n(Z))$, (c) $\mathbb{P}\{X_1 \leq x_1, \dots, X_n \leq x_n\} = \min\{\mathbb{P}\{X_1 \leq x_1\}, \dots, \mathbb{P}\{X_n \leq x_n\}\}$ for all $(x_1, \dots, x_n) \in \mathbb{R}^n$.

In addition, Theorem 5 of Dhaene et al. (2002) provides a characterization of the comonotonicity for multivariate normal distribution by its covariance matrix. Specifically, $\mathbf{X} = (X_1, \dots, X_n) \sim MVN(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ is comonotonic if and only if $corr(X_i, X_j) = 1$ for all i, j (i.e., $rank(\boldsymbol{\Sigma}) = 1$). Furthermore, if all marginal distributions have the same variance 1, then the comonotonicity of $\mathbf{X}$ is equivalent to $\boldsymbol{\Sigma} = \mathbf{1}_{n\times n}$.

As a matter of fact, this characterization can be generalized to elliptical distributions induced by Ψ_∞. Specifically, an elliptical distribution with $\psi \in \Psi_\infty$ is comonotonic if and only if its dispersion matrix has rank 1. McNeil et al. (2005) point out this fact for multivariate-t distribution (in Chapter 5). Below, we formally state the characterization and prove it in general case.

Proposition 2.10. *Let* $\mathbf{X} \sim EC_n(\boldsymbol{\mu}, \boldsymbol{\Sigma}, \psi)$ *with* $\psi \in \Psi_\infty$. $\mathbf{X}$ *is comonotonic if and only if* $rank(\boldsymbol{\Sigma}) = 1$.

Proof. Recalling the stochastic representation (2.3), there exists $\mathbf{Z} \sim MVN(\mathbf{0}, \boldsymbol{\Sigma})$ and $R \geq 0$ independent of $\mathbf{Z}$ such that $\mathbf{X} \stackrel{d}{=} \boldsymbol{\mu} + R\mathbf{Z}$.

The "if" part. Assume $rank(\boldsymbol{\Sigma}) = 1$, then $corr(Z_i, Z_j) = 1$ for all i, j. Therefore, there exists $Z \sim N(0, 1)$ such that $Z_i = a_i Z$ with $a_i \geq 0$ for all $i = 1, \dots, n$. It immediately follows that $\mathbf{X}$ is comonotonic from the stochastic representation (2.3) and the functional characterization of comonotonicity.

The "only if" part. Assume that $\mathbf{X}$ is comonotonic. Consider any $\mathbf{y}, \mathbf{z} \in supp(\mathbf{Z})$. Since R is independent of $\mathbf{Z}$, then $\boldsymbol{\mu} + r\mathbf{y}$, $\boldsymbol{\mu} + r\mathbf{z} \in supp(\mathbf{X})$ for any $0 < r \in supp(\mathbf{Z})$. From the comonotonicity of $\mathbf{X}$, it holds that $\boldsymbol{\mu} + r\mathbf{y} \leq \boldsymbol{\mu} + r\mathbf{z}$ or $\boldsymbol{\mu} + r\mathbf{y} \geq \boldsymbol{\mu} + r\mathbf{z}$, which implies that $\mathbf{y} \leq \mathbf{z}$ or $\mathbf{y} \geq \mathbf{z}$. Therefore, we conclude that $\mathbf{Z}$ is comonotonic and thus $rank(\boldsymbol{\Sigma}) = 1$. □

3. Ordering Gini indexes according to $\leq_{icx}$

Lemma 3.1. *Let* $G(\mathbf{x}) = G(x_1, \dots, x_n)$ *be defined as in* (2.1), *i.e.,* $G(\mathbf{x}) = \sum_{1\leq i,j\leq n} |x_i - x_j|$. *Then* $-G(\mathbf{x})$ *is supermodular.*

Proof. See Appendix. □

Lemma 3.2. *Let* $G : \mathbb{R}^3 \to \mathbb{R}$ *be defined as in* (2.1), *i.e.,* $G(x_1, x_2, x_3) = 2(|x_1 - x_2| + |x_2 - x_3| + |x_3 - x_1|)$. *Then* $-u(G(x_1, x_2, x_3))$ *is supermodular for any increasing convex* u.

Proof. See Appendix. □

Müller (2001) develops a sufficient and necessary condition for the supermodular order between multivariate normal distributions, cited in Lemma 3.3.

Lemma 3.3. *Let* $\mathbf{X} = (X_1, \dots, X_n) \sim MVN(\boldsymbol{\mu}_1, \boldsymbol{\Sigma}_1)$ *and* $\mathbf{Y} = (Y_1, \dots, Y_n) \sim MVN(\boldsymbol{\mu}_2, \boldsymbol{\Sigma}_2)$. *Then* $\mathbf{X} \leq_{sm} \mathbf{Y}$ *if and only if* $\boldsymbol{\mu}_1 = \boldsymbol{\mu}_2$ *and* $\boldsymbol{\Sigma}_1 \leq \boldsymbol{\Sigma}_2$ *with all diagonal elements equal.*

Based on Block and Sampson (1988, Corollary 2.3) together with Müller and Scarsini (2000), we get a similar sufficient and necessary condition for elliptical distributions.

Proposition 3.4. *Let* $\mathbf{X} \sim EC_n(\boldsymbol{\mu}_1, \boldsymbol{\Sigma}_1, \psi)$ *and* $\mathbf{Y} \sim EC_n(\boldsymbol{\mu}_2, \boldsymbol{\Sigma}_2, \psi)$ *with* $\psi \in \Psi_\infty$. *Then* $\mathbf{X} \leq_{sm} \mathbf{Y}$ *if and only if* $\boldsymbol{\mu}_1 = \boldsymbol{\mu}_2$ *and* $\boldsymbol{\Sigma}_1 \leq \boldsymbol{\Sigma}_2$ *with all diagonal elements equal.*

By combining Proposition 3.4 and Lemma 3.2, we obtain the following result.

Proposition 3.5. *Let* $\mathbf{X} = (X_1, X_2, X_3) \sim EC_3(\mathbf{0}, \mathbf{\Sigma}_1, \psi)$ *and* $\mathbf{Y} = (Y_1, Y_2, Y_3) \sim EC_3(\mathbf{0}, \mathbf{\Sigma}_2, \psi)$ *with* $\psi \in \Psi_\infty$. *If* $\mathbf{\Sigma}_1 \le \mathbf{\Sigma}_2$ *with all diagonal elements equal, then* $G(\mathbf{X}) \ge_{icx} G(\mathbf{Y})$.

Similarly, by combining Proposition 3.4 and Lemma 3.1, we obtain the following result for high dimensional risks.

Proposition 3.6. *Let* $\mathbf{X} \sim EC_n(\mathbf{0}, \mathbf{\Sigma}_1, \psi)$ *and* $\mathbf{Y} \sim EC_n(\mathbf{0}, \mathbf{\Sigma}_2, \psi)$ *with* $\psi \in \Psi_\infty$. *If* $\mathbf{\Sigma}_1 \le \mathbf{\Sigma}_2$ *with all diagonal elements equal, then* $\mathbb{E}[G(\mathbf{X})] \ge \mathbb{E}[G(\mathbf{Y})]$ *given the expectations exist.*

Our ultimate objective is to show that $G(\mathbf{X})$ decreases in the sense of usual stochastic order as the dispersion matrix increases with diagonal elements fixed for multivariate elliptical risk $\mathbf{X}$. Now Proposition 3.5 completes a significant step to this objective in 3-dimensional case. However, the arguments of proving Proposition 3.5 cannot be generalized to higher-dimensional case. The main problem is that, when it comes to higher dimension, the composite function $-u(G(\mathbf{x}))$ is not necessarily supermodular for any increasing function u as in the three dimension case, i.e., Lemma 3.2 cannot be generalized to higher dimension. The following example demonstrates this point. Therefore, we remark that ordering $G(\mathbf{X})$ in the increasing convex order for higher dimensional risk is still an open problem.

Example 3.7. Consider $n = 4$. Let $u(x) = x^2$. Then u is increasing convex in $x \ge 0$. We shall show that $-u(G(x_1, x_2, x_3, x_4))$ is not supermodular. To this end, set $\mathbf{x} = (3, 1, 0, 0)$, $\mathbf{y} = (2, 2, 0, 0)$. Then $\mathbf{x}\wedge\mathbf{y} = (2, 1, 0, 0)$ and $\mathbf{x}\vee\mathbf{y} = (3, 2, 0, 0)$. Direct calculations yield that $u(G(\mathbf{x})) = 20^2 = 400$, $u(G(\mathbf{y})) = 16^2 = 256$, $u(G(\mathbf{x} \wedge \mathbf{y})) = 14^2 = 196$, and $u(G(\mathbf{x} \vee \mathbf{y})) = 22^2 = 484$, which does not satisfy $-u(G(\mathbf{x}))-u(G(\mathbf{y})) \le -u(G(\mathbf{x}\wedge\mathbf{y}))-u(G(\mathbf{x}\vee\mathbf{y}))$.

4. Ordering Gini indexes according to $\le_{st}$

In this section, we will discuss the monotonicity of $G(\mathbf{X})$ in the usual stochastic order with imposing certain assumptions on the dispersion matrix.

4.1. Conditional exchangeable case

Proposition 4.1. *Let* $\mathbf{X} = (X_1, \dots, X_n)$ *follow a multivariate normal distribution with mean* $\mathbf{0}$ *and positive definite covariance matrix* $\mathbf{\Sigma} = (\sigma_{ij}) \in \mathbb{R}^{n\times n}$. *If* $\sigma_{1k} = \sigma_{2k}$ *for all* $k = 3, \dots, n$ *and* $\sigma_{11} = \sigma_{22}$, *then* $\mathbb{P}\{G(\mathbf{X}) \le t\}$ *is increasing in* σ_{12} *for any* $t \ge 0$.

Proof. See Appendix. □

Proposition 4.1 suggests that if we impose a conditional exchangeable structure on the multivariate normal random vector $\mathbf{X}$, i.e., (X_1, X_2) is exchangeable conditional on the remaining components, then $G(\mathbf{X})$ is stochastically decreasing in $Cov[X_1, X_2]$, which is one component of the covariance matrix. In the following, we derive an analogous result for elliptical distribution by using Proposition 2.6.

Proposition 4.2. *Let* $\mathbf{X} \sim EC_n(\mathbf{0}, \mathbf{\Sigma}, \psi)$ *with* $\mathbf{\Sigma} = (\sigma_{ij}) \in \mathbb{R}^{n\times n}$ *and* $\psi \in \Psi_\infty$. *If* $\sigma_{1k} = \sigma_{2k}$ *for all* $k = 3, \dots, n$ *and* $\sigma_{11} = \sigma_{22}$, $\mathbb{P}\{G(\mathbf{X}) \le t\}$ *is increasing in* σ_{12} *for any* $t \ge 0$.

Proof. Let $\mathbf{X}' \sim EC_n(\mathbf{0}, \mathbf{\Sigma}', \psi)$ with $\mathbf{\Sigma}' = (\sigma'_{ij}) \in \mathbb{R}^{n\times n}$, where $\sigma'_{12} \ge \sigma_{12}$ and $\sigma'_{ij} = \sigma_{ij}$ for all $1 \le i < j \le n$ and $(i, j) \ne (1, 2)$. According to Proposition 2.6, there exist $\mathbf{Y} \sim MVN(\mathbf{0}, \mathbf{\Sigma})$, $\mathbf{Y}' \sim MVN(\mathbf{0}, \mathbf{\Sigma}')$ and a random variable $R \ge 0$ independent of $\mathbf{Y}, \mathbf{Y}'$ such that $\mathbf{X} \stackrel{d}{=} R\,\mathbf{Y}$ and $\mathbf{X}' \stackrel{d}{=} R\,\mathbf{Y}'$.

Note that for any given $r > 0$, $r\mathbf{Y}$ and $r\mathbf{Y}'$ follow multivariate normal distributions with covariance matrices satisfying the condition in Proposition 4.1. Therefore, we have $\mathbb{P}\{G(r\mathbf{Y}) \le t\} \le \mathbb{P}\{G(r\mathbf{Y}') \le t\}$. Thus,

$$\begin{aligned}\mathbb{P}\{G(\mathbf{X}) \le t\} &= \mathbb{P}\{G(R\mathbf{Y}) \le t\} = \mathbb{E}[\mathbb{P}\{G(R\mathbf{Y}) \le t\}|R] \\ &\le \mathbb{E}[\mathbb{P}\{G(R\mathbf{Y}') \le t\}|R] = \mathbb{P}\{G(R\mathbf{Y}') \le t\} = \mathbb{P}\{G(\mathbf{X}') \le t\}. \quad \square\end{aligned}$$

4.2. Quasi dominance between covariance matrices

We first cite Theorem 4.1 of Eaton and Perlman (1991) below.

Lemma 4.3. *Let* $\mathbf{X}_i \sim MVN(\mathbf{0}, \mathbf{\Sigma}_i)$ *for* $i = 1, 2$. *If* $\mathbf{\Sigma}_2 - \mathbf{\Sigma}_1$ *is positive semidefinite, then* $\mathbb{P}\{\mathbf{X_2} \in C\} \le \mathbb{P}\{\mathbf{X_1} \in C\}$ *for any convex and centrally symmetric set* C *(i.e.,* $C = -C$*).*

A matrix P is said to dominate another matrix Q if $P - Q$ is positive semidefinite. In this sense, Condition (4.1) is referred to as "quasi" dominance. Intuitively, Lemma 4.3 indicates that the covariance matrix determines the degree of central concentration of a multivariate normal distribution. Specifically, the "smaller" the covariance matrix is, the more concentrated the normal random vector is on a convex and centrally symmetric region.

Proposition 4.4. *Let* $\mathbf{X} \sim MVN(\mathbf{0}, \mathbf{\Sigma}_X)$ *and* $\mathbf{Y} \sim MVN(\mathbf{0}, \mathbf{\Sigma}_Y)$. *If there exists* $a \in \mathbb{R}$ *such that*

$$a\mathbf{1}_{n\times n} + \mathbf{\Sigma}_X - \mathbf{\Sigma}_Y \quad \text{is positive semidefinite}, \tag{4.1}$$

where $\mathbf{1}_{n\times n}$ *denotes the* $n \times n$ *matrix with all entries equal to* 1, *then* $G(\mathbf{X}) \ge_{st} G(\mathbf{Y})$.

Proof. See Appendix. □

In this paper, we focus on the comparison of dependence structure without changing marginals. That means the dispersion matrices we compare have the same diagonal elements. When taking difference, the diagonal elements become 0. In this sense, we do not expect one dispersion matrix to dominate another since the difference matrix is not positive semidefinite. Therefore, the dominance condition in Lemma 4.3 is relaxed to "quasi" dominance condition (4.1) to deal with this situation.

Example 4.5. Examples satisfying condition (4.1).

(i) All the off-diagonal elements of the covariance matrix increase by the same amount, i.e., $\mathbf{\Sigma}_Y = \mathbf{\Sigma}_X + \sigma(\mathbf{1}_{n\times n} - \mathbf{I}_n)$ with $\sigma > 0$. This includes the case that $\mathbf{X}$ and $\mathbf{Y}$ are both exchangeable. The conclusion of Proposition 4.4 for exchangeable $\mathbf{X}$ and $\mathbf{Y}$ has been verified using other approaches, see for example, Theorem 6.25 of Tong (1990).

(ii) The off-diagonal elements of the covariance on the kth row and column increase by the same amount, i.e., $\mathbf{\Sigma}_Y = \mathbf{\Sigma}_X + \sigma\sum_{j\ne k}(\mathbf{\Delta}_{jk} + \mathbf{\Delta}_{kj})$ with $\sigma > 0$, where $\mathbf{\Delta}_{kj}$ denotes the matrix with 1 in the (k, j) position and 0 in others.

Using the same argument as in Proposition 4.2, it is easy to generalize the conclusion of Proposition 4.4 to elliptical distributions. The proof is similar to that of Proposition 4.2 and thus is omitted.

Proposition 4.6. *Let* $\mathbf{X} \sim EC_n(\mathbf{0}, \mathbf{\Sigma}_X, \psi)$ *and* $\mathbf{Y} \sim EC_n(\mathbf{0}, \mathbf{\Sigma}_Y, \psi)$ *with* $\psi \in \Psi_\infty$. *If there exists* $a \in \mathbb{R}$ *such that*

$$a\mathbf{1}_{n\times n} + \mathbf{\Sigma}_X - \mathbf{\Sigma}_Y \quad \text{is positive semidefinite},$$

where $\mathbf{1}_{n\times n}$ *denotes the* $n \times n$ *matrix with all entries equal to* 1, *then* $G(\mathbf{X}) \ge_{st} G(\mathbf{Y})$.

4.3. Comparison with comonotonicity

In the above, we manage to order Gini indexes of multivariate elliptical risks when the dispersion matrices follow special structures. On the other hand, it is difficult to deal with dispersion matrices with general structure. In this section, we shall prove that comonotonicity produces the stochastically smallest Gini index among all multivariate elliptical risks with common marginals. The geometric argument used in this section is motivated by Theorem A2 of Joe (1990), which establishes the concordance order between elliptical distributions.

Proposition 4.7. *Let $\mathbf{X} = (X_1, X_2, X_3)$ and $\mathbf{Y} = (Y_1, Y_2, Y_3)$ follow multivariate normal distributions with mean $\mathbf{0}$ and common marginal distributions. If (Y_1, Y_2, Y_3) is comonotonic, then $G(\mathbf{Y}) \le_{st} G(\mathbf{X})$.*

Proof. See Appendix. □

Similarly to Propositions 4.1 and 4.4, Proposition 4.7 can also be generalized to elliptical distributions. As before, the proof is omitted.

Proposition 4.8. *Let $\mathbf{X} = (X_1, X_2, X_3)$ and $\mathbf{Y} = (Y_1, Y_2, Y_3)$ follow elliptical distributions with mean vector $\mathbf{0}$, common marginal distributions and a common generator $\psi \in \Psi_\infty$. If $\mathbf{Y}$ is comonotonic, then $G(\mathbf{Y}) \le_{st} G(\mathbf{X})$.*

5. Discussion on the pd-1 order

The methods used to analyze the Gini index in the above sections can be used to generalize the pd-1 order proposed by Shaked and Tong (1985). The pd-1 order is a partial order which compares the degree of dispersion of exchangeable random vectors. We cite the definition of pd-1 order below.

Definition 5.1. Let $\mathbf{X} = (X_1, \ldots, X_n)$ and $\mathbf{Y} = (Y_1, \ldots, Y_n)$ be two exchangeable random vectors with common marginals. We write $\mathbf{X} \le_{pd-1} \mathbf{Y}$ if

$$\left|\sum_{i=1}^n c_i X_{(i)}\right| \ge_{st} \left|\sum_{i=1}^n c_i Y_{(i)}\right| \quad \text{whenever } \sum_{i=1}^n c_i = 0.$$

Shaked and Tong (1985) give several examples satisfying this order, such as multivariate normal, t, exponential distributions. For more details about this order and other related orders, readers are referred to Shaked and Tong (1985) and Shaked and Shanthikumar (2007).

Note that this order is restrictive in the sense that it only applies to exchangeable random vectors. In order to apply it to non-exchangeable random vectors, we propose a weaker order $\le_{wpd-1}$ and establish this order among elliptical random vectors.

Definition 5.2. Let $\mathbf{X} = (X_1, \ldots, X_n)$ and $\mathbf{Y} = (Y_1, \ldots, Y_n)$ be two random vectors with common marginals. We write $\mathbf{X} \le_{wpd-1} \mathbf{Y}$ if

$$\left|\sum_{i=1}^n c_i X_{(i)}\right| \ge_{st} \left|\sum_{i=1}^n c_i Y_{(i)}\right|,$$

for all $c_1 \le \cdots \le c_n$ such that $\{c_1, \ldots, c_n\} = \{-c_1, \ldots, -c_n\}$.

Clearly, $\mathbf{X} \le_{pd-1} \mathbf{Y}$ implies $\mathbf{X} \le_{wpd-1} \mathbf{Y}$ since $\{c_1, \ldots, c_n\} = \{-c_1, \ldots, -c_n\}$ implies $\sum_{i=1}^n c_i = 0$. Since the weak pd-1 order is defined for general random vector, the constraint $\{c_1, \ldots, c_n\} = \{-c_1, \ldots, -c_n\}$ is added to remedy the loss of symmetry by relaxing the assumption of exchangeability.

Proposition 5.3. *Let $\mathbf{X} \sim EC_n(\mathbf{0}, \mathbf{\Sigma}_X, \psi)$, $\mathbf{Y} \sim EC_n(\mathbf{0}, \mathbf{\Sigma}_Y, \psi)$ with $\psi \in \Psi_\infty$ and $\mathbf{X}$ and $\mathbf{Y}$ have common marginal distributions. If there exists $a \in \mathbb{R}$ such that*

$$a\mathbf{1}_{n\times n} + \mathbf{\Sigma}_X - \mathbf{\Sigma}_Y \quad \textit{is positive semidefinite},$$

where $\mathbf{1}_{n\times n}$ denotes the $n \times n$ matrix with all entries equal to 1, then $\mathbf{X} \le_{wpd-1} \mathbf{Y}$.

Proof. Note that the arguments in the proof of Proposition 4.4 still hold for any $\{c_1, \ldots, c_n\}$ satisfying the conditions in Definition 5.2. The conclusion is reached by simply combining Propositions 4.4 and 4.6. Details are omitted. □

Similar as Proposition 4.8, it can be proved that comonotonic elliptical risk is the lower bound of all multivariate elliptical risks with common marginals in the sense of the weak pd-1 order.

Proposition 5.4. *Let $\mathbf{Y} = (Y_1, Y_2, Y_3)$, $\mathbf{X} = (X_1, X_2, X_3)$ follow elliptical distributions with common marginals and a common generator $\psi \in \Psi_\infty$. If $\mathbf{Y}$ is comonotonic, then $\mathbf{X} \le_{wpd-1} \mathbf{Y}$.*

Proof. For any $c_1 \le c_2 \le c_3$ such that $\{c_1, c_2, c_3\} = \{-c_1, -c_2, -c_3\}$, we have $c_2 = 0$ and $c_3 = -c_1 \ge 0$. Therefore, $\sum_{i=1}^3 c_i X_{(i)} = c_3(X_{(3)} - X_{(1)}) = \frac{c_3}{4} G(\mathbf{X})$. According to Proposition 4.8, we know that $G(\mathbf{X}) \ge_{st} G(\mathbf{Y})$ and thus $\frac{c_3}{4} G(\mathbf{X}) \ge_{st} \frac{c_3}{4} G(\mathbf{Y})$, which implies that $\mathbf{X} \le_{wpd-1} \mathbf{Y}$. □

An interesting application of the weak pd-1 order is to measure how far a random vector is from the comonotonic dependence structure. For a random vector $\mathbf{X} = (X_1, \ldots, X_n)$, we introduce a new index: $H(\mathbf{X}) = \sum_{i=1}^{[\frac{n+1}{2}]} (X_{(n+1-i)} - X_{(i)})$. Clearly, the coefficients of the order statistics satisfy the conditions in Definition 5.2. Therefore, $\mathbf{X} \le_{wpd-1} \mathbf{Y}$ implies that $H(\mathbf{X}) \ge_{st} H(\mathbf{Y})$.

From the geometric perspective, the comonotonic dependence structure is represented by the line $l = \{(t, \ldots, t), t \in \mathbb{R}\}$, and the distance between the dependence structure of $\mathbf{X}$ and the comonotonic dependence structure can be measured by the geometric distance between $(X_1, \ldots, X_n)$ and the line l. If we use the Minkowski norm of order 1, that is $d_1((x_1, \ldots, x_n), (y_1, \ldots, y_n)) = \sum_{i=1}^n |x_i - y_i|$, then the distance between $(X_1, \ldots, X_n)$ and the line l is $d_1((X_1, \ldots, X_n), l) = \inf_t d_1((X_1, \ldots, X_n), (t, \ldots, t)) = \inf_t \sum_{i=1}^n |X_i - t|$. Note that

$$\begin{aligned}\sum_{i=1}^n |X_i - t| = \sum_{i=1}^n |X_{(i)} - t| &= \sum_{i=1}^{\left[\frac{n+1}{2}\right]-1} (|X_{(i)} - t| + |X_{(n+1-i)} - t|) \\ &\quad + \sum_{i=\left[\frac{n+1}{2}\right]}^{n+1-\left[\frac{n+1}{2}\right]} |X_{(i)} - t| \\ &\ge \sum_{i=1}^{\left[\frac{n+1}{2}\right]} (X_{(n+1-i)} - X_{(i)}) = H(\mathbf{X}).\end{aligned}$$

The equality is obtained at $t \in [X_{([\frac{n+1}{2}])}, X_{(n+1-[\frac{n+1}{2}])}]$. Therefore, $d_1(\mathbf{X}, l) = H(\mathbf{X})$. In this sense, the index $H(\mathbf{X})$ measures the distance of a random vector from the comonotonic dependence structure, or degree of comonotonicity. Intuitively, the smaller $H(\mathbf{X})$ is, the higher degree of comonotonicity the random vector has.

The dependence structure of comonotonicity has wide applications. In finance and insurance, comonotonicity usually serves as the most conservative assumption when the underlying dependence structure is unknown. On the other hand, it is still important to evaluate the appropriateness of the assumption of comonotonicity by investigating the distance between the underlying dependence structure and comonotonicity. Dhaene et al. (2012) studied the herd behavior in the stock market, which is essentially the degree of comonotonicity of certain stocks. They proposed the herd behavior index, defined to be the ratio of the variance of a portfolio and the variance of the same portfolio but with comonotonicity, to measure the degree of co-movement of stocks. The results presented in this section show that the index $H(\mathbf{X})$ or its related quantity (such as $\mathbb{E}[H(\mathbf{X})]$) also serves as a measure of degree of comonotonicity and thus supplements the studies in Dhaene et al. (2012).

6. Concluding remarks

The studies of Brazauskas et al. (2007) and Samanthi et al. (submitted for publication) motivate the comparison of Gini indexes of multivariate elliptical risks with common marginals but different dispersion matrices. In this paper, we first generally establish the increasing convex order between Gini indexes of multivariate elliptical risks. Then we manage to order Gini indexes in the usual stochastic order when the dispersion matrices follow special structures. Furthermore, we demonstrate that among all dependence structures, comonotonicity produces the smallest Gini index in the sense of usual stochastic order.

Apart from its usefulness in actuarial applications, the comparison of Gini indexes presents its own interests. First, it enriches the studies on the concentration of elliptical random vectors on convex centrally symmetric regions. Second, it generalizes the concept of pd-1 order and permits further applications in operations research.

On the other hand, this paper motivates open problems. For example, to what extent can Gini indexes of multivariate elliptical risks be ordered in the sense of usual stochastic order? Does the conclusion still hold for high dimensional risks with general elliptical distribution? Investigation of these problems will be presented in the future papers.

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Appendix

Proof of Lemma 3.1. According to Proposition 2.4, it suffices to show that $-G_{ij}(x_i, x_j) := -G(x_1, \ldots, x_n)$ is supermodular as a bivariate function of (x_i, x_j) for any fixed x_k, $k \neq i, j$ and any $1 \leq i < j \leq n$. Specifically, we want to show that for any $1 \leq i < j \leq n$ and any $x_i \leq y_i$ and $x_j \leq y_j$,

$$G_{ij}(x_i, x_j) + G_{ij}(y_i, y_j) \leq G_{ij}(y_i, x_j) + G_{ij}(x_i, y_j).$$

Noting that $G_{ij}(x_i, x_j) = 2|x_i - x_j| + 2\sum_{k \neq i,j}(|x_i - z_k| + |x_j - z_k|) + \sum_{k,l:\{k,l\}\cap\{i,j\}=\emptyset} |z_k - z_l|$, it is equivalent to show

$$2|x_i - x_j| + 2|y_i - y_j| \leq 2|y_i - x_j| + 2|x_i - y_j|,$$

which is easy to verify. □

Proof of Lemma 3.2. Noting that $u \circ G$ is permutational invariant, it is sufficient to show that

$$u(G(x_1, x_2, x_3)) + u(G(y_1, y_2, y_3)) \leq u(G(y_1, x_2, x_3)) + u(G(x_1, y_2, y_3)), \quad \text{(A.1)}$$

for any $(x_1, x_2, x_3) \leq (y_1, y_2, y_3)$.

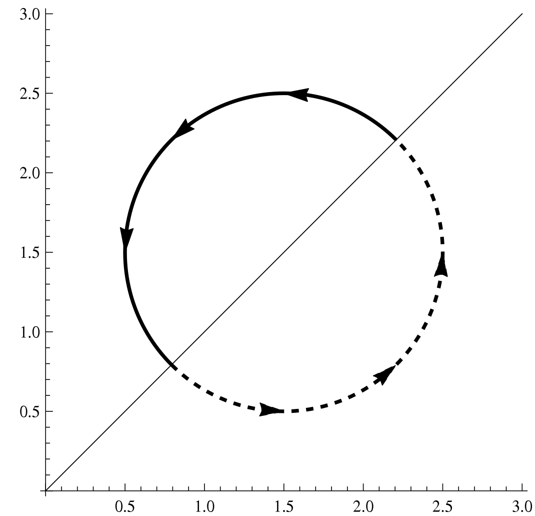

Fig. 1. ∂C_+ (solid) and ∂C_- (dashed).

According to Lemma 3.1, we have $G(x_1, x_2, x_3) + G(y_1, y_2, y_3) \leq G(y_1, x_2, x_3) + G(x_1, y_2, y_3)$. Recall that it holds for an increasing convex function that $u(a) + u(b) \leq u(c) + u(d)$ for any a, b, c, d such that $a + b \leq c + d$ and $\max\{a, b\} \leq \max\{c, d\}$. It suffices to show

$$\max\{G(x_1, x_2, x_3), G(y_1, y_2, y_3)\} \leq \max\{G(y_1, x_2, x_3), G(x_1, y_2, y_3)\}, \quad \text{(A.2)}$$

for any $(x_1, x_2, x_3) \leq (y_1, y_2, y_3)$.

Note that $G(x_1, x_2, x_3) = 4(\max\{x_1, x_2, x_3\} - \min\{x_1, x_2, x_3\})$. If $x_1 > \min\{x_2, x_3\}$, then $y_1 \geq x_1 > \min\{x_2, x_3\}$, and thus

$$G(x_1, x_2, x_3) = 4(\max\{x_1, x_2, x_3\} - \min\{x_1, x_2, x_3\}) \leq 4(\max\{y_1, x_2, x_3\} - \min\{y_1, x_2, x_3\}) = G(y_1, x_2, x_3).$$

Otherwise, if $x_1 \leq \min\{x_2, x_3\}$, then $x_1 \leq \min\{y_2, y_3\}$, and thus

$$\begin{aligned} G(x_1, x_2, x_3) &= 4(\max\{x_1, x_2, x_3\} - \min\{x_1, x_2, x_3\}) \\ &= 4(\max\{x_2, x_3\} - x_1) \\ &\leq 4(\max\{y_2, y_3\} - x_1) \\ &= 4(\max\{x_1, y_2, y_3\} - \min\{x_1, y_2, y_3\}) \\ &= G(x_1, y_2, y_3). \end{aligned}$$

Therefore, we conclude that $G(x_1, x_2, x_3) \leq \max\{G(y_1, x_2, x_3), G(x_1, y_2, y_3)\}$. Similarly, we can show that $G(y_1, y_2, y_3) \leq \max\{G(y_1, x_2, x_3), G(x_1, y_2, y_3)\}$. □

In order to prove Proposition 4.1, we propose the following lemma.

Lemma A.1. *Assume (X, Y) have an exchangeable bivariate normal random vector with correlation coefficient ρ. Let $C \subset \mathbb{R}^2$ be any convex set such that $\{(x, y)|(y, x) \in C\} = C$. Then $\mathbb{P}\{(X, Y) \in C\}$ is increasing in ρ.*

Proof. For simplicity, assume C is bounded. The unbounded case can be approached by limiting argument. Denote by ∂C the boundary of C with positive (counterclockwise) orientation. Then ∂C is piecewise smooth due to the convexity of C. Furthermore, denote $\partial C_+ = \{(x, y) \in \partial C | y \geq x\}$ and $\partial C_- = \{(x, y) \in \partial C | y \leq x\}$, then $\partial C = \partial C_+ \cup \partial C_-$. Let $\partial C'_+$ be same as ∂C_+ but with the opposite (clockwise) orientation. Then $\partial C'_+$ and ∂C_- are reflections to each other with respect to the line $y = x$. Figs. 1 and 2 provide an illustration (not an accurate representation) of these orientated curves.

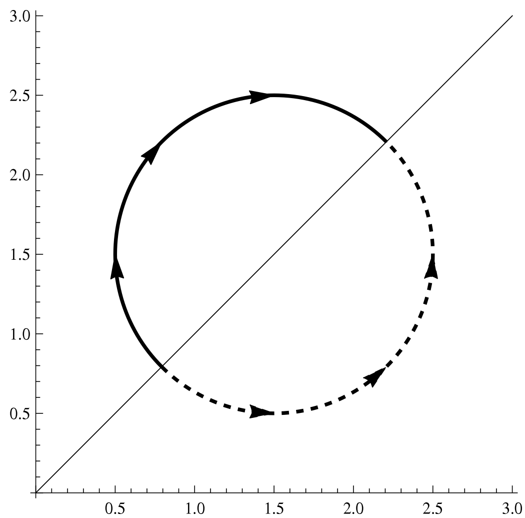

Fig. 2. $\partial C_+'$ (solid) and ∂C_- (dashed).

Without loss of generality, assume $\mathbb{E}[X] = \mathbb{E}[Y] = 0$ and $\mathrm{Var}[X] = \mathrm{Var}[Y] = 1$. Then the density function of (X, Y) by $f(x, y) = \frac{1}{2\pi\sqrt{1-\rho^2}} \exp\left\{-\frac{x^2+y^2-2\rho xy}{2(1-\rho^2)}\right\}$. Plackett's identity (Plackett, 1954) states that $\frac{\partial}{\partial\rho} f(x, y) = \frac{\partial^2}{\partial x \partial y} f(x, y)$. According to Fubini's theorem, we have

$$\begin{aligned}\frac{\partial}{\partial\rho}\mathbb{P}\{(X, Y) \in C\} &= \frac{\partial}{\partial\rho}\int_C f(x, y)dxdy = \int_C \frac{\partial}{\partial\rho} f(x, y)dxdy \\ &= \int_C \frac{\partial^2}{\partial x\partial y} f(x, y)dxdy \overset{(*)}{=} \oint_{\partial C} \frac{\partial}{\partial y} f(x, y)dy \\ &= \left(\int_{\partial C_+} + \int_{\partial C_-}\right)\frac{\partial}{\partial y} f(x, y)dy \\ &= \left(-\int_{\partial C_+'} + \int_{\partial C_-}\right)\frac{\partial}{\partial y} f(x, y)dy, \qquad \text{(A.3)}\end{aligned}$$

where Equality $(*)$ follows from Green's theorem.

Note that $\frac{\partial}{\partial y} f(x, y) = \frac{\rho x - y}{1-\rho^2} f(x, y)$. Following (A.3), we have

$$\begin{aligned}\frac{\partial}{\partial\rho}\mathbb{P}\{(X, Y) \in C\} &= -\int_{\partial C_+'} \frac{\rho x - y}{1-\rho^2} f(x, y)dy \\ &\quad + \int_{\partial C_-} \frac{\rho x - y}{1-\rho^2} f(x, y)dy \\ &= -\int_{\partial C_+'} f(x, y)dy \times \int_{\partial C_+'} \frac{\rho x - y}{1-\rho^2}\frac{f(x, y)}{\int_{\partial C_+'} f(x, y)dy}dy \\ &\quad + \int_{\partial C_-} f(x, y)dy \times \int_{\partial C_-} \frac{\rho x - y}{1-\rho^2}\frac{f(x, y)}{\int_{\partial C_-} f(x, y)dy}dy \\ &= -\int_{\partial C_+'} f(x, y)dy \times \mathbb{E}\left[\frac{\rho X - Y}{1-\rho^2}\middle| (X, Y) \in \partial C_+'\right] \\ &\quad + \int_{\partial C_-} f(x, y)dy \times \mathbb{E}\left[\frac{\rho X - Y}{1-\rho^2}\middle| (X, Y) \in \partial C_-\right].\end{aligned}$$

Due to the symmetry between ∂C_+ and $\partial C_-'$ and the exchangeability of (X, Y) (or $f(x, y)$), we know that $\int_{\partial C_+'} f(x, y)dy = \int_{\partial C_-} f(x, y)dy$ and

$$\begin{aligned}\mathbb{E}\left[\frac{\rho X - Y}{1-\rho^2}\middle| (X, Y) \in \partial C_+'\right] &= \mathbb{E}\left[\frac{\rho Y - X}{1-\rho^2}\middle| (Y, X) \in \partial C_+'\right] \\ &= \mathbb{E}\left[\frac{\rho Y - X}{1-\rho^2}\middle| (X, Y) \in \partial C_-\right].\end{aligned}$$

Therefore,

$$\begin{aligned}\frac{\partial}{\partial\rho}\mathbb{P}\{(X, Y) \in C\} &= \int_{\partial C_-} f(x, y)dy \\ &\quad \times \left(-\mathbb{E}\left[\frac{\rho X - Y}{1-\rho^2}\middle| (X, Y) \in \partial C_+'\right]\right. \\ &\quad \left. + \mathbb{E}\left[\frac{\rho X - Y}{1-\rho^2}\middle| (X, Y) \in \partial C_-\right]\right) \\ &= \int_{\partial C_-} f(x, y)dy \\ &\quad \times \left(-\mathbb{E}\left[\frac{\rho Y - X}{1-\rho^2}\middle| (X, Y) \in \partial C_-\right]\right. \\ &\quad \left. + \mathbb{E}\left[\frac{\rho X - Y}{1-\rho^2}\middle| (X, Y) \in \partial C_-\right]\right) \\ &= \int_{\partial C_-} f(x, y)dy \times \mathbb{E}\left[\frac{(1+\rho)(X - Y)}{1-\rho^2}\middle| \right. \\ &\quad \left. (X, Y) \in \partial C_-\right] \geq 0.\end{aligned}$$

The last inequality holds because $(X, Y) \in \partial C_-$ implies $X \geq Y$. It immediately follows that $\mathbb{P}\{(X, Y) \in C\}$ is increasing in ρ. □

Proof of Proposition 4.1. We shall use conditioning argument. According to the property of multivariate normal distribution, we know that conditioning on $\{X_3 = x_3, \ldots, X_n = x_n\}$, (X_1, X_2) has an exchangeable bivariate normal distribution with covariance $\sigma_{12}^* = \sigma_{12} - s$, where s is determined by other components of the covariance matrix.

For any fixed $x_3, \ldots, x_n$ and $t \geq 0$, denote $C = \{(x_1, x_2)| G(x_1, x_2, x_3, \ldots, x_n) \leq t\}$. Note that $C \subset \mathbb{R}^2$ is a convex polygon and symmetric with respect to the line $x_1 = x_2$. According to Lemma A.1, $\mathbb{P}\{G(\mathbf{X}) \leq t | X_3 = x_3, \ldots, X_n = x_n\} = \mathbb{P}\{(X_1, X_2) \in C | X_3 = x_3, \ldots, X_n = x_n\}$ is increasing in σ_{12}^* and thus in σ_{12}. Therefore, $\mathbb{P}\{G(\mathbf{X}) \leq t\} = \mathbb{E}[\mathbb{P}\{G(\mathbf{X}) \leq t | X_3, \ldots, X_n\}]$ is also increasing in σ_{12}. □

Proof of Proposition 4.4. Recall expression (2.2),

$$G(\mathbf{X}) = \sum_{k=1}^{n}(4k - 2n - 2)X_{(k)} \triangleq \sum_{k=1}^{n} c_k X_{(k)},$$

where $c_k = 4k - 2n - 2$. Noting that $\{c_k, k = 1, 2, \ldots, n\}$ is an increasing sequence, according to arrangement inequality, we have

$$G(\mathbf{X}) = \max_{\pi\in\mathcal{P}}\left\{\sum_{k=1}^{n} c_k X_{\pi(k)}\right\} = \max_{\pi\in\mathcal{P}}\left\{\sum_{k=1}^{n} c_{\pi(k)} X_k\right\},$$

where $\mathcal{P}$ denotes the collection of all permutations of $(1, 2, \ldots, n)$. Let $\mathbf{C} \in \mathbb{R}^{n!\times n}$ be the matrix generated by all different permutations of $(c_1, \ldots, c_n)$. Then $G(\mathbf{X})$ and $G(\mathbf{Y})$ are the largest order statistics of random vectors $\mathbf{C}\mathbf{X}^T$ and $\mathbf{C}\mathbf{Y}^T$, respectively. On the other hand, since $\mathbf{X}$ and $\mathbf{Y}$ follow multivariate normal distributions, so do $\mathbf{C}\mathbf{X}^T$ and $\mathbf{C}\mathbf{Y}^T$. Specifically, $\mathbf{C}\mathbf{X}^T \sim MVN(\mathbf{0}, \mathbf{C}\boldsymbol{\Sigma}_X\mathbf{C}^T)$ and $\mathbf{C}\mathbf{Y}^T \sim MVN(\mathbf{0}, \mathbf{C}\boldsymbol{\Sigma}_Y\mathbf{C}^T)$.

Noting that $\mathbf{C}\mathbf{1}_{n\times n}\mathbf{C}^T = \mathbf{0}$ since $\sum_{k=1}^{n} c_k = 0$, comparing covariance matrices of $\mathbf{C}\mathbf{X}^T$ and $\mathbf{C}\mathbf{Y}^T$ yields that

$$\begin{aligned}\mathbf{C}\boldsymbol{\Sigma}_X\mathbf{C}^T - \mathbf{C}\boldsymbol{\Sigma}_Y\mathbf{C}^T &= \mathbf{C}\boldsymbol{\Sigma}_X\mathbf{C}^T - \mathbf{C}\boldsymbol{\Sigma}_Y\mathbf{C}^T + a\mathbf{C}\mathbf{1}_{n\times n}\mathbf{C}^T \\ &= \mathbf{C}(\boldsymbol{\Sigma}_X - \boldsymbol{\Sigma}_Y + a\mathbf{1}_{n\times n})\mathbf{C}^T,\end{aligned}$$

which is positive semidefinite since $\boldsymbol{\Sigma}_X - \boldsymbol{\Sigma}_Y + a\mathbf{1}_{n\times n}$ is positive semidefinite.

Recall that $G(\mathbf{X}) = \max\{row_i\mathbf{C}.\mathbf{X}^T, i = 1, \dots, n!\}$. Since $\{c_1, \dots, c_n\} = \{-c_1, \dots, -c_n\}$, then $\{row_i\mathbf{C}.\mathbf{X}^T, i = 1, \dots, n!\} = \{-row_i\mathbf{C}.\mathbf{X}^T, i = 1, \dots, n!\}$. Therefore,

$$\begin{aligned}\mathbb{P}\{G(\mathbf{X}) \le t\} &= \mathbb{P}\{\mathbf{C}\mathbf{X}^T \le (t, \dots, t)^T\} = \mathbb{P}\{-\mathbf{C}\mathbf{X}^T \le (t, \dots, t)^T\} \\ &= \mathbb{P}\{(-t, \dots, -t)^T \le \mathbf{C}\mathbf{X}^T \le (t, \dots, t)^T\} \\ &\triangleq \mathbb{P}\{\mathbf{C}\mathbf{X}^T \in Q_t\},\end{aligned}$$

where Q_t is a super cube centered at origin with side length of $2t$. It is clear that Q_t is convex and centrally symmetric. According to Lemma 4.3,

$$\mathbb{P}\{G(\mathbf{X}) \le t\} = \mathbb{P}\{\mathbf{C}\mathbf{X}^T \in Q_t\} \le \mathbb{P}\{\mathbf{C}\mathbf{Y}^T \in Q_t\} = \mathbb{P}\{G(\mathbf{Y}) \le t\},$$

for any t, which implies that $G(\mathbf{X}) \ge_{st} G(\mathbf{Y})$ from Definition 2.1. □

Proof of Proposition 4.7. Denote the variances of the marginal distributions by $\sigma_1^2 \le \sigma_2^2 \le \sigma_3^2$. There exists $\{Z_1, Z_2, Z_3\} \overset{i.i.d.}{\sim} N(0, 1)$, such that $(Y_1, Y_2, Y_3) \overset{d}{=} (\sigma_1 Z_1, \sigma_2 Z_1, \sigma_3 Z_1)$ and

$$(X_1, X_2, X_3) \overset{d}{=} \begin{pmatrix}\sigma_1 & 0 & 0\\ 0 & \sigma_2 & 0\\ 0 & 0 & \sigma_3\end{pmatrix}\begin{pmatrix}l_{11} & 0 & 0\\ l_{21} & l_{22} & 0\\ l_{31} & l_{32} & l_{33}\end{pmatrix}\begin{pmatrix}Z_1\\ Z_2\\ Z_3\end{pmatrix}$$

where $\mathbf{L}_X = \begin{pmatrix}l_{11} & 0 & 0\\ l_{21} & l_{22} & 0\\ l_{31} & l_{32} & l_{33}\end{pmatrix}$ is the Cholesky decomposition of the correlation matrix of (X_1, X_2, X_3), which means that $l_{11} = 1$, $l_{21}^2 + l_{22}^2 = 1$ and $l_{31}^2 + l_{32}^2 + l_{33}^2 = 1$.

Therefore, $\mathbb{P}\{G(\mathbf{X}) \le t\} = \mathbb{P}\{(Z_1, Z_2, Z_3) \in R_X(t)\}$ and $\mathbb{P}\{G(\mathbf{Y}) \le t\} = \mathbb{P}\{(Z_1, Z_2, Z_3) \in R_Y(t)\}$, where

$$R_Y(t) = \{(z_1, z_2, z_3) : 4(\sigma_3 - \sigma_1)|z_1| \le t\}$$

$$\begin{aligned}R_X(t) &= \{(z_1, z_2, z_3) : 2|\sigma_1 z_1' - \sigma_2 z_2'| \\ &\quad + 2|\sigma_2 z_2' - \sigma_3 z_3'| + 2|\sigma_3 z_3' - \sigma_1 z_1'| \le t\} \\ &= \{(z_1, z_2, z_3) : 4|\sigma_1 z_1' - \sigma_2 z_2'| \le t\} \\ &\quad \cap \{(z_1, z_2, z_3) : 4|\sigma_2 z_2' - \sigma_3 z_3'| \le t\} \\ &\quad \cap \{(z_1, z_2, z_3) : 4|\sigma_3 z_3' - \sigma_1 z_1'| \le t\},\end{aligned}$$

with $z_1' = z_1$, $z_2' = l_{21}z_1 + l_{22}z_2$ and $z_3' = l_{31}z_1 + l_{32}z_2 + l_{33}z_3$.

Now we compare the two regions $R_Y(t)$ and $R_X^1(t) = \{(z_1, z_2, z_3) : 4|\sigma_3 z_3' - \sigma_1 z_1'| \le t\}$. Note that both of them are regions between a pair of parallel planes. For $R_Y(t)$, the distance between the boundary planes is $\frac{t}{2(\sigma_3 - \sigma_1)}$. For $R_X^1(t)$, the distance between the boundary planes is $\frac{t}{2\sqrt{\sigma_1^2 + \sigma_3^3 - 2l_{31}\sigma_1\sigma_3}} \le \frac{t}{2(\sigma_3 - \sigma_1)}$. Since both $R_Y(t)$ and $R_X^1(t)$ are centered at the origin, we conclude that $R_X^1(t)$, and thus $R_X(t)$ as a subset of $R_X^1(t)$, can be moved inside $R_Y(t)$ through certain rational transformations. Since the distribution of (Z_1, Z_2, Z_3) is rational invariant, it immediately follows that $\mathbb{P}\{(Z_1, Z_2, Z_3) \in R_X(t)\} \le \mathbb{P}\{(Z_1, Z_2, Z_3) \in R_Y(t)\}$, i.e., $\mathbb{P}\{G(\mathbf{X}) \le t\} \le \mathbb{P}\{G(\mathbf{Y}) \le t\}$ for any $t \ge 0$, which implies that $G(\mathbf{X}) \ge_{st} G(\mathbf{Y})$. □

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