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Some new notions of dependence with applications in optimal allocation problems

Jun Cai [a,*], Wei Wei [b]

[a] Department of Statistics and Actuarial Science, University of Waterloo, Waterloo, Ontario, Canada N2L 3G1

[b] Department of Mathematical Sciences, University of Wisconsin-Milwaukee, Milwaukee, WI, 53211, USA

HIGHLIGHTS

- We propose new dependence notions of RWSAI, COUAI, and UOAI.
- We develop the properties and relationships of these dependence notions.
- We show how these notions of dependence can be constructed by copulas.
- We discuss the applications of these dependence notions in insurance.

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ABSTRACT

Dependence structures of multiple risks play an important role in optimal allocation problems for insurance, quantitative risk management, and finance. However, in many existing studies on these problems, risks or losses are often assumed to be independent or comonotonic or exchangeable. In this paper, we propose several new notions of dependence to model dependent risks and give their characterizations through the probability measures or distributions of the risks or through the expectations of the transformed risks. These characterizations are related to the properties of arrangement increasing functions and the proposed notions of dependence incorporate many typical dependence structures studied in the literature for optimal allocation problems. We also develop the properties of these dependence structures. We illustrate the applications of these notions in the optimal allocation problems of deductibles and policy limits and in capital reserves problems. These applications extend many existing researches to more general dependent risks.

1. Introduction

Optimal allocation problems appear in many fields such as insurance, quantitative risk management, finance, and so on. In the study of insurance, Cheung (2007) considered some interesting questions and models for optimal allocations of deductibles and policy limits. The models and questions of Cheung (2007) have been further generalized and studied in Hua and Cheung (2008), Zhuang et al. (2009), Hu and Wang (2010), Lu and Meng (2011), Li and You (2012), and references therein. The questions and models for optimal allocations of deductibles and policy limits can be formulated as follows.

Let $X_1, \ldots, X_n$ be n losses/risks to be incurred by a policyholder in his n policies and T_i be the occurrence time of loss X_i. The losses vector $(X_1, \ldots, X_n)$ is assumed to be independent of the occurrence times vector $(T_1, \ldots, T_n)$. Through an insurance arrangement of deductibles (policy limits) with an insurer, the policyholder is granted a total deductible (limit) of $d > 0$ over the n policies and the policyholder is allowed to allocate an arbitrary deductible (limit) of d_i with $0 \le d_i \le d$ on risk X_i. If $d_1, \ldots, d_n$ are the allocated deductibles (limits), then $d_i \ge 0$ for all $i = 1, \ldots, n$ and $d_1 + \cdots + d_n = d$. Denote all the admissible allocations of deductibles or limits by D_n, namely, $D_n = \{(d_1, \ldots, d_n) | d_1 + \cdots + d_n = d, d_i \ge 0, i = 1, \ldots, n\}$. Thus, with the insurance arrangements

* Corresponding author.
E-mail address: jcai@uwaterloo.ca (J. Cai).

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of deductibles and policy limits, the total discounted retained loss of the policyholder is $\sum_{i=1}^{n} e^{-\delta T_i}(X_i \wedge d_i)$ and $\sum_{i=1}^{n} e^{-\delta T_i}(X_i - d_i)_+$, respectively, where $\delta \geq 0$ is the force of interest. One interesting question is what the optimal deductibles or limits $(d_1^*, \ldots, d_n^*) \in D_n$ are for the policyholder. The policyholder may choose the optimal deductibles or limits $(d_1^*, \ldots, d_n^*)$ to maximize his expected utility of the discounted wealth, namely

$$\max_{(d_1,\ldots,d_n)\in D_n} \mathbb{E}\left[u\left(\omega - \sum_{i=1}^{n} e^{-\delta T_i}(X_i \wedge d_i)\right)\right] \tag{1.1}$$

or

$$\max_{(d_1,\ldots,d_n)\in D_n} \mathbb{E}\left[u\left(\omega - \sum_{i=1}^{n} e^{-\delta T_i}(X_i - d_i)_+\right)\right], \tag{1.2}$$

and the policyholder may also choose the optimal deductibles or limits $(d_1^*, \ldots, d_n^*)$ to minimize his expected discounted total retained loss, namely

$$\min_{(d_1,\ldots,d_n)\in D_n} \mathbb{E}\left[\sum_{i=1}^{n} e^{-\delta T_i}(X_i \wedge d_i)\right] \tag{1.3}$$

or

$$\min_{(d_1,\ldots,d_n)\in D_n} \mathbb{E}\left[\sum_{i=1}^{n} e^{-\delta T_i}(X_i - d_i)_+\right], \tag{1.4}$$

where ω is the initial wealth of the policyholder after premiums are paid, and u is an increasing and/or concave utility function. Note that if $u(x)$ is increasing and/or concave, then $u^*(x) = -u(\omega - x)$ is increasing and/or convex. Therefore, the optimal allocation problems (1.1)–(1.4) are reduced to the following two types of optimal allocation problems:

$$\min_{(d_1,\ldots,d_n)\in D_n} \mathbb{E}\left[u\left(\sum_{i=1}^{n} e^{-\delta T_i}(X_i \wedge d_i)\right)\right], \tag{1.5}$$

$$\min_{(d_1,\ldots,d_n)\in D_n} \mathbb{E}\left[u\left(\sum_{i=1}^{n} e^{-\delta T_i}(X_i - d_i)_+\right)\right], \tag{1.6}$$

where u is an increasing and/or convex function.

In the existing study of the optimal allocation problems (1.5) and (1.6) such as Cheung (2007), Hua and Cheung (2008), Zhuang et al. (2009), Hu and Wang (2010), Lu and Meng (2011), and references therein, the losses $X_1, \ldots, X_n$ were often assumed to have the following independent or comonotonic structures: (i) $X_1, \ldots, X_n$ are mutually independent and $X_1 \leq_{hr} \cdots \leq_{hr} X_n$ or $X_1 \leq_{lr} \cdots \leq_{lr} X_n$, and (ii) $X_1, \ldots, X_n$ are comonotonic and $X_1 \leq_{st} \cdots \leq_{st} X_n$, where the stochastic orders of $\leq_{st}$, $\leq_{hr}$, and $\leq_{lr}$ are defined in Section 2. The similar assumptions on the losses occurrence times $T_1, \ldots, T_n$ were made as well. Recently, Li and You (2012) have studied the optimal allocation problems (1.5) and (1.6) under the same comonotonicity assumptions on the losses $X_1, \ldots, X_n$ as before but a dependence assumption on the losses occurrence times $T_1, \ldots, T_n$. They assumed that $(T_1, \ldots, T_n)$ is linked by a certain Archimedean copula, which implies that the discounted vector $(e^{-\delta T_1}, \ldots, e^{-\delta T_n})$ has an arrangement increasing joint density function, where the definition of an arrangement function will be given in Section 2. These special dependence structures, together with exchangeable losses, are also assumed in other optimal allocation problems such as Cheung and Yang (2004). These restrictions on dependence structures for losses or risks motivate us to consider more general dependence structures.

To determine an optimal allocation, we use the criterion of minimizing the traditional convex risk measure in this paper. For a comprehensive review of other criteria for capital allocations, readers are referred to Dhaene et al. (2012). Recent applications of these allocation principles can be seen in Cheung et al. (2013) and Zaks and Tsanakas (2013).

This paper aims to develop more general dependence structures and to study their applications in optimal allocation problems with dependent risks. The rest of the paper is organized as follows. In Section 2, we present some preliminaries on arrangement increasing functions and stochastic orders. In Section 3, we define the dependence notions of SAI and RWSAI and develop the properties and equivalent characterizations of the two notions. In Section 4, we introduce the dependence notions of UOAI and CUOAI and derive their properties. We conclude that these notions have the implications of SAI $\Longrightarrow$ RWSAI $\Longrightarrow$ CUOAI $\Longrightarrow$ UOAI. In Section 5, we present the properties of marginal distributions of the random vectors with the dependence structures proposed in this paper. We also show how to construct dependent random vectors with these dependence structures through copulas. As applications of these notions of dependence, in Section 6, we consider the optimal allocation problems (1.5) and (1.6) with dependent losses and dependent losses occurrence times. These applications generalize the studies of Cheung (2007), Zhuang et al. (2009), Li and You (2012) to more general dependent risks. Many of their results are special cases of these applications. We also give an application in the allocation problem of capital reserves with RWSAI dependent risks. Section 7 gives some concluding remarks.

2. Preliminaries

In this section, we recall the concept of arrangement increasing functions and the definitions of some stochastic orders, which will be used in this paper.

Throughout the paper, we refer an n-dimensional real-valued vector $(x_1, \ldots, x_n)$ as $\mathbf{x}$ and an n-dimensional random vector $(X_1, \ldots, X_n)$ as $\mathbf{X}$. Accordingly, $\mathbf{X} > (<)\, \mathbf{x}$ means $X_i > (<)\, x_i$ for all $i = 1, \ldots, n$. For any set $K = \{i_1, \ldots, i_k\} \subset \{1, \ldots, n\}$ where $1 \leq i_1 < \cdots < i_k \leq n$ and $k = 1, \ldots, n$, we denote $\mathbf{x}_K = (x_{i_1}, \ldots, x_{i_k})$ and $\mathbf{X}_K = (X_{i_1}, \ldots, X_{i_k})$. For the sake of convenience, we refer the vector $\mathbf{x}$ as $(\mathbf{x}_K, \mathbf{x}_{\bar{K}})$ and $\mathbf{X}$ as $(\mathbf{X}_K, \mathbf{X}_{\bar{K}})$, where $\bar{K} = \{1, \ldots, n\} \setminus K$ is the complement of the set K. In particular, for any $1 \leq i < j \leq n$, if $K = \{i, j\}$, we write $\overline{ij} = \{1, \ldots, n\} \setminus \{i, j\}$, $\mathbf{X}_K = \mathbf{X}_{ij}$, $\mathbf{X}_{\bar{K}} = \mathbf{X}_{\overline{ij}}$, $\mathbf{x}_K = \mathbf{x}_{ij}$, and $\mathbf{x}_{\bar{K}} = \mathbf{x}_{\overline{ij}}$. For example, $\mathbf{X}_{12} = (X_1, X_2)$ and $\mathbf{X}_{\overline{12}} = (X_3, \ldots, X_n)$.

Let $\pi = (\pi(1), \ldots, \pi(n))$ be any permutation of $\{1, \ldots, n\}$, we define $\pi(\mathbf{x}) = (x_{\pi(1)}, \ldots, x_{\pi(n)})$. For any $1 \leq i \neq j \leq n$, we denote the special permutation of transposition by $\pi_{ij} = (\pi_{ij}(1), \ldots, \pi_{ij}(n))$, where $\pi_{ij}(k) = k$ for $k \neq i, j$ and $\pi_{ij}(i) = j$, $\pi(j) = i$.

Furthermore, throughout the paper, 'increasing (decreasing)' means 'non-decreasing (non-increasing)'; all random variables are defined on the common probability space $(\Omega, \mathcal{F}, \mathbb{P})$; expectations under $\mathbb{P}$ are assumed to be finite whenever we write them; the notation of '$\leq_{a.s.} (\geq_{a.s.})$' means the inequality '$\leq$ $(\geq)$' holds almost surely on the common probability space $(\Omega, \mathcal{F}, \mathbb{P})$; and the notation of '$=^d$' means the equality holds in distribution. In addition, we denote the supports of a random variable X and a random vector $\mathbf{X}$ by $S(X)$ and $S(\mathbf{X})$, respectively.

Definition 2.1. A multivariate function $f(\mathbf{x}) = f(x_1, \ldots, x_n)$ is said to be arrangement increasing (AI) if $f(\mathbf{x}) \geq f(\pi_{ij}(\mathbf{x}))$ for any $\mathbf{x} \in \mathbb{R}^n$ and any $1 \leq i < j \leq n$ such that $x_i \leq x_j$. □

Note that a multivariate function $f(\mathbf{x}) = f(x_1, \ldots, x_n)$ is arrangement increasing if and only if $(x_i - x_j)[f(x_1, \ldots, x_i, \ldots, x_j, \ldots, x_n) - f(x_1, \ldots, x_j, \ldots, x_i, \ldots, x_n)] \leq 0$ for any $1 \leq i < j \leq n$ and that the arrangement increasing property is preserved under marginalization as stated below. More properties about arrangement increasing functions can be found in Marshall and Oklin (1979).

Remark 2.2. If $f(\mathbf{x}) = f(x_1, \ldots, x_n)$ is arrangement increasing, then $g(\mathbf{x}_K) := f(\mathbf{x}_K, \mathbf{x}_{\bar{K}})$ with $|K| \geq 2$ is arrangement increasing for any fixed $\mathbf{x}_{\bar{K}} \in \mathbb{R}^{n-|K|}$. □

In the following, we recall the definitions of some stochastic orders, which will be used in this paper. A detailed study on these orders and other stochastic orders is given by Shaked and Shanthikumar (2007). Applications of stochastic orders in insurance can be found in Denuit et al. (2006).

Definition 2.3. Let X and Y be two random variables with survival functions $\bar{F}_X(x) = \mathbb{P}\{X > x\}$ and $\bar{F}_Y(y) = \mathbb{P}\{Y > y\}$, and distribution functions $F_X(x) = \mathbb{P}\{X \leq x\}$ and $F_Y(y) = \mathbb{P}\{Y \leq y\}$.

(i) We say that X is smaller than Y in *usual stochastic order*, denoted as $X \leq_{st} Y$, if $\bar{F}_X(x) \leq \bar{F}_Y(x)$ for all $x \in \mathbb{R}$.

(ii) We say that X is smaller than Y in *hazard rate order*, denoted as $X \leq_{hr} Y$, if $\bar{F}_Y(x)/\bar{F}_X(x)$ is increasing in $x \in \{x : \bar{F}_X(x) > 0\}$ or equivalently $X \leq_{hr} Y$ if $\bar{F}_Y(x)\,\bar{F}_X(y) \leq \bar{F}_Y(y)\,\bar{F}_X(x)$ for all $-\infty < x < y < \infty$.

(iii) We say that X is smaller than Y in *reverse hazard rate order*, denoted as $X \leq_{rh} Y$, if $F_Y(x)/F_X(x)$ is increasing in $x \in \{x : F_X(x) > 0\}$.

(iv) We say that X is smaller than Y in *likelihood ratio order*, denoted as $X \leq_{lr} Y$, if

$$\mathbb{P}\{X \in A\}\,\mathbb{P}\{Y \in B\} \geq \mathbb{P}\{X \in B\}\,\mathbb{P}\{Y \in A\}$$

for all $A, B \in \mathcal{B}(\mathbb{R})$ such that $\sup A \leq \inf B$. □

These stochastic orders are defined by their distributions of random variables. Each of the stochastic orders has a functional characterization by the expectations of transformations of random variables.

For any bivariate function $g(x, y)$, denote $\Delta g(x, y) = g(x, y) - g(y, x)$. Define two classes of functions by

$$\mathcal{G}_{lr} = \{g(x, y) : \Delta g(x, y) \geq 0 \text{ for any } y \geq x\},$$
$$\mathcal{G}_{hr} = \{g(x, y) : \Delta g(x, y) \text{ is increasing in } y \text{ for any } y \geq x\}.$$

The following functional characterizations of $\leq_{hr}$, and $\leq_{lr}$ can be found in Shaked and Shanthikumar (2007).

Proposition 2.4. *For two random variables X and Y, let (X^*, Y^*) be the independent copy of (X, Y), namely $X^* =^d X$, $Y^* =^d Y$ and X^* is independent of Y^*, then*

(i) *$X \leq_{hr} Y$ if and only if $\mathbb{E}[g(X^*, Y^*)] \geq \mathbb{E}[g(Y^*, X^*)]$ for all $g \in \mathcal{G}_{hr}$,*

(ii) *$X \leq_{lr} Y$ if and only if $\mathbb{E}[g(X^*, Y^*)] \geq \mathbb{E}[g(Y^*, X^*)]$ for all $g \in \mathcal{G}_{lr}$.* □

Remark 2.5. These orders have the implications of $X \leq_{lr} Y \implies X \leq_{hr} Y (X \leq_{rh} Y) \implies X \leq_{st} Y$. Furthermore, $X \leq_{hr} Y \iff -Y \leq_{rh} -X$ and if X and Y have density functions $f_X(x), f_Y(x)$, then $X \leq_{lr} Y$ if and only if $f_Y(x)/f_X(x)$ is increasing in $x \in \{x : f_X(x) > 0\}$.

3. Dependence notions of SAI and RWSAI

For a random vector $\mathbf{X} = (X_1, \ldots, X_n)$, there are two common ways to define or describe its dependence notion. One way is to characterize the expectations of the transformations of the random vector and the other is to characterize the distribution or probability measure of the random vector. Shanthikumar and Yao (1991) has described the dependence of a bivariate random vector (X, Y) by characterizing the expectation of $g(X, Y)$ with $g \in \mathcal{G}_{lr}$ or $\mathcal{G}_{hr}$.

Motivated by Proposition 2.4 and the work of Shanthikumar and Yao (1991), in this section, we first define two classes of multivariate functions and use them to define the notions of dependence for multi-dimensional random vectors.

For any $1 \leq i < j \leq n$, define

$$\mathcal{G}_{sai}^{ij}(n) = \{g(x_1, \ldots, x_n) : \Delta_{ij} g(x_1, \ldots, x_n) \geq 0 \text{ for any } x_j \geq x_i\},$$
$$\mathcal{G}_{rwsai}^{ij}(n) = \{g(x_1, \ldots, x_n) : \Delta_{ij} g(x_1, \ldots, x_n) \text{ is increasing in } x_j \text{ for any } x_j \geq x_i\},$$

where $\Delta_{ij} g(\mathbf{x}) = g(\mathbf{x}) - g(\pi_{ij}(\mathbf{x}))$ for a multivariate function $g(x_1, \ldots, x_n)$.

It is easy to verify that $\mathcal{G}_{sai}^{ij}(n) \subset \mathcal{G}_{rwsai}^{ij}(n)$ and that $\mathcal{G}_{sai}^{ij}(2) = \mathcal{G}_{lr}$ and $\mathcal{G}_{rwsai}^{ij}(2) = \mathcal{G}_{hr}$. We then revisit the notion of dependence of a random vector considered by Shanthikumar and Yao (1991) using the proposed functional classes $\mathcal{G}_{sai}^{ij}(n)$ and name the notion as the stochastically arrangement increasing (SAI) notion in next subsection. We also give the distributional characterization of SAI and many new properties of SAI. Then, we introduce a weaker notion of dependence, called RWSAI notion, by the functional class $\mathcal{G}_{rwsai}^{ij}(n)$ and consider the properties of RWSAI and its distributional characterization as well.

3.1. Dependence notion of SAI and its properties

Definition 3.1. A random vector $\mathbf{X} = (X_1, \ldots, X_n)$ or its joint distribution is said to be stochastically arrangement increasing (SAI) if $\mathbb{E}[g(\mathbf{X})] \geq \mathbb{E}[g(\pi_{ij}(\mathbf{X}))]$ for any $1 \leq i < j \leq n$ and any $g(x_1, \ldots, x_n) \in \mathcal{G}_{sai}^{ij}(n)$ such that the expectations exist. □

The following Proposition 3.3 shows that the SAI property is preserved under marginalization, conditioning, and increasing transformation. To prove the proposition, we need the following lemma.

Lemma 3.2. *Let $\mathbf{X} = (X_1, \ldots, X_n)$ be a random vector defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and $f(\mathbf{x}) = f(x_1, \ldots, x_n)$ be a multivariate function. If $\mathbb{E}[f(\mathbf{X})\,\mathbb{I}(A)] \leq 0$ for all $A \in \mathcal{F}$, then $f(\mathbf{X}) \leq_{a.s.} 0$.*

Proof. Define $A = \{\omega \in \Omega : f(\mathbf{X}(\omega)) > 0\}$, we want to show that $\mathbb{P}(A) = 0$. Otherwise, assume $\mathbb{P}(A) > 0$. Denote $A_n = \{\omega \in \Omega : f(\mathbf{X}(\omega)) \geq 1/n\}$, then the sets sequence $\{A_n, n = 1, 2, \ldots\}$ is increasing and converges to the set A, therefore $\lim_{n\to\infty} \mathbb{P}(A_n) = \mathbb{P}(A) > 0$. Then there exists $N \in \mathbb{N}^+$ such that $\mathbb{P}(A_N) > 0$, and thus $\mathbb{E}[f(\mathbf{X}(\omega))\,\mathbb{I}\{\omega \in A_N\}] \geq \mathbb{E}\left[\frac{1}{N}\,\mathbb{I}\{\omega \in A_N\}\right] = \frac{1}{N}\mathbb{P}(A_N) > 0$, which contradicts the fact that $\mathbb{E}[f(\mathbf{X})\,\mathbb{I}(A)] \leq 0$ for all $A \in \mathcal{F}$. □

Proposition 3.3. *If random vector $\mathbf{X} = (X_1, \ldots, X_n)$ is SAI, then*

(i) *$\mathbf{X}_K = (X_k, k \in K)$ is SAI for any $K \subseteq \{1, \ldots, n\}$ with $|K| \geq 2$,*

(ii) *the conditional distribution of $\mathbf{X}_K$ given $\mathbf{X}_{\bar{K}} = \mathbf{x}_{\bar{K}}$ or $\mathbf{X}_K|\mathbf{X}_{\bar{K}} = \mathbf{x}_{\bar{K}}$ is SAI for any $K \subset \{1, \ldots, n\}$ with $|K| \geq 2$ and any $\mathbf{x}_{\bar{K}} \in S(\mathbf{X}_{\bar{K}})$,*

(iii) *$(f(X_1), \ldots, f(X_n))$ is SAI for any increasing function $f(x)$.*

Proof. For (i) and (ii), without loss of generality, assume $K = \{1, \ldots, |K|\}$. For any $1 \leq i < j \leq |K|$, consider any function $g(x_1, \ldots, x_{|K|}) : \mathbb{R}^{|K|} \to \mathbb{R}$ such that $g \in \mathcal{G}_{sai}^{ij}(|K|)$. For (i), we define function $h(\mathbf{x}) = h(x_1, \ldots, x_n) \equiv g(x_1, \ldots, x_{|K|})$, then it is easy to verify that $h(\mathbf{x}) \in \mathcal{G}_{sai}^{ij}(n)$ for any $1 \leq i < j \leq |K|$. Since $(X_1, \ldots, X_n)$ is SAI, we have $\mathbb{E}[g(\mathbf{X}_K)] = \mathbb{E}[h(\mathbf{X})] \geq \mathbb{E}[h(\pi_{ij}(\mathbf{X}))] = \mathbb{E}[g(\pi_{ij}(\mathbf{X}_K))]$ for any $1 \leq i < j \leq |K|$, which implies that $\mathbf{X}_K$ is SAI by Definition 3.1. For (ii), we define function $h(\mathbf{x}) = g(x_1, \ldots, x_{|K|})\,\mathbb{I}\{(x_{|K|+1}, \ldots, x_n) \in A\}$, where $A \in \sigma(X_{|K|+1}, \ldots, X_n)$. It is easy to verify that $h(\mathbf{x}) \in \mathcal{G}_{sai}^{ij}(n)$ for any $1 \leq i < j \leq |K|$.

Following the definition of SAI, we have $\mathbb{E}[h(\mathbf{X})] \ge \mathbb{E}[h(\pi_{ij}(\mathbf{X}))]$, namely $\mathbb{E}[g(\mathbf{X}_K)\,\mathbb{I}\{\mathbf{X}_{\bar{K}} \in A\}] \ge \mathbb{E}[g(\pi_{ij}(\mathbf{X}_K))\,\mathbb{I}\{(\mathbf{X}_{\bar{K}}) \in A\}]$, which is equivalent to $\mathbb{E}[\mathbb{E}[g(\mathbf{X}_K)|\mathbf{X}_{\bar{K}}]\mathbb{I}\{\mathbf{X}_{\bar{K}} \in A\}] \ge \mathbb{E}[\mathbb{E}[g(\pi_{ij}(\mathbf{X}_K))|\mathbf{X}_{\bar{K}}]\,\mathbb{I}\{\mathbf{X}_{\bar{K}} \in A\}]$ by the property of conditional expectations. Therefore, by Lemma 3.2, we have $\mathbb{E}[g(\mathbf{X}_K)|\mathbf{X}_{\bar{K}}] \ge_{a.s.} \mathbb{E}[g(\pi_{ij}(\mathbf{X}_K))|\mathbf{X}_{\bar{K}}]$, which means that $\mathbf{X}_K|\mathbf{X}_{\bar{K}} = \mathbf{x}_{\bar{K}}$ is SAI for any $\mathbf{x}_{\bar{K}} \in S(\mathbf{X}_{\bar{K}})$.

(iii) For any $1 \le i < j \le n$, consider any function $g(x_1, \ldots, x_n) \in \mathcal{G}^{ij}_{sai}(n)$. Since $f(x)$ is increasing, we have $h(x_1, \ldots, x_n) = g(f(x_1), \ldots, f(x_n)) \in \mathcal{G}^{ij}_{sai}(n)$. Thus, by the definition of SAI, we have $\mathbb{E}[h(X_1, \ldots, X_n)] \ge \mathbb{E}[h(\pi_{ij}(X_1, \ldots, X_n))]$ or equivalently $\mathbb{E}[g(f(X_1), \ldots, f(X_n))] \ge \mathbb{E}[g(\pi_{ij}(f(X_1), \ldots, f(X_n)))]$, which implies that $(f(X_1), \ldots, f(X_n))$ is SAI. $\square$

Proposition 3.4. *A random vector $\mathbf{X} = (X_1, \ldots, X_n)$ is SAI if and only if the conditional distribution of (X_i, X_j) given $\mathbf{X}_{\overline{ij}} = \mathbf{x}_{\overline{ij}}$ or $(X_i, X_j)|\mathbf{X}_{\overline{ij}} = \mathbf{x}_{\overline{ij}}$ is SAI for any $1 \le i < j \le n$ and any $\mathbf{x}_{\overline{ij}} \in S(\mathbf{X}_{\overline{ij}})$.*

Proof. If $\mathbf{X}$ is SAI, then $(X_i, X_j)|\mathbf{X}_{\overline{ij}} = \mathbf{x}_{\overline{ij}}$ is SAI by Proposition 3.3(ii). Conversely, assume that $(X_i, X_j)|\mathbf{X}_{\overline{ij}} = \mathbf{x}_{\overline{ij}}$ is SAI for any $1 \le i < j \le n$, consider any $1 \le i < j \le n$ and any function $g \in \mathcal{G}^{ij}_{sai}(n)$, we need to show $\mathbb{E}[g(\mathbf{X})] \ge \mathbb{E}[g(\pi_{ij}(\mathbf{X}))]$. Note that for any fixed $\mathbf{x}_{\overline{ij}} \in S(\mathbf{X}_{\overline{ij}})$, $g(x_1, \ldots, x_n) \in \mathcal{G}^{12}_{sai}(2)$ as a bivariate function of (x_i, x_j). Since $(X_i, X_j)|\mathbf{X}_{\overline{ij}} = \mathbf{x}_{\overline{ij}}$ is SAI, we have $\mathbb{E}[g(\mathbf{X})|\mathbf{X}_{\overline{ij}} = \mathbf{x}_{\overline{ij}}] \ge \mathbb{E}[g(\pi_{ij}(\mathbf{X}))|\mathbf{X}_{\overline{ij}} = \mathbf{x}_{\overline{ij}}]$, which implies $\mathbb{E}[g(\mathbf{X})] \ge \mathbb{E}[g(\pi_{ij}(\mathbf{X}))]$ by taking expectation on both sides of the above inequality. $\square$

The following theorem gives the distributional characterizations of a bivariate SAI random vector.

Theorem 3.5. *For a bivariate random vector (X, Y), the following statements are equivalent.*

(i) *(X, Y) is SAI.*
(ii) *$\mathbb{P}\{(X, Y) \in I \times J\} \ge \mathbb{P}\{(X, Y) \in J \times I\}$ for all measurable sets $I, J \subset \mathbb{R}$ such that $\sup I \le \inf J$.*
(iii) *$\mathbb{P}\{(X, Y) \in A\} \ge \mathbb{P}\{(Y, X) \in A\}$ for any measurable $A \subseteq \{(x, y)|x \le y\}$.*

Proof. The proof of (i) $\Longrightarrow$ (iii) is obvious since $h(x, y) = \mathbb{I}\{(x, y) \in A\}$ is arrangement increasing for any measurable $A \subset \{(x, y)|x \le y\}$.

The proof of (iii) $\Longrightarrow$ (ii) is straightforward since $I \times J \subseteq \{(x, y)|x \le y\}$ for all measurable sets $I, J \subset \mathbb{R}$ such that $\sup I \le \inf J$.

For the proof of (ii) $\Longrightarrow$ (i), consider any arrangement increasing function $g(x, y)$, we first assume $g(x, y) \ge 0$. For positive integer n, define

$$\mathcal{A}_n = \left\{\left[\frac{i}{2^n}, \frac{i+1}{2^n}\right) \times \left[\frac{j}{2^n}, \frac{j+1}{2^n}\right), i, j \in \mathbb{Z} \text{ and } -n2^n \le j < i \le n2^n - 1\right\}$$

and $g_n(x, y) = \sum_{A \in \mathcal{A}_n} \inf_{(s,t)\in A} g(s, t) \times \mathbb{I}\{(x, y) \in A\}$, where the infimum $\inf_{(x,y)\in A} g(x, y)$ always exists since $g(x, y) \ge 0$. It is easy to see that $\{g_n(x, y)\}$ is an increasing series and converges to $g(x, y) \times \mathbb{I}\{x > y\}$ as $n \to \infty$. Therefore, by the monotone convergence theorem, we have $\mathbb{E}[g(X, Y)\,\mathbb{I}\{X > Y\}] = \lim_{n\to\infty} \mathbb{E}[g_n(X, Y)]$.

For any set $A \subset \mathbb{R}^2$, define its symmetric set as $A^s = \{(y, x)|(x, y) \in A\}$. Furthermore, define $\mathcal{B}_n = \{A^s|A \in \mathcal{A}_n\}$ and $h_n(x, y) = \sum_{B\in\mathcal{B}_n} \inf_{(s,t)\in B} g(s, t) \times \mathbb{I}\{(x, y) \in B\}$, then $\mathbb{E}[g(X, Y)\,\mathbb{I}\{X < Y\}] = \lim_{n\to\infty} \mathbb{E}[h_n(X, Y)]$. Therefore,

$$\mathbb{E}[g(X, Y)] = \lim_{n\to\infty} \mathbb{E}[g_n(X, Y) + h_n(X, Y)] + \mathbb{E}[g(X, X)\,\mathbb{I}\{X = Y\}], \tag{3.1}$$

$$\mathbb{E}[g(Y, X)] = \lim_{n\to\infty} \mathbb{E}[g_n(Y, X) + h_n(Y, X)] + \mathbb{E}[g(X, X)\,\mathbb{I}\{X = Y\}]. \tag{3.2}$$

In order to show $\mathbb{E}[g(X, Y)] \ge \mathbb{E}[g(Y, X)]$, it is sufficient to show that for all $n \ge 1$,

$$\mathbb{E}[g_n(X, Y) + h_n(X, Y)] \ge \mathbb{E}[g_n(Y, X) + h_n(Y, X)]. \tag{3.3}$$

We have

$$\mathbb{E}[g_n(X, Y)] = \sum_{A\in\mathcal{A}_n} \inf_{(x,y)\in A} g(x, y) \times \mathbb{P}\{(X, Y) \in A\};$$

$$\mathbb{E}[h_n(X, Y)] = \sum_{A\in\mathcal{A}_n} \inf_{(x,y)\in A^s} g(x, y) \times \mathbb{P}\{(X, Y) \in A^s\};$$

$$\mathbb{E}[g_n(Y, X)] = \sum_{A\in\mathcal{A}_n} \inf_{(x,y)\in A} g(x, y) \times \mathbb{P}\{(X, Y) \in A^s\};$$

$$\mathbb{E}[h_n(Y, X)] = \sum_{A\in\mathcal{A}_n} \inf_{(x,y)\in A^s} g(x, y) \times \mathbb{P}\{(X, Y) \in A\}.$$

Note that for any $A \in \mathcal{A}_n$, A has the form of $J \times I$ with $I, J \subset \mathbb{R}$ and $\sup I \le \inf J$, and its symmetric set A^s has the form of $I \times J$. By (ii), we have $\mathbb{P}\{(X, Y) \in A\} \le \mathbb{P}\{(X, Y) \in A^s\}$ for any $A \in \mathcal{A}_n$. Recall that $g(x, y) \le g(y, x)$ since g is arrangement increasing, then $\inf_{(x,y)\in A} g(x, y) \le \inf_{(x,y)\in A} g(y, x) = \inf_{(x,y)\in A^s} g(x, y)$. Thus, recall that the inequality $ab + cd \ge ad + bc$ holds for any constants a, b, c, d such that $a \le c$ and $b \le d$, we have

$$\begin{aligned} &\inf_{(x,y)\in A} g(x, y) \times \mathbb{P}\{(X, Y) \in A\} + \inf_{(x,y)\in A^s} g(x, y) \times \mathbb{P}\{(X, Y) \in A^s\} \\ &\quad \ge \inf_{(x,y)\in A} g(x, y) \times \mathbb{P}\{(X, Y) \in A^s\} \\ &\qquad + \inf_{(x,y)\in A^s} g(x, y) \times \mathbb{P}\{(X, Y) \in A\}, \end{aligned}$$

which implies (3.3) immediately.

In the case that $g(x, y) \le 0$, define $f(x, y) = -g(y, x)$, then $f(x, y)$ is nonnegative and arrangement increasing. According to the above conclusion, we have $\mathbb{E}[f(X, Y)] \ge \mathbb{E}[f(Y, X)]$, i.e. $\mathbb{E}[g(X, Y)] \ge \mathbb{E}[g(Y, X)]$.

For a general arrangement increasing function $g(x, y)$, denote $g_+(x, y) = \max\{g(x, y), 0\}$ and $g_-(x, y) = \min\{g(x, y), 0\}$. Then $g_+(x, y)$ and $g_-(x, y)$ are both arrangement increasing, and $g(x, y) = g_+(x, y) + g_-(x, y)$. According to the above result, we have $\mathbb{E}[g_+(X, Y)] \ge \mathbb{E}[g_+(Y, X)]$ and $\mathbb{E}[g_-(X, Y)] \ge \mathbb{E}[g_-(Y, X)]$, which imply $\mathbb{E}[g(X, Y)] \ge \mathbb{E}[g(Y, X)]$. $\square$

Based on Proposition 3.4 and Theorem 3.5, we derive a distributional characterization of SAI for a multivariate random vector.

Theorem 3.6. *A random vector $\mathbf{X} = (X_1, \ldots, X_n)$ is SAI if and only if*

$$\mathbb{P}\{\mathbf{X} \in A\} \ge \mathbb{P}\{\pi_{ij}(\mathbf{X}) \in A\}, \tag{3.4}$$

for any $1 \le i < j \le n$ and any set $A \subset \{(x_1, \ldots, x_n)|x_i \le x_j\}$.

Proof. The proof of "$\Longrightarrow$" is straightforward by noting that the function $h(\mathbf{x}) = \mathbb{I}\{\mathbf{x} \in A\} \in \mathcal{G}^{ij}_{sai}(n)$ for any $1 \le i < j \le n$.

For the proof of "$\Longleftarrow$", assume (3.4) holds. First let $i = 1$ and $j = 2$, consider any set $A_{12} \subset \{(x_1, x_2)|x_1 \le x_2\}$ and any set $A_{\overline{12}} \subset \mathbb{R}^{n-2}$. Then $A = A_{12} \times A_{\overline{12}} \subset \{(x_1, \ldots, x_n)|x_1 \le x_2\}$. By (3.4), we have $\mathbb{P}\{(X_1, X_2) \in A_{12}, \mathbf{X}_{\overline{12}} \in A_{\overline{12}}\} \ge \mathbb{P}\{(X_2, X_1) \in A_{12}, \mathbf{X}_{\overline{12}} \in A_{\overline{12}}\}$, which can be rewritten as

$$\begin{aligned} &\mathbb{E}[\mathbb{E}[\mathbb{I}\{(X_1, X_2) \in A_{12}\}|\mathbf{X}_{\overline{12}}] \times \mathbb{I}\{\mathbf{X}_{\overline{12}} \in A_{\overline{12}}\}] \\ &\quad \ge \mathbb{E}[\mathbb{E}[\mathbb{I}\{(X_2, X_1) \in A_{12}\}|\mathbf{X}_{\overline{12}}] \times \mathbb{I}\{\mathbf{X}_{\overline{12}} \in A_{\overline{12}}\}]. \end{aligned}$$

Thus, combining this inequality with Lemma 3.2, we have $\mathbb{E}[\mathbb{I}\{(X_i, X_j) \in A_{12}\}|\mathbf{X}_{\overline{12}}] \le_{a.s.} \mathbb{E}[\mathbb{I}\{(X_j, X_i) \in A_{12}\}|\mathbf{X}_{\overline{12}}]$. Then, $(X_1, X_2)|\mathbf{X}_{\overline{12}} = \mathbf{x}_{\overline{12}}$ is SAI according to Theorem 3.5(iii). Similarly, we can prove that $(X_i, X_j)|\mathbf{X}_{\overline{ij}} = \mathbf{x}_{\overline{ij}}$ is SAI for any $1 \le i < j \le n$ and any $\mathbf{x}_{\overline{ij}} \in S(\mathbf{X}_{\overline{ij}})$. Therefore $\mathbf{X}$ is SAI by Proposition 3.4. $\square$

Remark 3.7. Assume that $\mathbf{X}$ has a joint density function $f(\mathbf{x}) = f(x_1, \ldots, x_n)$, we can prove that $\mathbf{X}$ is SAI if and only if $f(\mathbf{x})$ is arrangement increasing. Indeed, this fact has been pointed out by Shanthikumar and Yao (1991).

3.2. Dependence notion of RWSAI and its properties

Note that $\mathcal{G}^{ij}_{sai}(n) \subset \mathcal{G}^{ij}_{rwsai}(n)$. It is natural to define a weaker notion of dependence than SAI by the functional class $\mathcal{G}^{ij}_{rwsai}$.

Definition 3.8. Random vector $\mathbf{X} = (X_1, \ldots, X_n)$ or its joint distribution is said to be *weakly stochastic arrangement increasing through right tail probability* (RWSAI) if $\mathbb{E}[g(\mathbf{X})] \geq \mathbb{E}[g(\pi_{ij}(\mathbf{X}))]$ for any $1 \leq i < j \leq n$ and any $g(x_1, \ldots, x_n) \in \mathcal{G}^{ij}_{rwsai}(n)$ such that the expectations exist. □

It follows from $\mathcal{G}^{ij}_{sai}(n) \subset \mathcal{G}^{ij}_{rwsai}(n)$ that SAI $\Longrightarrow$ RWSAI. Furthermore, by the definition of RWSAI and the same arguments used in the proofs for Propositions 3.3 and 3.4, we can show that the RWSAI property is preserved under marginalization, conditioning, increasing transformation and that a random vector is RWSAI if and only if the joint conditional distribution of any pair of the random vector conditioning on the rest of the random vector is RWSAI. We omit the proofs of the following two propositions since their proofs are similar to the proofs of Propositions 3.3 and 3.4.

Proposition 3.9. *If random vector $\mathbf{X} = (X_1, \ldots, X_n)$ is RWSAI, then*

(i) *$\mathbf{X}_K = (X_k, k \in K)$ is RWSAI for any $K \subseteq \{1, \ldots, n\}$ with $|K| \geq 2$,*
(ii) *the conditional distribution of $\mathbf{X}_K$ given $\mathbf{X}_{\bar{K}} = \mathbf{x}_{\bar{K}}$ or $\mathbf{X}_K | \mathbf{X}_{\bar{K}} = \mathbf{x}_{\bar{K}}$ is RWSAI for any $K \subseteq \{1, \ldots, n\}$ with $|K| \geq 2$ and any $\mathbf{x}_{\bar{K}} \in S(\mathbf{X}_{\bar{K}})$,*
(iii) *$(f(X_1), \ldots, f(X_n))$ is RWSAI for any increasing function $f(x)$.* □

Proposition 3.10. *A random vector $\mathbf{X} = (X_1, \ldots, X_n)$ is RWSAI if and only if the conditional distribution of (X_i, X_j) given $\mathbf{X}_{\overline{ij}} = \mathbf{x}_{\overline{ij}}$ or $(X_i, X_j)|\mathbf{X}_{\overline{ij}} = \mathbf{x}_{\overline{ij}}$ is RWSAI for any $1 \leq i < j \leq n$ and any $\mathbf{x}_{\overline{ij}} \in S(\mathbf{X}_{\overline{ij}})$.* □

Now, we consider the distributional characterization of RWSAI. In doing so, we give the following definition.

Definition 3.11. A subset A of $\mathbb{R}^n$ is said to be an *upper set* if the indicator function $\mathbb{I}\{(x_1, \ldots, x_n) \in A\}$ is increasing in $x_1, \ldots, x_n$ and is said to be a *partial upper set* with respect to a non-empty subset $K \subset \{1, \ldots, n\}$, or to be a K-upper set, if $\mathbb{I}\{(x_1, \ldots, x_n) \in A\}$ is increasing in x_k for all $k \in K$. □

Theorem 3.12. *A bivariate random vector (X, Y) is RWSAI if and only if*

$$\mathbb{P}\{(X, Y) \in A\} \geq \mathbb{P}\{(Y, X) \in A\} \tag{3.5}$$

for all $\{2\}$-upper set $A \subset \{(x, y)|x \leq y\}$.

Proof. First, by the definition of RWSAI and the fact $\mathcal{G}^{ij}_{rwsai}(2) = \mathcal{G}_{hr}$, we have that (X, Y) is RWSAI if and only if $\mathbb{E}[g(X, Y)] \geq \mathbb{E}[g(Y, X)]$ for all $g(x, y) \in \mathcal{G}_{hr}$. Thus, assume that (X, Y) is RWSAI, for any $\{2\}$-upper set $A \subset \{(x, y)|x \leq y\}$, consider the indicator function $h(x, y) = \mathbb{I}\{(x, y) \in A\}$. Obviously, $h(x, y)$ in increasing in y since A is a $\{2\}$-upper set. Note that $\Delta h(x, y) = h(x, y)$ if $y > x$, and $\Delta h(x, y) = 0$ if $y = x$, then $\Delta h(x, y) = h(x, y)$ is increasing in $y \geq x$. Therefore $h(x, y) \in \mathcal{G}_{hr}$, which implies that $\mathbb{P}\{(X, Y) \in A\} = \mathbb{E}[h(X, Y)] \geq \mathbb{E}[h(Y, X)] = \mathbb{P}\{(Y, X) \in A\}$.

Conversely, consider any function $g(x, y) \in \mathcal{G}_{hr}$. Note that $\Delta g(y, x) = -\Delta g(x, y)$ and $\Delta g(x, y) = 0$ if $x = y$, we have

$$\begin{aligned}\mathbb{E}[\Delta g(X, Y)] &= \mathbb{E}[\Delta g(X, Y)\,\mathbb{I}\{Y \geq X\}] + \mathbb{E}[\Delta g(X, Y)\,\mathbb{I}\{Y \leq X\}] \\ &= \mathbb{E}[\Delta g(X, Y)\,\mathbb{I}\{Y \geq X\}] - \mathbb{E}[\Delta g(Y, X)\,\mathbb{I}\{X \geq Y\}].\end{aligned} \tag{3.6}$$

Since $\Delta g(X, Y)\,\mathbb{I}\{Y \geq X\} \geq 0$, we have

$$\begin{aligned}\mathbb{E}[\Delta g(X, Y)\,\mathbb{I}\{Y \geq X\}] &= \int_0^\infty \mathbb{P}\{\Delta g(X, Y)\,\mathbb{I}\{Y \geq X\} > z\}dz \\ &= \int_0^\infty \mathbb{P}\{\Delta g(X, Y) > z,\ Y \geq X\}dz \\ &= \int_0^\infty \mathbb{P}\{(X, Y) \in A_z\}dz,\end{aligned} \tag{3.7}$$

where $A_z = \{(x, y)|\Delta g(x, y) > z, y \geq x\}$. Similarly,

$$\mathbb{E}[\Delta g(Y, X)\,\mathbb{I}\{X \geq Y\}] = \int_0^\infty \mathbb{P}\{(Y, X) \in A_z\}dz. \tag{3.8}$$

Recall that $g(x, y) \in \mathcal{G}_{hr}$, it is easy to verify that $A_z \subset \{(x, y)|x \leq y\}$ and A_z is a $\{2\}$-upper set for any fixed $z \geq 0$, thus $\mathbb{P}\{(X, Y) \in A_z\} \geq \mathbb{P}\{(Y, X) \in A_z\}$ for any $z \geq 0$. Combining with (3.7) and (3.8), we have $\mathbb{E}[\Delta g(X, Y)\,\mathbb{I}\{Y \geq X\}] \geq \mathbb{E}[\Delta g(Y, X)\,\mathbb{I}\{X \geq Y\}]$, which implies $\mathbb{E}[g(X, Y)] \geq \mathbb{E}[g(Y, X)]$ according to (3.6) and thus (X, Y) is RWSAI. □

For general multivariate random vectors, we have the following distributional characterization for RWSAI.

Theorem 3.13. *A random vector $\mathbf{X} = (X_1, \ldots, X_n)$ is RWSAI if and only if*

$$\mathbb{P}\{\mathbf{X} \in A\} \geq \mathbb{P}\{\pi_{ij}(\mathbf{X}) \in A\} \tag{3.9}$$

for any $1 \leq i < j \leq n$ and any $\{j\}$-upper set $A \subset \{(x_1, \ldots, x_n)|x_i \leq x_j\}$.

Proof. Assume that $\mathbf{X} = (X_1, \ldots, X_n)$ is RWSAI. For any $1 \leq i < j \leq n$ and any $\{j\}$-upper set $A \subset \{(x_1, \ldots, x_n)|x_i \leq x_j\}$, define $h(x_1, \ldots, x_n) = \mathbb{I}\{(x_1, \ldots, x_n) \in A\}$. It is easy to verify that $h(x_1, \ldots, x_n) \in \mathcal{G}^{ij}_{rwsai}(n)$. Then $\mathbb{P}\{\mathbf{X} \in A\} = \mathbb{E}[h(\mathbf{X})] \geq \mathbb{E}[h(\pi_{ij}(\mathbf{X}))] = \mathbb{P}\{\pi_{ij}(\mathbf{X}) \in A\}$.

Conversely, assume that (3.9) holds. Following similar arguments used in the proof of Theorem 3.12, we can prove that $\mathbb{E}[g(\mathbf{X})] \geq \mathbb{E}[g(\pi_{ij}(\mathbf{X}))]$ for any $1 \leq i < j \leq n$ and any $g(x_1, \ldots, x_n) \in \mathcal{G}^{ij}_{rwsai}(n)$. □

Notice from Remark 3.7 that if a random vector has a joint density function, then the SAI property can be characterized through its density function. For RWSAI, if a random vector has a joint density function, we can derive a characterization of RWSAI through the partial derivatives of joint conditional survival functions of any pair of the random vector conditioning on the rest of the random vector.

Theorem 3.14. *Assume random vector $(X_1, \ldots, X_n)$ has a joint density function. Then $(X_1, \ldots, X_n)$ is RWSAI if and only if*

$$\begin{aligned}&\frac{\partial}{\partial x_i}\mathbb{P}\{X_i > x_i, X_j > x_j|\mathbf{X}_{\overline{ij}} = \mathbf{x}_{\overline{ij}}\} \\ &\quad \leq \frac{\partial}{\partial x_i}\mathbb{P}\{X_i > x_j, X_j > x_i|\mathbf{X}_{\overline{ij}} = \mathbf{x}_{\overline{ij}}\},\end{aligned} \tag{3.10}$$

for any $1 \leq i < j \leq n$ and $x_i \leq x_j$, and any $\mathbf{x}_{\overline{ij}} \in S(\mathbf{X}_{\overline{ij}})$.

Proof. Assume $(X_1, \ldots, X_n)$ is RWSAI. To prove (3.10), without loss of generality, it is sufficient to show that

$$\frac{\partial}{\partial x_1}\mathbb{P}\{X_1 > x_1, X_2 > x_2|\mathbf{X}_{\overline{12}} = \mathbf{x}_{\overline{12}}\}$$
$$\le \frac{\partial}{\partial x_1}\mathbb{P}\{X_1 > x_2, X_2 > x_1|\mathbf{X}_{\overline{12}} = \mathbf{x}_{\overline{12}}\}, \tag{3.11}$$

for any $x_1 \le x_2$ and any fixed $\mathbf{x}_{\overline{12}} \in S(\mathbf{X}_{\overline{12}})$.

For any $x_1 \le x_2$, define $A_t = (x_1 - t, x_1] \times (x_2, \infty) \times A_{\overline{12}}$, where $t > 0$ and $A_{\overline{12}} \in \sigma(\mathbf{X}_{\overline{12}})$. Then A_t is a $\{2\}$-upper set and $A_t \subset \{(x_1, \ldots, x_n) \in \mathbb{R}^n | x_1 \le x_2\}$. Therefore, $\mathbb{P}\{(X_1, X_2, \ldots, X_n) \in A_t\} \ge \mathbb{P}\{(X_2, X_1, \ldots, X_n) \in A_t\}$ or

$$\mathbb{P}\{x_1 - t < X_1 \le x_1, X_2 > x_2, \mathbf{X}_{\overline{12}} \in A_{\overline{12}}\}$$
$$\ge \mathbb{P}\{x_1 - t < X_2 \le x_1, X_1 > x_2, \mathbf{X}_{\overline{12}} \in A_{\overline{12}}\},$$

which is equivalent to

$$\mathbb{E}[\mathbb{E}[\mathbb{I}\{x_1 - t < X_1 \le x_1, X_2 > x_2\}|\mathbf{X}_{\overline{12}}] \times \mathbb{I}\{\mathbf{X}_{\overline{12}} \in A_{\overline{12}}\}]$$
$$\ge \mathbb{E}[\mathbb{E}[\mathbb{I}\{x_1 - t < X_2 \le x_1, X_1 > x_2\}|\mathbf{X}_{\overline{12}}] \times \mathbb{I}\{\mathbf{X}_{\overline{12}} \in A_{\overline{12}}\}].$$

According to Lemma 3.2, we have

$$\mathbb{E}[\mathbb{I}\{x_1 - t < X_1 \le x_1, X_2 > x_2\}|\mathbf{X}_{\overline{12}}]$$
$$\ge_{a.s.} \mathbb{E}[\mathbb{I}\{x_1 - t < X_2 \le x_1, X_1 > x_2\}|\mathbf{X}_{\overline{12}}]$$

or

$$\mathbb{P}\{x_1 - t < X_1 \le x_1, X_2 > x_2|\mathbf{X}_{\overline{12}} = \mathbf{x}_{\overline{12}}\}$$
$$\ge \mathbb{P}\{x_1 - t < X_2 \le x_1, X_1 > x_2|\mathbf{X}_{\overline{12}} = \mathbf{x}_{\overline{12}}\},$$

for any $\mathbf{x}_{\overline{12}} \in S(\mathbf{X}_{\overline{12}})$.

By dividing $t > 0$ on both sides of the above inequality and letting $t \searrow 0$, we have

$$\frac{\partial}{\partial x_1}\mathbb{P}\{X_1 > x_1, X_2 > x_2|\mathbf{X}_{\overline{12}} = \mathbf{x}_{\overline{12}}\}$$
$$\le \frac{\partial}{\partial x_1}\mathbb{P}\{X_1 > x_2, X_2 > x_1|\mathbf{X}_{\overline{12}} = \mathbf{x}_{\overline{12}}\},$$

which implies (3.11).

Conversely, assume (3.10) holds, we want to show that $(X_1, \ldots, X_n)$ is RWSAI. The proof for the case of $n = 2$ follows from Theorem 3.17 of Shanthikumar and Yao (1991). As for the case of $n \ge 3$, according to the result on the case of $n = 2$, (3.10) implies that the conditional distribution of (X_i, X_j) given $\mathbf{X}_{\overline{ij}} = \mathbf{x}_{\overline{ij}}$ or $(X_i, X_j)|\mathbf{X}_{\overline{ij}} = \mathbf{x}_{\overline{ij}}$ is RWSAI for any $1 \le i < j \le n$ and any fixed $\mathbf{x}_{\overline{ij}} \in S(\mathbf{X}_{\overline{ij}})$, which means that $(X_1, \ldots, X_n)$ is RWSAI according to Proposition 3.10. □

4. Dependence notions of UOAI and CUOAI

In Section 3, we defined the SAI and RWSAI notions of dependence by the functional classes and give their distributional characterizations as well. If random vectors have density functions, the SAI and RWSAI notions of dependence can be characterized by their density functions and the partial derivatives of the conditional survival functions, respectively. However, in general, the density functions or the partial derivatives of conditional survival functions of random vectors may not exist. This fact motivates us to consider if we can define weaker notions of dependence by the joint survival functions or the joint conditional survival functions of random vectors, which always hold for any random vectors. In this section, we define two weaker notions of dependence, called UOAI and CUOAI, through the joint survival functions or the joint conditional survival functions of random vectors.

Definition 4.1. Random vector $\mathbf{X} = (X_1, \ldots, X_n)$ or its joint distribution is said to be upper orthant increasing (UOAI) if its joint survival function $\bar{F}(x_1, \ldots, x_n) = \mathbb{P}\{X_1 > x_1, \ldots, X_n > x_n\}$ is arrangement increasing. □

Definition 4.2. Random vector $\mathbf{X} = (X_1, \ldots, X_n)$ or its joint distribution is said to be conditionally upper orthant arrangement increasing (CUOAI) if the conditional distribution of (X_i, X_j) given $\mathbf{X}_{\overline{ij}} = \mathbf{x}_{\overline{ij}}$ or $(X_i, X_j)|\mathbf{X}_{\overline{ij}} = \mathbf{x}_{\overline{ij}}$ is UOAI for any $1 \le i < j \le n$ and any fixed $\mathbf{x}_{\overline{ij}} \in S(\mathbf{X}_{\overline{ij}})$. □

The following proposition shows the implications of RWSAI $\Longrightarrow$ CUOAI $\Longrightarrow$ UOAI.

Proposition 4.3. (i) *If random vector* $\mathbf{X} = (X_1, \ldots, X_n)$ *is CUOAI, then* $(X_1, \ldots, X_n)$ *is UOAI.* (ii) *If random vector* $\mathbf{X} = (X_1, \ldots, X_n)$ *is RWSAI, then* $(X_1, \ldots, X_n)$ *is CUOAI.*

Proof. (i) From Definition 4.2, for any $1 \le i < j \le n$ and $x_i \le x_j$, we have, for any $\mathbf{x}_{\overline{ij}} \in S(\mathbf{X}_{\overline{ij}})$, $\mathbb{P}\{X_i > x_i, X_j > x_j|\mathbf{X}_{\overline{ij}} = \mathbf{x}_{\overline{ij}}\} \ge \mathbb{P}\{X_i > x_j, X_j > x_i|\mathbf{X}_{\overline{ij}} = \mathbf{x}_{\overline{ij}}\}$, which means $\mathbb{E}[\mathbb{I}\{X_i > x_i, X_j > x_j\}|\mathbf{X}_{\overline{ij}}] \ge_{a.s.} \mathbb{E}[\mathbb{I}\{X_i > x_j, X_j > x_i\}|\mathbf{X}_{\overline{ij}}]$. Therefore,

$$\mathbb{P}\{X_i > x_i, X_j > x_j, \mathbf{X}_{\overline{ij}} > \mathbf{x}_{\overline{ij}}\} = \mathbb{E}[\mathbb{I}\{X_i > x_i, X_j > x_j\} \times \mathbb{I}\{\mathbf{X}_{\overline{ij}} > \mathbf{x}_{\overline{ij}}\}]$$
$$= \mathbb{E}[\mathbb{E}[\mathbb{I}\{X_i > x_i, X_j > x_j\}|\mathbf{X}_{\overline{ij}}] \times \mathbb{I}\{\mathbf{X}_{\overline{ij}} > \mathbf{x}_{\overline{ij}}\}]$$
$$\ge \mathbb{E}[\mathbb{E}[\mathbb{I}\{X_i > x_j, X_j > x_i\}|\mathbf{X}_{\overline{ij}}] \times \mathbb{I}\{\mathbf{X}_{\overline{ij}} > \mathbf{x}_{\overline{ij}}\}]$$
$$= \mathbb{P}\{X_i > x_j, X_j > x_i, \mathbf{X}_{\overline{ij}} > \mathbf{x}_{\overline{ij}}\},$$

which means that $\mathbf{X}$ is UOAI.

(ii) Without loss of generality, it suffices to show that $(X_1, X_2)|\mathbf{X}_{\overline{12}} = \mathbf{x}_{\overline{12}}$ is UOAI for any $\mathbf{x}_{\overline{12}} \in S(\mathbf{X}_{\overline{12}})$, or

$$\mathbb{P}\{X_1 > x_1, X_2 > x_2|\mathbf{X}_{\overline{12}} = \mathbf{x}_{\overline{12}}\}$$
$$\ge \mathbb{P}\{X_1 > x_2, X_2 > x_1|\mathbf{X}_{\overline{12}} = \mathbf{x}_{\overline{12}}\}, \tag{4.1}$$

for any $x_1 \le x_2$ and any $\mathbf{x}_{\overline{12}} \in S(\mathbf{X}_{\overline{12}})$.

For any fixed $x_1 \le x_2$ and any $A_{\overline{12}} \in \sigma(\mathbf{X}_{\overline{12}})$, define function $h(y_1, \ldots, y_n) = \mathbb{I}\{y_1 > x_1, y_2 > x_2\} \times \mathbb{I}\{\mathbf{y}_{\overline{12}} \in A_{\overline{12}}\}$. It is easy to verify that $h(y_1, \ldots, y_n) \in \mathcal{G}^{12}_{rwsai}(n)$. Since $(X_1, \ldots, X_n)$ is RWSAI, we have $\mathbb{E}[h(X_1, X_2, \ldots, X_n)] \ge \mathbb{E}[h(X_2, X_1, \ldots, X_n)]$, or equivalently,

$$\mathbb{E}[\mathbb{I}\{X_1 > x_1, X_2 > x_2\}\,\mathbb{I}\{\mathbf{X}_{\overline{12}} \in A_{\overline{12}}\}]$$
$$\ge \mathbb{E}[\mathbb{I}\{X_1 > x_2, X_2 > x_1\}\,\mathbb{I}\{\mathbf{X}_{\overline{12}} \in A_{\overline{12}}\}],$$

for any $A_{\overline{12}} \in \sigma(\mathbf{X}_{\overline{12}})$. By Lemma 3.2, we have $\mathbb{E}[\mathbb{I}\{X_1 > x_1, X_2 > x_2\}|\mathbf{X}_{\overline{12}}] \ge_{a.s.} \mathbb{E}[\mathbb{I}\{X_1 > x_2, X_2 > x_1\}|\mathbf{X}_{\overline{12}}]$, which implies (4.1). □

This proposition, together with SAI $\Longrightarrow$ RWSAI, means that the SAI, RWSAI, CUOAI, UOAI notions of dependence have the following implications:

$$\text{SAI} \Longrightarrow \text{RWSAI} \Longrightarrow \text{CUOAI} \Longrightarrow \text{UOAI}. \tag{4.2}$$

We point out that all the above implications are strict. In other words, any reverse of these implications does not hold. Here, we give examples of UOAI $\not\Longrightarrow$ CUOAI and CUOAI $\not\Longrightarrow$ RWSAI.

Let (X_1, X_2, X_3) be a discrete random vector with the following joint probability mass function: $\mathbb{P}\{(X_1, X_2, X_3) = (1, 2, 3)\} = p_1$, $\mathbb{P}\{(X_1, X_2, X_3) = (2, 1, 4)\} = p_2$, $\mathbb{P}\{(X_1, X_2, X_3) = (2, 3, 5)\} = p_3$ with $p_1 + p_2 + p_3 = 1$ and $p_1 > p_2$. Then it is easy to verify that (X_1, X_2, X_3) is UOAI. But $(X_1, X_2)|X_3 = 4$ is not UOAI, which means that (X_1, X_2, X_3) is not CUOAI. Furthermore, let (X, Y) be a discrete random vector with the following joint probability mass function: $p_{00} = p_{11} = p_{22} = p_{01} = p_{10} = 0, p_{02} = 0.1, p_{12} = 0.4, p_{20} = 0.2$, and $p_{21} = 0.3$, where $p_{ij} = \mathbb{P}\{X = i, Y = j\}$ for $i, j = 0, 1, 2$. Then, it is easy to verify that (X, Y) is CUOAI but not RWSAI.

It is not hard to show by the definition of UOAI that the UOAI property is preserved under marginalization and increasing transformation. The proof of the following proposition is straightforward but tedious and thus is omitted.

Proposition 4.4. *If random vector* $\mathbf{X} = (X_1, \ldots, X_n)$ *is UOAI, then*

(i) $(X_{i_1}, \ldots, X_{i_k})$ *is UOAI for any* $1 \le i_1 < \cdots < i_k \le n$ *and* $2 \le k \le n$,
(ii) $(f(X_1), \ldots, f(X_n))$ *is UOAI for any increasing function* $f(x)$. □

5. Marginal distributions and constructions of SAI/RWSAI/CUOAI/UOAI random vectors

Due to the arrangement increasing property of joint survival functions of UOAI random vectors, it is expected that marginal distributions of UOAI random vectors can be ordered in some stochastic orders. In this section, we show that marginal distributions of UOAI random vectors can be ordered in the stochastic order of $\le_{st}$, independent SAI random vectors are characterized by the stochastic order of $\le_{lr}$, independent RWSAI/CUOAI/UOAI random vectors are characterized by the stochastic order of $\le_{hr}$, and comonotonic SAI/RWSAI/CUOAI/UOAI random vectors are characterized by the stochastic order of $\le_{st}$. Furthermore, we illustrate that the class of UOAI is a large class of dependent random vectors in the sense that UOAI random vectors can be constructed by a class of copulas.

Proposition 5.1. *If random vector* $(X_1, \ldots, X_n)$ *is UOAI, then* $X_i \le_{st} X_{i+1}$ *for all* $1 \le i \le n-1$.

Proof. We only give the proof for $X_1 \le_{st} X_2$, other cases can be similarly proved. Consider any $x \in \mathbb{R}$, since $(X_1, \ldots, X_n)$ is UOAI, we have $\bar{F}(y, x, x_3, \ldots, x_3) \ge \bar{F}(x, y, x_3, \ldots, x_n)$ for any $y < x$ and any $x_3, \ldots, x_n$. Let $y \to -\infty, x_3 \to -\infty, \ldots, x_n \to -\infty$, we get $\bar{F}_2(x) \ge \bar{F}_1(x)$ or $X_1 \le_{st} X_2$. □

Proposition 5.2. *Assume* $\mathbf{X} = (X_1, \ldots, X_n)$ *is mutually independent. Then* $(X_1, \ldots, X_n)$ *is SAI if and only if* $X_i \le_{lr} X_{i+1}$ *for all* $i = 1, \ldots, n-1$.

Proof. If $(X_1, \ldots, X_n)$ is mutually independent, by Proposition 3.4, we know that $(X_1, \ldots, X_n)$ is SAI $\Longleftrightarrow$ $(X_i, X_j)|\mathbf{X}_{\overline{ij}} = \mathbf{x}_{\overline{ij}}$ is SAI for any $1 \le i < j \le n$ and any $\mathbf{x}_{\overline{ij}} \in S(\mathbf{X}_{\overline{ij}}) \Longleftrightarrow (X_i, X_j)$ is SAI for any $1 \le i < j \le n$, which is equivalent to $X_i \le_{lr} X_j$ for any $1 \le i < j \le n$ by Proposition 2.4(ii). □

Lemma 5.3. *If* X *and* Y *are comonotonic and* $X \le_{st} Y$, *then* $X \le_{a.s.} Y$.

Proof. Since X and Y are comonotonic, for any x, we have $\mathbb{P}\{X \le x, Y \le x\} = \min\{\mathbb{P}\{X \le x\}, \mathbb{P}\{Y \le x\}\}$, which, together with $X \le_{st} Y$, implies that $\mathbb{P}\{X \le x, Y \le x\} = \mathbb{P}\{Y \le x\}$. Therefore, $\mathbb{P}\{X > x, Y \le x\} = 0$ for any x, which means $X \le_{a.s.} Y$. □

Proposition 5.4. *Assume random vector* $(X_1, \ldots, X_n)$ *is mutually independent. Then the following statements are equivalent:* (i) $X_1 \le_{hr} \cdots \le_{hr} X_n$; (ii) $(X_1, \ldots, X_n)$ *is UOAI;* (iii) $(X_1, \ldots, X_n)$ *is CUOAI;* (iv) $(X_1, \ldots, X_n)$ *is RWSAI.*

Proof. "(i) ⇒ (iv)". According to Proposition 3.10, if suffices to show that $(X_i, X_j)|\mathbf{X}_{\overline{ij}} = \mathbf{x}_{\overline{ij}}$ is RWSAI for any $1 \le i < j \le n$ and any $\mathbf{x}_{\overline{ij}} \in S(\mathbf{X}_{\overline{ij}})$, or equivalently (X_i, X_j) is RWSAI for any $1 \le i < j \le n$, which holds from Proposition 2.4(i).

The implications of "(iv) ⇒ (iii) ⇒ (ii)" are obvious from (4.2).

"(ii) ⇒ (i)". From Proposition 4.4, we know that (X_i, X_{i+1}) is UOAI for any $1 \le i \le n-1$. Then, for any $x_i \le x_{i+1}$, we have

$$\begin{aligned}\bar{F}_i(x_i)\bar{F}_{i+1}(x_{i+1}) &= \mathbb{P}\{X_i > x_i, X_{i+1} > x_{i+1}\} \\ &\ge \mathbb{P}\{X_i > x_{i+1}, X_{i+1} > x_i\} = \bar{F}_i(x_{i+1})\bar{F}_{i+1}(x_i),\end{aligned}$$

which means $X_i \le_{hr} X_{i+1}$. □

Proposition 5.5. *Assume random vector* $(X_1, \ldots, X_n)$ *is comonotonic. Then the following statements are equivalent:* (i) $X_1 \le_{st} \cdots \le_{st} X_n$; (ii) $(X_1, \ldots, X_n)$ *is UOAI;* (iii) $(X_1, \ldots, X_n)$ *is CUOAI;* (iv) $(X_1, \ldots, X_n)$ *is RWSAI;* (v) $(X_1, \ldots, X_n)$ *is SAI.*

Proof. The implications of "(v) ⇒ (iv) ⇒ (iii)⇒ (ii)" are obvious from (4.2). "(ii) ⇒ (i)" holds from Proposition 5.1. It suffices to show (i) ⇒ (v).

Since $(X_1, \ldots, X_n)$ is comonotonic, according to Lemma 5.3, $X_1 \le_{st} \cdots \le_{st} X_n$ implies $X_i \le_{a.s.} X_{i+1}$ for $i = 1, \ldots, n-1$. Therefore, we have $g(X_1, \ldots, X_n) \ge_{a.s.} g(\pi_{ij}(X_1, \ldots, X_n))$ for any $1 \le i < j \le n$ and any $g \in \mathcal{G}^{ij}_{sai}(n)$. Taking expectations on both sides, we have $\mathbb{E}[g(X_1, \ldots, X_n)] \ge \mathbb{E}[g(\pi_{ij}(X_1, \ldots, X_n))]$, which means that $(X_1, \ldots, X_n)$ is SAI. □

Propositions 5.4 and 5.5 indicate that the special dependence structures of the losses vector $(X_1, \ldots, X_n)$ studied in Cheung (2007), Zhuang et al. (2009), and Li and You (2012) are the special cases of SAI or RWSAI. Furthermore, it is easy to see from the definition of SAI that an exchangeable random vector is SAI.

At the end of this section, we point out that it is possible to construct continuous UOAI random vectors by a class of copulas. Recall the implications of SAI $\Longrightarrow$ RWSAI $\Longrightarrow$ CUOAI $\Longrightarrow$ UOAI. It is sufficient to construct continuous SAI random vectors by a class of copulas. Indeed, Li and You (2012) have constructed continuous SAI random vectors by the class of Archimedean copulas under some assumptions on Archimedean copulas. In the following Theorem 5.7, under weaker assumptions on Archimedean copulas, we construct continuous RWSAI random vectors. We point out that under more general copulas with certain conditions, we can construct continuous UOAI random vectors. Here, we omit the detailed discussions about how to construct continuous UOAI random vectors under more general copulas.

An Archimedean copula function is defined by

$$C(u_1, \ldots, u_n) = \Lambda\left(\sum_{k=1}^{n} \Psi(u_k)\right), \quad u_1, \ldots, u_n \in [0, 1], \tag{5.1}$$

where $\Psi : (0, 1] \to [0, \infty)$ is invertible and satisfies that (i) $\Psi(1) = 0, \lim_{x \downarrow 0} \Psi(x) = \infty$, and (ii) $\Lambda(x) = \Psi^{-1}(x)$ is completely monotonic, namely, $(-1)^k \Lambda^{(k)}(x) = (-1)^k \frac{d^k}{dx^k}\Lambda(x) \ge 0$ for all $k = 0, 1, \ldots$.

Remark 5.6. From the complete monotonicity of $\Lambda(x)$, one concludes that $(-1)^k \Lambda^{(k)}(x)$ is decreasing for all $k = 1, 2, \ldots$, since $(-1)^k \Lambda^{(k+1)}(x) = -(-1)^{k+1}\Lambda^{(k+1)}(x) \le 0$. □

Theorem 5.7. *Assume random vector* $(X_1, \ldots, X_n)$ *has a positive joint density function and* $X_1 \le_{hr} \cdots \le_{hr} X_n$. *If the joint survival function of* $(X_1, \ldots, X_n)$ *is linked by an Archimedean copula* $C(u_1, \ldots, u_n) = \Psi^{-1}(\sum_{k=1}^{n} \Psi(u_k))$ *and* $x\Psi'(x)$ *is increasing in* $x \in [0, 1]$, *then* $(X_1, \ldots, X_n)$ *is RWSAI.*

Proof. By Theorem 3.13, without loss of generality, it suffices to show that

$$\begin{aligned}&\frac{\partial}{\partial x_1}\mathbb{P}\{X_1 > x_1, X_2 > x_2|\mathbf{X}_{\overline{12}} = \mathbf{x}_{\overline{12}}\} \\ &\quad \le \frac{\partial}{\partial x_1}\mathbb{P}\{X_1 > x_2, X_2 > x_1|\mathbf{X}_{\overline{12}} = \mathbf{x}_{\overline{12}}\},\end{aligned} \tag{5.2}$$

for any $x_1 \le x_2$.

Let $\bar{F}(x_1, \ldots, x_n)$ be the joint survival function of $(X_1, \ldots, X_n)$, $f_{(X_3,\ldots,X_n)}(x_3, \ldots, x_n)$ be the joint density function of $(X_3, \ldots, X_n)$, and $f_k(x_k)$ be the marginal density function of X_k. Then $\bar{F}(x_1, \ldots, x_n)$

$= \Psi^{-1}\left(\sum_{k=1}^{n}\Psi(\bar{F}_k(x_k))\right)$ and $\sum_{k=1}^{n}\Psi(\bar{F}_k(x_k)) = \Psi(\bar{F}(x_1,\ldots,x_n))$. Note that

$$f(x_1,\ldots,x_n) = (-1)^n \frac{\partial^n}{\partial x_1\ldots\partial x_n}\bar{F}(x_1,\ldots,x_n).$$

Then,

$$\begin{aligned}
&\mathbb{P}\{X_1 > x_1, X_2 > x_2 | \mathbf{X}_{\overline{12}} = \mathbf{x}_{\overline{12}}\}\\
&= \int_{x_1}^{\infty}\int_{x_2}^{\infty} \frac{(-1)^n \frac{\partial^n}{\partial y_1 \partial y_2 \partial x_3 \ldots \partial x_n}\bar{F}(y_1,y_2,x_3,\ldots,x_n)}{f_{(X_3,\ldots,X_n)}(x_3,\ldots,x_n)} dy_1 dy_2\\
&= (-1)^{n-2}\frac{\partial^{n-2}}{\partial x_3\ldots\partial x_n}\bar{F}(x_1,\ldots,x_n)\times\frac{1}{f_{(X_3,\ldots,X_n)}(x_3,\ldots,x_n)}\\
&= (-1)^{n-2}\Lambda^{(n-2)}\left(\sum_{k=1}^{n}\Psi(\bar{F}_k(x_k))\right)\times\prod_{k=3}^{n}(-\Psi'(\bar{F}_k(x_k)))\\
&\quad\times\frac{\prod_{k=3}^{n} f_k(x_k)}{f_{(X_3,\ldots,X_n)}(x_3,\ldots,x_n)}\\
&= (-1)^{n-2}\Lambda^{(n-2)}\left(\Psi(\bar{F}(x_1,\ldots,x_n))\right)\times h(x_3,\ldots,x_n),
\end{aligned}$$

where $h(x_3,\ldots,x_n) = \prod_{k=3}^{n}(-\Psi'(\bar{F}_k(x_k)))\times\frac{\prod_{k=3}^{n}f_k(x_k)}{f_{(X_3,\ldots,X_n)}(x_3,\ldots,x_n)} \ge 0$.

Therefore,

$$\begin{aligned}
&-\frac{\partial}{\partial x_1}\mathbb{P}\{X_1 > x_1, X_2 > x_2 | \mathbf{X}_{\overline{12}} = \mathbf{x}_{\overline{12}}\}\\
&= (-1)^{n-1}\Lambda^{(n-1)}\left(\Psi(\bar{F}(x_1,x_2,\ldots,x_n))\right)\\
&\quad\times\Psi'(\bar{F}_1(x_1))(-f_1(x_1))\times h(x_3,\ldots,x_n) \qquad (5.3)
\end{aligned}$$

and

$$\begin{aligned}
&-\frac{\partial}{\partial x_1}\mathbb{P}\{X_1 > x_2, X_2 > x_1 | \mathbf{X}_{\overline{12}} = \mathbf{x}_{\overline{12}}\}\\
&= (-1)^{n-1}\Lambda^{(n-1)}\left(\Psi(\bar{F}(x_2,x_1,\ldots,x_n))\right)\\
&\quad\times\Psi'(\bar{F}_2(x_1))(-f_2(x_1))\times h(x_3,\ldots,x_n). \qquad (5.4)
\end{aligned}$$

Recall that $\Psi'(x)x$ is increasing and $\bar{F}_1(s)\le\bar{F}_2(s)$, we have

$$-\bar{F}_1(s)\times\Psi'(\bar{F}_1(s)) \ge -\bar{F}_2(s)\times\Psi'(\bar{F}_2(s)). \qquad (5.5)$$

Note that $X_1 \le_{hr} X_2$, we have $f_1(s)/\bar{F}_1(s) \ge f_2(s)/\bar{F}_2(s)$. Combining with (5.5), we get

$$\Psi'(\bar{F}_1(s))(-f_1(s)) \ge \Psi'(\bar{F}_2(s))(-f_2(s)). \qquad (5.6)$$

By (5.6), we have $\Psi(\bar{F}_1(x_2)) - \Psi(\bar{F}_1(x_1)) = \int_{x_1}^{x_2}\Psi'(\bar{F}_1(s))(-f_1(s))ds \ge \int_{x_1}^{x_2}\Psi'(\bar{F}_2(s))(-f_2(s))ds = \Psi(\bar{F}_2(x_2)) - \Psi(\bar{F}_2(x_1))$, which implies $\Psi(\bar{F}_1(x_1)) + \Psi(\bar{F}_2(x_2)) \le \Psi(\bar{F}_1(x_2)) + \Psi(\bar{F}_2(x_1))$. Therefore, $\Psi(\bar{F}(x_1,x_2,\ldots,x_n)) = \Psi(\bar{F}_1(x_1)) + \Psi(\bar{F}_2(x_2)) + \sum_{k=3}^{n}\Psi(\bar{F}_k(x_k)) \le \Psi(\bar{F}_1(x_2)) + \Psi(\bar{F}_2(x_1)) + \sum_{k=3}^{n}\Psi(\bar{F}_k(x_k)) = \Psi(\bar{F}(x_2,x_1,\ldots,x_n))$. Note that $(-1)^{n-1}\Lambda^{(n-1)}(x)$ is decreasing from Remark 5.6, we have

$$\begin{aligned}
&(-1)^{n-1}\Lambda^{(n-1)}\left(\Psi(\bar{F}(x_1,x_2,\ldots,x_n))\right)\\
&\quad\ge(-1)^{n-1}\Lambda^{(n-1)}\left(\Psi(\bar{F}(x_2,x_1,\ldots,x_n))\right). \qquad (5.7)
\end{aligned}$$

Combining (5.3), (5.4), (5.6) and (5.7), we get (5.2). □

We point out that there are many examples of Archimedean copulas that satisfy the conditions of Theorem 5.7. For example, Gumbel copulas with $\Psi(x) = (-\log x)^{\alpha}$, $\alpha \ge 1$ and Clayton copulas with $\Psi(x) = x^{-\theta} - 1, \theta > 0$. Furthermore, the conditions of Theorem 5.7 are weaker than those of Theorem 1 of Li and You (2012).

6. Applications in some allocations problems

As applications of the dependence notions studied in Sections 3–5, in this section, we will study the optimization problems (1.5)–(1.6) and capital reserves problems with dependent risks. To do so, we need to generalize some existing results from independent or comonotonic risks to SAI/RWSAI/CUOAI/UOAI dependent risks.

Theorem 6.1. *A bivariate random vector (X, Y) is SAI if and only if $\mathbb{E}[g_1(X,Y)] \le \mathbb{E}[g_2(X,Y)]$ for any functions $g_1(x,y)$ and $g_2(x,y)$ such that $g_2(x,y) \ge g_1(x,y)$ for all $x \le y$ and $x \in S(X), y \in S(Y)$ and $g_2(x,y) + g_2(y,x) \ge g_1(x,y) + g_1(y,x)$ for all $x \le y$ and $x \in S(X)$, $y \in S(Y)$.*

Proof. Without loss of generality, we assume $S(X) = S(Y) = \mathbb{R}$. The proof of "$\Longrightarrow$". For any arrangement increasing function $g(x,y)$, let $g_1(x,y) = g(y,x)$ and $g_2(x,y) = g(x,y)$, then $g_2(x,y) \ge g_1(x,y)$ for all $x \le y$ and $g_2(x,y) + g_2(y,x) = g_1(x,y) + g_1(y,x)$ for all $x \le y$. Therefore, $\mathbb{E}[g(X,Y)] = \mathbb{E}[g_2(X,Y)] \ge \mathbb{E}[g_1(X,Y)] = \mathbb{E}[g(Y,X)]$.

The proof of "$\Longleftarrow$". Define $h(x,y) = (g_2(x,y) - g_1(x,y)) \times \mathbb{I}\{x < y\}$, then $h(y,x) = (g_2(y,x) - g_1(y,x)) \times \mathbb{I}\{y < x\}$, thus $h(x,y) \ge 0 = h(y,x)$ for all $x \le y$, which means $h(x,y)$ is arrangement increasing. Since (X,Y) is SAI, we have $\mathbb{E}[h(X,Y)] \ge \mathbb{E}[h(Y,X)]$ or

$$\begin{aligned}
&\mathbb{E}[(g_2(X,Y) - g_1(X,Y))\,\mathbb{I}\{X < Y\}]\\
&\quad\ge\mathbb{E}[(g_2(Y,X) - g_1(Y,X))\,\mathbb{I}\{Y < X\}]. \qquad (6.1)
\end{aligned}$$

On the other hand, recall that $g_2(x,y) + g_2(y,x) \ge g_1(x,y) + g_1(y,x)$ for all $x < y$, or $g_2(x,y) - g_1(x,y) \ge g_1(y,x) - g_2(y,x)$ for all $x < y$. Therefore, we have $(g_2(Y,X) - g_1(Y,X))\,\mathbb{I}\{Y < X\} \ge_{a.s.} (g_1(X,Y) - g_2(X,Y))\,\mathbb{I}\{Y < X\}$, which implies

$$\begin{aligned}
&\mathbb{E}[(g_2(Y,X) - g_1(Y,X))\,\mathbb{I}\{Y < X\}]\\
&\quad\ge\mathbb{E}[(g_1(X,Y) - g_2(X,Y))\,\mathbb{I}\{Y < X\}]. \qquad (6.2)
\end{aligned}$$

Combining (6.1), (6.2) and the fact that $\mathbb{E}[(g_2(X,Y) - g_1(X,Y)) \times \mathbb{I}\{X = Y\}] \ge 0$, we get $\mathbb{E}[g_2(X,Y)] \ge \mathbb{E}[g_1(X,Y)]$. □

Theorem 6.1 generalizes Theorem 1(i) of Righter and Shanthikumar (1992), in which they developed a functional characterization of $X \le_{lr} Y$ when X and Y are independent or a functional characterization of an independent bivariate SAI random vector. The following Lemma 6.2 is a generalization of Lemma 4.6 of Zhuang et al. (2009), in which they assumed that random variables X_1 and X_2 are independent and $X_1 \le_{lr} X_2$. The proof of Lemma 6.2 follows from the properties of UOAI and SAI and the proof of Lemma 4.6 of Zhuang et al. (2009) and is omitted here.

Lemma 6.2. *Let (X_1, X_2) be a bivariate random vector and $u(x)$ be an increasing convex function. For any $d_1 \ge d_2$, define functions $g_1(\omega_1,\omega_2) = \mathbb{E}[u(\omega_1(X_1\wedge d_1) + \omega_2(X_2\wedge d_2))]$ and $g_2(\omega_1,\omega_2) = \mathbb{E}[u(\omega_1(X_1\wedge d_2) + \omega_2(X_2\wedge d_1))]$. If (X_1,X_2) is nonnegative and UOAI, then* (i) *$g_2(\omega_1,\omega_2) \ge g_1(\omega_1,\omega_2)$ for all $0 \le \omega_1 \le \omega_2$ and* (ii) *$g_2(\omega_1,\omega_2) + g_2(\omega_2,\omega_1) \ge g_1(\omega_1,\omega_2) + g_1(\omega_2,\omega_1)$ for all $0 \le \omega_1 \le \omega_2$. If (X_1,X_2) is SAI, then* (i) *$g_2(\omega_1,\omega_2) \ge g_1(\omega_1,\omega_2)$ for all $\omega_1 \le \omega_2$ and* (ii) *$g_2(\omega_1,\omega_2) + g_2(\omega_2,\omega_1) \ge g_1(\omega_1,\omega_2) + g_1(\omega_2,\omega_1)$ for all $\omega_1 \le \omega_2$.* □

Theorem 6.3. *In the optimal deductibles problem (1.5), if $\mathbf{X} = (X_1,\ldots,X_n)$ is a nonnegative CUOAI random vector or a SAI random vector, $(e^{-\delta T_1},\ldots,e^{-\delta T_n})$ is SAI, and $u(x)$ is an increasing convex function, then the optimal solutions to (1.5) satisfy $d_1^* \ge \cdots \ge d_n^*$.*

Proof. Denote $\mathbf{W} = (W_1,\ldots,W_n) = (e^{-\delta T_1},\ldots,e^{-\delta T_n})$. We first give the proof for the case that $\mathbf{X}$ is nonnegative and CUOAI and $n = 2$. It is sufficient to show that

$$\mathbb{E}[u(W_1(X_1 \wedge d_1) + W_2(X_2 \wedge d_2))] \le \mathbb{E}[u(W_1(X_1 \wedge d_2) + W_2(X_2 \wedge d_1))], \quad (6.3)$$

holds for all $d_1 \ge d_2$.

Using the notations in Lemma 6.2, we have $\mathbb{E}[u(W_1(X_1 \wedge d_1) + W_2(X_2 \wedge d_2))] = \mathbb{E}[\mathbb{E}[u(W_1(X_1 \wedge d_1) + W_2(X_2 \wedge d_2))]|(W_1, W_2)] = \mathbb{E}[g_1(W_1, W_2)]$, and $\mathbb{E}[u(W_1(X_1 \wedge d_2) + W_2(X_2 \wedge d_1))] = \mathbb{E}[\mathbb{E}[u(W_1(X_1 \wedge d_2) + W_2(X_2 \wedge d_1))]|(W_1, W_2)] = \mathbb{E}[g_2(W_1, W_2)]$. Combining Lemma 6.2 and Theorem 6.1, we get $\mathbb{E}[g_1(W_1, W_2)] \le \mathbb{E}[g_2(W_1, W_2)]$ or (6.3) holds.

As for the case $n \ge 3$, it is sufficient to show that $\mathbb{E}[u(I_{\mathbf{X},\mathbf{W}}(\mathbf{d}))] \le \mathbb{E}[u(I_{\mathbf{X},\mathbf{W}}(\pi_{ij}(\mathbf{d})))]$ for any $1 \le i < j \le n$ and $d_i \ge d_j$, where $I_{\mathbf{X},\mathbf{W}}(\mathbf{d}) = \sum_{i=1}^n W_i(X_i \wedge d_i)$ and $\mathbf{d} = (d_1, \ldots, d_n)$.

Without loss of generality, assume $i = 1, j = 2$ and $d_1 \ge d_2$. Note that

$$\mathbb{E}[u(I_{\mathbf{X},\mathbf{W}}(\mathbf{d}))] = \mathbb{E}[\mathbb{E}[u(I_{\mathbf{X},\mathbf{W}}(\mathbf{d}))|(\mathbf{X}_{\overline{12}}, \mathbf{W}_{\overline{12}})]] = \mathbb{E}[\mathbb{E}[u(W_1(X_1 \wedge d_1) + W_2(X_2 \wedge d_2) + c)|(\mathbf{X}_{\overline{12}}, \mathbf{W}_{\overline{12}})]], \quad (6.4)$$

where $c = \sum_{k=3}^n W_k(X_k \wedge d_k)$.

Note that $\mathbf{X}$ is CUOAI, $\mathbf{W}$ is SAI, and $\mathbf{X}$ and $\mathbf{W}$ are independent. Thus, it is easy to verify that for any fixed $(\mathbf{x}_{\overline{12}}, \mathbf{w}_{\overline{12}}) \in S(\mathbf{X}_{\overline{12}}, \mathbf{W}_{\overline{12}})$, $(X_1, X_2)|(\mathbf{X}_{\overline{12}}, \mathbf{W}_{\overline{12}}) = (\mathbf{x}_{\overline{12}}, \mathbf{w}_{\overline{12}})$ is CUOAI, $(W_1, W_2)|(\mathbf{X}_{\overline{12}}, \mathbf{W}_{\overline{12}}) = (\mathbf{x}_{\overline{12}}, \mathbf{w}_{\overline{12}})$ is SAI. Consider the increasing convex function $u_1(x) = u(x + c)$, by (6.3), we have $\mathbb{E}[u_1(W_1(X_1 \wedge d_1) + W_2(X_2 \wedge d_2))|(\mathbf{X}_{\overline{12}}, \mathbf{W}_{\overline{12}})] \le_{a.s.} \mathbb{E}[u_1(W_1(X_1 \wedge d_2) + W_2(X_2 \wedge d_1))|(\mathbf{X}_{\overline{12}}, \mathbf{W}_{\overline{12}})]$. By taking expectation on both sides of the above inequality and (6.4), we have $\mathbb{E}[u(I_{\mathbf{X},\mathbf{W}}(\mathbf{d}))] \le \mathbb{E}[u(I_{\mathbf{X},\mathbf{W}}(\pi_{12}(\mathbf{d})))]$.

The proof for the case that $(X_1, \ldots, X_n)$ is SAI is similar and thus is omitted. □

The following Lemma 6.4 is a generalization of Lemma 4.2 of Zhuang et al. (2009), in which they assumed that random variables X_1 and X_2 are independent and $X_1 \le_{lr} X_2$. The proof of Lemma 6.4 follows from the properties of SAI and the proof of Lemma 4.2 of Zhuang et al. (2009) and thus is omitted here.

Lemma 6.4. *Let (X_1, X_2) be a bivariate random vector and $u(x)$ be an increasing convex function. For any $d_1 \le d_2$, define functions $h_1(\omega_1, \omega_2) = \mathbb{E}[u(\omega_1(X_1 - d_1)_+ + \omega_2(X_2 - d_2)_+)]$ and $h_2(\omega_1, \omega_2) = \mathbb{E}[u(\omega_1(X_1 - d_2)_+ + \omega_2(X_2 - d_1)_+)]$. If (X_1, X_2) is SAI, then $h_2(\omega_1, \omega_2) \ge h_1(\omega_1, \omega_2)$ for all $\omega_1 \le \omega_2$ and $h_2(\omega_1, \omega_2) + h_2(\omega_2, \omega_1) \ge h_1(\omega_1, \omega_2) + h_1(\omega_2, \omega_1)$ for all $\omega_1 \le \omega_2$.* □

Theorem 6.5. *In the optimal limits problem (1.6), if $\mathbf{X} = (X_1, \ldots, X_n)$ is SAI, $(e^{-\delta T_1}, \ldots, e^{-\delta T_n})$ is SAI, and $u(x)$ is an increasing convex function, then the optimal solutions to (1.6) satisfy $d_1^* \le \cdots \le d_n^*$.*

Proof. Denote $\mathbf{W} = (W_1, \ldots, W_n) = (e^{-\delta T_1}, \ldots, e^{-\delta T_n})$. We first give the proof for the case $n = 2$. It is sufficient to show that

$$\mathbb{E}[u(W_1(X_1 - d_1)_+ + W_2(X_2 - d_2)_+)] \le \mathbb{E}[u(W_1(X_1 - d_2) + W_2(X_2 - d_1)_+)], \quad (6.5)$$

holds for all $d_1 \le d_2$.

Using the notations in Lemma 6.4, we have $\mathbb{E}[u(W_1(X_1 - d_1)_+ + W_2(X_2 - d_2)_+)] = \mathbb{E}[\mathbb{E}[u(W_1(X_1 - d_1)_+ + W_2(X_2 - d_2)_+)]|(W_1, W_2)] = \mathbb{E}[h_1(W_1, W_2)]$, and $\mathbb{E}[u(W_1(X_1 - d_2)_+ + W_2(X_2 - d_1)_+)] = \mathbb{E}[\mathbb{E}[u(W_1(X_1 - d_2)_+ + W_2(X_2 - d_1)_+)]|(W_1, W_2)] = \mathbb{E}[h_2(W_1, W_2)]$. Combining Lemma 6.4 and Theorem 6.1, we get $\mathbb{E}[h_1(W_1, W_2)] \le \mathbb{E}[h_2(W_1, W_2)]$ or (6.5) holds.

As for the case $n \ge 3$, the proof is similar to the proof of Theorem 6.3 for the case $n \ge 3$ and thus is omitted. □

Theorems 6.3 and 6.5 imply that in an insurance arrangement of deductibles, the policyholder should allocate larger deductibles in the polices with smaller losses, while in an insurance arrangement of limits, the policyholder should allocate smaller limits in the polices with smaller losses. Furthermore, Theorems 6.3 and 6.5 generalize the studies of Cheung (2007), Zhuang et al. (2009), and Li and You (2012) from independent or comonotonic losses to more general dependent losses. In particular, Theorems 4.7 and 4.3 of Zhuang et al. (2009) are the special cases of Theorems 6.3 and 6.5, respectively, and Theorem 2 of Li and You (2012) is the special case of Theorem 6.5.

In addition, Theorems 4.4 and 4.8 of Zhuang et al. (2009) studied problems (1.5) and (1.6) in the case that $(X_1, \ldots, X_n)$ is comonotonic with $X_1 \le_{st} \cdots \le_{st} X_n$ and $(T_1, \ldots, T_n)$ is mutually independent with $T_1 \ge_{rh} \cdots \ge_{rh} T_n$. We point out that it is easy to extend their results to more general dependent losses occurrence times $T_1, \ldots, T_n$. The following theorem follows from the properties of RWSAI and the proofs for Theorems 4.4 and 4.8 of Zhuang et al. (2009) and thus are omitted here.

Theorem 6.6. *In the optimal deductibles problems (1.5) and (1.6), if $\mathbf{X} = (X_1, \ldots, X_n)$ is comonotonic with $X_1 \le_{st} \cdots \le_{st} X_n$, $(e^{-\delta T_1}, \ldots, e^{-\delta T_n})$ is RWSAI, and $u(x)$ is an increasing convex function, then the optimal solutions to (1.5) satisfy $d_1^* \ge \cdots \ge d_n^*$ and the optimal solutions to (1.6) satisfy $d_1^* \le \cdots \le d_n^*$.* □

Remark 6.7. Note from Remark 2.5 that if $T_1 \ge_{rh} \cdots \ge_{rh} T_n$, then $-T_1 \le_{hr} \cdots \le_{hr} -T_n$. Furthermore, if $(T_1, \ldots, T_n)$ is independent, then by Proposition 5.4, $(-T_1, \ldots, -T_n)$ is RWSAI, which implies that $(e^{-\delta T_1}, \ldots, e^{-\delta T_n})$ is RWSAI from Proposition 3.9 (iii). Hence, Theorems 4.8 and 4.4 of Zhuang et al. (2009) are the special cases of Theorem 6.6.

At the end of this section, we consider another application of the dependence notions in the allocation problem of capitals reserves. Let $X_1, \ldots, X_n$ be n random variables, which represent losses or profits from n lines of business of an investor. Under certain regulations, the investor is required to reserve certain amount of risk capitals to each line of business to cope with the future uncertainty. A commonly used principle is Euler's principle, from which the risk capital for each line of business is determined by $\rho_i = \mathbb{E}[X_i \mid S > \mathrm{VaR}_\alpha(S)]$, $i = 1, \ldots, n$, where $S = \sum_{k=1}^n X_k$ is the aggregate losses and $\mathrm{VaR}_\alpha(S)$ is the value at risk of S at the confidence level α. However, it is difficult to calculate the conditional expectations without full information about the joint distribution of $(X_1, \ldots, X_n)$. In this case, the qualitative analysis on the capital reserves is needed. For example, Asimit et al. (2011) have derived some asymptotic results about the capital reserves under regular varying assumptions of the distribution of $(X_1, \ldots, X_n)$. Here, we order the risk capitals for different lines of business under RWSAI dependence structures.

Proposition 6.8. *If the risk vector $(X_1, \ldots, X_n)$ is RWSAI, then $\rho_1 \le \cdots \le \rho_n$.*

Proof. Define $g(x_1, \ldots, x_n) = x_i \times \mathbb{I}\{\sum_{k=1}^n x_k > s\}$ for any fixed s and j. Note that, for any $1 \le i < j \le n$, $\Delta g_{ij}(x_1, \ldots, x_n) = (x_j - x_i) \times \mathbb{I}\{\sum_{k=1}^n x_k > s\}$ is increasing in $x_j \ge x_i$, which means $g \in \mathcal{G}_{rwsai}^{ij}(n)$. Therefore, by the definition of RWSAI, we have $\mathbb{E}[g(X_1, \ldots, X_n)] \ge \mathbb{E}[g(\pi_{ij}(X_1, \ldots, X_n))]$ or $\mathbb{E}\left[X_j \times \mathbb{I}\{S > s\}\right] \ge \mathbb{E}[X_i \times \mathbb{I}\{S > s\}]$. Let $s = \mathrm{VaR}_\alpha S$, we have $\mathbb{E}\left[X_j \times \mathbb{I}\{S > \mathrm{VaR}_\alpha S\}\right] \ge \mathbb{E}\left[X_j \times \mathbb{I}\{S > \mathrm{VaR}_\alpha S\}\right]$ for any $1 \le i < j \le n$. Therefore, $\mathbb{E}[X_i \mid S > \mathrm{VaR}_\alpha S] \le \mathbb{E}\left[X_j \mid S > \mathrm{VaR}_\alpha S\right]$ for any $1 \le i < j \le n$. □

Note that for a RWSAI risk vector $(X_1, \ldots, X_n)$, the losses are ordered as $X_1 \le_{st} \cdots \le_{st} X_n$. Hence, the result of Proposition 6.8 is consistent with the common sense that the more capitals should be reserved for the riskier lines of business.

7. Concluding remarks

In this paper, we revisit the SAI dependence notion considered by Shanthikumar and Yao (1991) and Righter and Shanthikumar

(1992) and propose new dependence notions of RWSAI, COUAI, UOAI. We consider the properties and relationships of these dependence notions and discuss their applications in optimal deductibles/policy limits and capital reserves problems. Our applications generalize the studies of Cheung (2007), Zhuang et al. (2009) and Li and You (2012) from independent or comonotonic risks to more general dependent risks. The dependence notions developed in this paper have potential applications in many other fields and they are particularly useful for optimal allocation problems. We will preset their applications in other optimal allocation problems in coming researches. In this paper, we give both functional and distributional characterizations of SAI and RWSAI dependence notions. The UOAI and COUAI dependence notions are defined through the properties of the joint distribution of a random vector, in other words, they are characterized by distributions or probability measures of random vectors. However, we point out that it is difficult to give functional characterizations of UOAI and COUAI dependence notions. We leave the functional characterizations of UOAI and CUOAI as open questions.

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