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Optimal reinsurance with positively dependent risks

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ABSTRACT

In the individual risk model, one is often concerned about positively dependent risks. Several notions of positive dependence have been proposed to describe such dependent risks. In this paper, we assume that the risks in the individual risk model are positively dependent through the stochastic ordering (PDS). The PDS risks include independent, comonotonic, conditionally stochastically increasing (CI) risks, and other interesting dependent risks. By proving the convolution preservation of the convex order for PDS random vectors, we show that in individualized reinsurance treaties, to minimize certain risk measures of the retained loss of an insurer, the excess-of-loss treaty is the optimal reinsurance form for an insurer with PDS dependent risks among a general class of individualized reinsurance contracts. This extends the study in Denuit and Vermandele (1998) on individualized reinsurance treaties to dependent risks. We also derive the explicit expressions for the retentions in the optimal excess-of-loss treaty in a two-line insurance business model.

1. Introduction

Let $\{X_i, i \geq 1\}$ be random variables. Assume that an insurer has n lines of business or the insurance portfolio of an insurer has n policy holders. The loss or claim in line i or for policy holder i is X_i, $i = 1, \ldots, n$. Without reinsurance, the total loss/claim of the insurer is $S_n = \sum_{i=1}^{n} X_i$, which is called the individual risk model. However, each line of business or each policy holder may produce a large claim. To protect from a potential huge loss, the insurer applies reinsurance strategy I_i to the loss in line i. With the reinsurance strategy I_i, the insurer retains the part of the loss in line i, which is $I_i(X_i)$, and a reinsurer covers the rest of the loss, which is $X_i - I_i(X_i)$, where the function $I_i(x)$ is increasing in $x \geq 0$ and satisfies $0 \leq I_i(x) \leq x$ for $i = 1, 2, \ldots, n$. Thus, the total retained loss of the insurer is $S_n^I = I_1(X_1) + I_2(X_2) + \cdots + I_n(X_n)$ and the total loss covered by the reinsurer is $S_n - S_n^I$, where we use $I = (I_1, \ldots, I_n)$ to denote the n-dimensional reinsurance policy. Such a policy I is called an individualized reinsurance treaty.

In the reinsurance contract I, the insurer needs to pay a reinsurance premium to the reinsurer. As in Denuit and Vermandele (1998) and Van Heerwaarden et al. (1989), we assume that the reinsurance premium is charged by the expected value principle and is fixed to a constant \$$P$, which means that the reinsurance premium is equal to $(1+\theta_R)\mathbb{E}[S_n - S_n^I] = P$, where $\theta_R > 0$ is called the security loading of the reinsurer. In this way, the insurer can control his cost or budget for the reinsurance contract at the amount of P. Note that $(1+\theta_R)\mathbb{E}[S_n - S_n^I] = P$ is equivalent to assuming that $\mathbb{E}[S_n^I]$ is fixed and equal to $p = \mathbb{E}[S_n] - P/(1+\theta_R)$ or that the expected retained loss of the insurer is fixed. We are interested in the following class of admissible reinsurance strategies:

$$\mathcal{D}_n^p = \left\{ I = (I_1, \ldots, I_n) \,\middle|\, \begin{array}{l} I_i(x) \text{ is increasing in } x \geq 0 \text{ with} \\ 0 \leq I_i(x) \leq x \quad \text{for } i = 1, \ldots, n \\ \text{and } \mathbb{E}[S_n^I] = p > 0 \end{array} \right\}. \quad (1.1)$$

In particular, when $I_i(x) = x \wedge d_i$ for $i = 1, \ldots, n$, the reinsurance $I = (I_1, \ldots, I_n)$ is called the excess-of-loss treaty and $(d_1, \ldots, d_n)$ is called the retention vector of the excess-of-loss treaty.

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In this paper, we will study what the optimal reinsurance strategy $I^* = (I_1^*, \ldots, I_n^*) \in \mathcal{D}_n^p$ is for the insurer under certain optimization criteria. We use a unified criterion and study the following optimization problem:

$$\inf_{I \in \mathcal{D}_n^p} \mathbb{E}[u(S_n^I)] \tag{1.2}$$

for a convex function u.

This optimization criterion (1.2) includes the criteria of minimizing the variance of the total retained loss of the insurer; maximizing the expected exponential utility for the insurer; maximizing the expected concave utility function for the insurer; and so on.

When $X_1, \ldots, X_n$ are exchangeable random variables, Denuit and Vermandele (1998) showed that the optimal reinsurance strategy for problem (1.2) is the excess-of-loss reinsurance with the equal retention for each line of business. A further study of Denuit and Vermandele (1998) about optimal reinsurance with exchangeable risks can be found in Denuit and Vermandele (1999).

However, in individualized reinsurance treaties, one is often concerned about dependent risks, and in particular positively dependent risks. For example, in a two-line insurance business with life insurance and non-life insurance, the property losses and the numbers of dead people in earthquakes, tornadoes, and hurricanes are usually positively dependent. Roughly speaking, two risks are positively dependent if a large value of one risk will result in a large value of the other. Several notions of positive dependence have been proposed to describe such dependent risks in the literature.

In this paper, we assume that the risks in the individual risk model are positively dependent through the stochastic ordering (PDS), which will be defined in Section 2. We show that when $X_1, \ldots, X_n$ are PDS dependent risks, the optimal reinsurance strategy for problem (1.2) is the excess-of-loss reinsurance. To do so, we denote $\mathcal{D}_n^{p*}$ by all excess-of-loss treaties in $\mathcal{D}_n^p$, namely

$$\mathcal{D}_n^{p*} = \Big\{ I^d = (I^{d_1}, \ldots, I^{d_n}) \mid I^d \in \mathcal{D}_n^p, I^{d_i}(x) = x \wedge d_i, d_i \geq 0, \\ i = 1, \ldots, n \Big\}.$$

This subclass $\mathcal{D}_n^{p*}$ is determined uniquely by the retention vector $(d_1, \ldots, d_n)$ and there is a one-to-one mapping between D_n^{p*} and L_n^p that is defined as

$$L_n^p = \Big\{ (d_1, \ldots, d_n) \mid d_i \geq 0,\ i = 1, \ldots, n \text{ and } \\ \mathbb{E}\Big[\sum_{i=1}^n (X_i \wedge d_i)\Big] = p > 0 \Big\}. \tag{1.3}$$

We will show that for the PDS dependent risks $X_1, \ldots, X_n$ and a convex function u,

$$\inf_{I \in \mathcal{D}_n^p} \mathbb{E}[u(S_n^I)] = \inf_{(d_1, \ldots, d_n) \in L_n^p} \mathbb{E}\Big[u\Big(\sum_{i=1}^n (X_i \wedge d_i)\Big)\Big], \tag{1.4}$$

which means that the optimal strategies for problem (1.2) are the excess-of-loss treaties and that the infinite-dimensional optimization problem (1.2) is reduced to the feasible finite-dimensional optimization problem:

$$\inf_{(d_1, \ldots, d_n) \in L_n^p} \mathbb{E}\Big[u\Big(\sum_{i=1}^n (X_i \wedge d_i)\Big)\Big]. \tag{1.5}$$

Throughout this paper, 'increasing' means 'non-decreasing' and 'decreasing' means 'non-increasing'.

The rest of the paper is organized as follows. In Section 2, we recall the notions of several positive dependence including the stochastically increasing (SI) and the positive dependence through the stochastic ordering (PDS). In Section 3, we first prove that the convolution preservation of the convex order for PDS random vectors. We then show when $X_1, \ldots, X_n$ are PDS dependent risks, for any $I = (I_1, \ldots, I_n) \in \mathcal{D}_n^p$, there exists a retention vector $(d_1, \ldots, d_n) \in L_n^p$ such that $\sum_{i=1}^n (X_i \wedge d_i) \leq_{cx} \sum_{i=1}^n I_i(X_i)$, which means that (1.4) holds or the excess-of-loss treaty is the optimal reinsurance form for the insurer with PDS dependent risks. This extends the study in Denuit and Vermandele (1998) on individualized reinsurance treaties to dependent risks. In Section 4, we use a two-line insurance business model to illustrate how to derive the explicit expressions for the retention vector $(d_1^*, d_2^*) \in L_2^p$ in the optimal excess-of-loss treaty such that $\mathbb{E}[u(X_1 \wedge d_1^* + X_2 \wedge d_2^*)] = \inf_{(d_1, d_2) \in L_2^p} \mathbb{E}[u(X_1 \wedge d_1 + X_2 \wedge d_2)]$.

2. The notions of several positive dependence

In this section, we only recall the notions of SI and PDS, which will be used in this paper. For other notions of positive dependence, we refer to Colangelo et al. (2005, 2008), Denuit et al. (2005), Müller and Stoyan (2002), Shaked and Shanthikumar (2007), and references therein.

We recall that for a random variable Y, a support of Y, denoted by $S(Y)$, is a Borel set of $\mathbb{R}$ such that $\mathbb{P}\{Y \in S(Y)\} = 1$.

Definition 2.1. Random variable X is said to be stochastically increasing (SI) in random variable Y, denoted as $X \uparrow_{SI} Y$, if for any $x \in \mathbb{R}$, $\mathbb{P}\{X > x \mid Y = y\}$ is increasing in $y \in S(Y)$, or equivalently, $X \uparrow_{SI} Y$ if and only if $\mathbb{E}[u(X) | Y = y]$ is increasing in $y \in S(Y)$ for all increasing function u such that the expectation exists. □

Definition 2.2. Random vector $(X_1, \ldots, X_n)$ is said to be stochastically increasing in random variable Y, denoted as $(X_1, \ldots, X_n) \uparrow_{SI} Y$, if $\mathbb{E}[u(X_1, \ldots, X_n) \mid Y = y]$ is increasing in $y \in S(Y)$ for any increasing function $u : \mathbb{R}^n \to \mathbb{R}$ such that the conditional expectation exists. Furthermore, random vector $(X_1, \ldots, X_n)$ is said to be positively dependent through the stochastic ordering (PDS) if $(X_1, \ldots, X_{i-1}, X_{i+1}, \ldots, X_n) \uparrow_{SI} X_i$ for any $i = 1, \ldots, n$. □

The notion of the PDS is interesting to model dependent risks. The PDS risk includes independent, comonotonic, conditionally stochastically increasing (CI) risks, and other interesting dependent risks.

The following property will be used in Section 3.

Proposition 2.3. *Let* $(X_1, \ldots, X_n)$ *be random vector and* Y *be random variable and assume* $(X_1, \ldots, X_n) \uparrow_{SI} Y$. *Then the following hold.*

(1) $(Y, X_1, \ldots, X_n) \uparrow_{SI} Y$.
(2) $u(X_1, \ldots, X_n) \uparrow_{SI} Y$ *for any increasing function* $u : \mathbb{R}^n \to \mathbb{R}^k$ *where* $k \in \mathbb{N}$.

Proof. (1) Denote $\mathbf{X} = (X_1, \ldots, X_n)$ and let $u : \mathbb{R}^{n+1} \mapsto \mathbb{R}$ be an increasing function. For any $y_1, y_2 \in S(Y)$ with $y_1 \leq y_2$, we have

$$\begin{aligned} \mathbb{E}[u(Y, \mathbf{X}) | Y = y_1] &= \mathbb{E}[u(y_1, \mathbf{X}) | Y = y_1] \\ &\leq \mathbb{E}[u(y_2, \mathbf{X}) | Y = y_1] \\ &\leq \mathbb{E}[u(y_2, \mathbf{X}) | Y = y_2] = \mathbb{E}[u(Y, \mathbf{X}) | Y = y_2], \end{aligned}$$

where the second inequality holds since $\mathbf{X} \uparrow_{SI} Y$ and $u(y_2, x_1, \ldots, x_n)$ is an increasing function. Hence, $(Y, \mathbf{X}) \uparrow_{SI} Y$ by Definition 2.2.

(2) For any increasing function $h : \mathbb{R}^k \to \mathbb{R}$, the function $h \circ u : \mathbb{R}^n \to \mathbb{R}$ is also increasing. By Definition 2.2, we know that $\mathbb{E}[h \circ u(X_1, \ldots, X_n) \mid Y = y]$ is increasing in $y \in S(Y)$, which means $u(X_1, \ldots, X_n) \uparrow_{SI} Y$. □

We refer to Block et al. (1985), Joe (1997), Lehmann (1966), and Shaked (1977) for more properties of SI and PDS.

3. Optimality of excess-of-loss reinsurance strategies with dependent risks

In this section, we first prove that two convolution preservation results of the convex order for SI and PDS random vectors in Lemma 3.3 and Theorem 3.4. Then, we can determine the optimal reinsurance forms with the PDS dependent risks in Propositions 3.7 and 3.8 for the individual risk model and the collective risk model, respectively.

Definition 3.1. Random variable X is said to be smaller than random variable Y in convex order, denoted as $X \le_{cx} Y$, if $\mathbb{E}[u(X)] \le \mathbb{E}[u(Y)]$ for any convex function u such that the expectations exist. Furthermore, X is said to be smaller than Y in stop-loss order, denoted as $X \le_{sl} Y$, if $\mathbb{E}[(X-t)_+] \le \mathbb{E}[(Y-t)_+]$ for all $t \in \mathbb{R}$. □

The following result is a useful criterion for the convex order, the proof can be found in Lemma 3 of Ohlin (1969).

Lemma 3.2. *Let X be a random variable, h_1 and h_2 be increasing functions such that $\mathbb{E}[h_1(X)] = \mathbb{E}[h_2(X)]$. If there exists $\alpha \in \mathbb{R} \cup \{+\infty\}$ such that $h_1(x) \ge h_2(x)$ for all $x < \alpha$ and $h_1(x) \le h_2(x)$ for all $x > \alpha$, then $h_1(X) \le_{cx} h_2(X)$.* □

Lemma 3.3. *Let X and Y be random variables. If $Y \uparrow_{SI} X$, then $h_1(X) + Y \le_{cx} h_2(X) + Y$ for any increasing functions h_1 and h_2 such that $h_1(X) \le_{cx} h_2(X)$.*

Proof. It is sufficient to show that $h_1(X) + Y \le_{sl} h_2(X) + Y$, or equivalently, to show that $\mathbb{E}[(h_1(X) + Y - t)_+] \le \mathbb{E}[(h_2(X) + Y - t)_+]$ for any $t \in \mathbb{R}$.

It is easy to verify that $(x-t)_+ - (y-t)_+ \le \mathbb{I}\{x > t\} \times (x-y)$ for any $x, y, t \in \mathbb{R}$, then

$$\begin{aligned}
&\mathbb{E}[(h_1(X) + Y - t)_+ - (h_2(X) + Y - t)_+] \\
&\quad \le \mathbb{E}[\mathbb{I}\{(h_1(X) + Y) > t\} \times (h_1(X) - h_2(X))] \\
&\quad = \mathbb{E}[\mathbb{E}[\mathbb{I}\{(h_1(X) + Y) > t\} | X] \times (h_1(X) - h_2(X))] \\
&\quad = \mathbb{E}[p_t(X)(h_1(X) - h_2(X))],
\end{aligned} \tag{3.1}$$

where the function $p_t(x) = \mathbb{E}[\mathbb{I}\{(h_1(x) + Y) > t\} \mid X = x] \ge 0$ is well defined since $0 \le \mathbb{I}\{x > t\} \le 1$.

From Proposition 2.3, we know that $(Y, X) \uparrow_{SI} X$ and $h_1(X) + Y \uparrow_{SI} X$. Therefore, the function $p_t(x) = \mathbb{E}[\mathbb{I}\{h_1(x) + Y > t\} \mid X = x]$ is increasing in x since $\mathbb{I}\{x > t\}$ is increasing in x. Thus, both $(p_t(X), h_1(X))$ and $(p_t(X), h_2(X))$ are comonotonic vectors. Hence, by Lemma 3.12.13 of Müller and Stoyan (2002), we know that $\mathbb{E}[\phi(p_t(X), h_1(X))] \le \mathbb{E}[\phi(p_t(X), h_2(X))]$ holds for any directional convex function $\phi(x, y)$ such that the expectations exist. Note that $\phi(x, y) = xy$ is a directional convex function, we have $\mathbb{E}[p_t(X)h_1(X)] \le \mathbb{E}[p_t(X)h_2(X)]$, which completes the proof by (3.1). □

Lemma 3.3 is an interesting result and will be used to prove the following Theorem 3.4. Also, Lemma 3.3 generalizes Theorems 1 and 2 of Aboudi and Thon (1995), in which they presented the optimal insurance policies when the insurance risk has positively dependent relationships with the random initial wealth.

Theorem 3.4. *Let $(X_1, \ldots, X_n)$ be a PDS random vector, and f_i, g_i be increasing functions such that $f_i(X_i) \le_{cx} g_i(X_i)$ for $i = 1, \ldots, n$. Then $\sum_{k=1}^n f_k(X_k) \le_{cx} \sum_{k=1}^n g_k(X_k)$.*

Proof. According to Proposition 2.3, we have $\sum_{i=1}^{k-1} f_i(X_i) + \sum_{i=k+1}^n g_i(X_i) \uparrow_{SI} X_k$ for any $k = 1, \ldots, n$, where $\sum_{k=i}^j a_k$ is defined to be 0 for $i > j$. Applying Lemma 3.3, we have for any $k = 1, \ldots, n$,

$$\begin{aligned}
&\sum_{i=1}^{k-1} f_i(X_i) + \sum_{i=k+1}^n g_i(X_i) + f_k(X_k) \\
&\quad \le_{cx} \sum_{i=1}^{k-1} f_i(X_i) + \sum_{i=k+1}^n g_i(X_i) + g_k(X_k),
\end{aligned}$$

or equivalently,

$$\sum_{i=1}^{k} f_i(X_i) + \sum_{i=k+1}^n g_i(X_i) \le_{cx} \sum_{i=1}^{k-1} f_i(X_i) + \sum_{i=k}^n g_i(X_i). \tag{3.2}$$

By applying the relationship (3.2) repeatedly from $k = n$ to $k = 1$ and using the transitive property of the convex order, we have

$$\begin{aligned}
\sum_{i=1}^n f_i(X_i) &\le_{cx} \sum_{i=1}^{n-1} f_i(X_i) + \sum_{i=n}^n g_i(X_i) \\
&\le_{cx} \sum_{i=1}^{n-2} f_i(X_i) + \sum_{i=n-1}^n g_i(X_i) \\
&\le_{cx} \cdots \le_{cx} \sum_{i=1}^{1} f_i(X_i) + \sum_{i=2}^n g_i(X_i) \le_{cx} \sum_{i=1}^n g_i(X_i).
\end{aligned}$$

It completes the proof. □

Using Theorem 3.4, we can prove the convolution preservation of the convex order for two random vectors with the same PDS copula in the following corollary.

Corollary 3.5. *Assume that random vectors $(Y_1, \ldots, Y_n)$ and $(Z_1, \ldots, Z_n)$ have the same PDS copula. If $Y_k \le_{cx} Z_k$ for $k = 1, \ldots, n$, then $\sum_{k=1}^n Y_k \le_{cx} \sum_{k=1}^n Z_k$.*

Proof. Let F_i and G_i be the distributions of Y_i and Z_i, respectively. Let the common PDS copula be $C(u_1, \ldots, u_n) = \Pr\{U_1 \le u_1, \ldots, U_n \le u_n\}$ for some uniform random vector $(U_1, \ldots, U_n)$ defined on $[0, 1]^n$. Then, $(U_1, \ldots, U_n)$ is a PDS random vector. From the last paragraph of the proof for Theorem 5.3 of McNeil et al. (2005), we know that $(Y_1, \ldots, Y_n) =_{\mathrm{st}} (F_1^{-1}(U_1), \ldots, F_n^{-1}(U_n))$ and $(Z_1, \ldots, Z_n) =_{\mathrm{st}} (G_1^{-1}(U_1), \ldots, G_n^{-1}(U_n))$, where F_i^{-1} and G_i^{-1} are the left-continuous generalized inverses of F_i and G_i and they are increasing. Thus, $\sum_{k=1}^n Y_k \le_{cx} \sum_{k=1}^n Z_k$ by Theorem 3.4. □

Remark 3.6. We point out that for all non-negative constants $\alpha_1, \ldots, \alpha_n$, $(\alpha_1 Y_1, \ldots, \alpha_n Y_n)$ and $(Y_1, \ldots, Y_n)$ have the same copula, and $(\alpha_1 Z_1, \ldots, \alpha_n Z_n)$ and $(Z_1, \ldots, Z_n)$ have the same copula. Thus, if $(Y_1, \ldots, Y_n)$ and $(Z_1, \ldots, Z_n)$ have the same PDS copula, and $Y_k \le_{cx} Z_k$ for $k = 1, \ldots, n$, then by Corollary 3.5, we have $\sum_{k=1}^n \alpha_k Y_k \le_{cx} \sum_{k=1}^n \alpha_k Z_k$ since $Y_k \le_{cx} Z_k \Longrightarrow \alpha_k Y_k \le_{cx} \alpha_k Z_k$ for $k = 1, \ldots, n$. Hence, Corollary 3.5 extends Corollary 3.12.15 of Müller and Stoyan (2002) about the preservation of the convex order under non-negative linear combinations of CI random variables since CI $\Longrightarrow$ PDS. □

Then, using Theorem 3.4, we can show in the following proposition that the optimal reinsurance for the optimization problem (1.2) is the excess-of-loss treaty or the relationship (1.4) holds.

Proposition 3.7. *Assume random vector $(X_1, \ldots, X_n)$ is PDS, then for any reinsurance policy $I = (I_1, \ldots, I_n) \in \mathcal{D}_n^p$, there exists retention vector $(d_1, \ldots, d_n) \in L_n^p$ such that*

$$\sum_{i=1}^n (X_i \wedge d_i) \le_{cx} \sum_{i=1}^n I_i(X_i),$$

where d_i is determined by $\mathbb{E}[X_i \wedge d_i] = \mathbb{E}[I_i(X_i)],\ i = 1, \ldots, n$.

Proof. Since $0 \le \mathbb{E}[I_k(X_k)] \le \mathbb{E}[X_k]$ and the function $g(x) = \mathbb{E}[X_k \wedge x]$ is continuous and increasing in $x \in [0, \infty)$ with $g(0) = 0$ and $g(\infty) = \mathbb{E}[X_k]$, there exists $d_k \in [0, \infty]$ such that $g(d_k) = \mathbb{E}[X_k \wedge d_k] = \mathbb{E}[I_k(X_k)]$. Note that $0 \le I_k(x) \le x$ for all $x \ge 0$. Thus, according to Lemma 3.2, we have $X_k \wedge d_k \le_{cx} I_k(X_k)$ for $k = 1, \ldots, n$. Therefore, $\sum_{i=1}^n (X_i \wedge d_i) \le_{cx} \sum_{i=1}^n I_i(X_i)$ from Theorem 3.4. □

Now, we apply the above result to consider the optimal reinsurance in a collective risk model. In this model, we assume that the number of claims in the insurance portfolio of an insurer is a counting random variable N and the amount of claim i is X_i, $i = 1, 2, \ldots$ and that the reinsurance strategy I_i is applied to claim i for $i = 1, 2, \ldots$, where $I_i(x)$ satisfies the same conditions assumed in the individual risk mode, namely $I_i(x)$ is increasing in $x \geq 0$ and $0 \leq I_i(x) \leq x$ for $i = 1, 2, \ldots$. In this case, the total retained loss for the insurer is $\sum_{i=1}^{N} I_i(X_i)$.

Proposition 3.8. *Let $\{X_1, X_2, \ldots\}$ be a sequence of random variables and N be a counting random variable independent of $\{X_1, X_2, \ldots\}$. If for any $n = 2, 3, \ldots$, the random vector $(X_1, \ldots, X_n)$ is PDS, then for any $I_i(x)$, $i = 1, 2, \ldots$, there exist $d_i \in [0, \infty]$, $i = 1, 2, \ldots$, such that*

$$\sum_{i=1}^{N}(X_i \wedge d_i) \leq_{cx} \sum_{i=1}^{N} I_i(X_i),$$

where d_i is determined by $\mathbb{E}[X_i \wedge d_i] = \mathbb{E}[I_i(X_i)]$, $i = 1, 2, \ldots$.

Proof. According to Proposition 3.7, $\sum_{i=1}^{n}(X_i \wedge d_i) \leq_{cx} \sum_{i=1}^{n} I_i(X_i)$ for any fixed n. Thus for any convex function u, we have

$$\mathbb{E}\left[u\left(\sum_{i=1}^{n}(X_i \wedge d_i)\right)\right] \leq \mathbb{E}\left[u\left(\sum_{i=1}^{n} I_i(X_i)\right)\right].$$

Therefore,

$$\begin{aligned}\mathbb{E}&\left[u\left(\sum_{i=1}^{N}(X_i \wedge d_i)\right)\right] \\ &= \sum_{n=0}^{\infty}\mathbb{P}\{N = n\}\,\mathbb{E}\left[u\left(\sum_{i=1}^{n}(X_i \wedge d_i)\right)\right] \\ &\leq \sum_{n=0}^{\infty}\mathbb{P}\{N = n\}\,\mathbb{E}\left[u\left(\sum_{i=1}^{n} I_i(X_i)\right)\right] = \mathbb{E}\left[u\left(\sum_{i=1}^{N} I_i(X_i)\right)\right],\end{aligned}$$

which means $\sum_{i=1}^{N}(X_i \wedge d_i) \leq_{cx} \sum_{i=1}^{N} I_i(X_i)$. □

If $X_1, X_2, \ldots$ are a sequence of independent random variables, then for any $n = 2, 3, \ldots$, the random vector $(X_1, X_2, \ldots, X_n)$ is PDS. Furthermore, if $X_1, X_2, \ldots$ are a sequence of comonotonic random variables or there exist a random variable Z and a sequence of increasing functions $\{f_i,\ i = 1, 2, \ldots\}$ such that $X_i = f_i(Z)$, $i = 1, 2, \ldots$, then for any $n = 2, 3, \ldots$, the random vector $(X_1, X_2, \ldots, X_n)$ is PDS. Propositions 3.7 and 3.8 mean that the excess-of-loss reinsurance is the optimal strategy for an insurer to minimize the certain risk measures of the retained loss.

4. Explicit expressions for the retentions in the optimal excess-of-loss treaty

In this section, we illustrate how to derive the explicit expressions for the retentions in the optimal excess-of-loss treaty. In general, it is difficult to derive such expressions due to the complexity of dependent risks. Here, we consider the bivariate case and assume that the company has two lines of business or $n = 2$ in the individual risk model. We assume that X_1 and X_2 are nonnegative random variables with distribution functions F_1 and F_2, respectively.

To avoid tedious arguments, throughout this section, we assume $\overline{F}_1(d_1) = 1 - F_1(d_1) > 0$ and $\overline{F}_2(d_2) = 1 - F_2(d_2) > 0$ for any $d_1, d_2 \in \mathbb{R}$. We will derive the explicit expressions for $(d_1^*, d_2^*) \in L$ such that

$$\mathbb{E}[u(X_1 \wedge d_1^* + X_2 \wedge d_2^*)] = \inf_{(d_1,d_2)\in L} \mathbb{E}\left[u\big(X_1 \wedge d_1 + X_2 \wedge d_2\big)\right], \tag{4.1}$$

where

$$L = L_2^p = \left\{(d_1, d_2)\,\middle|\, \int_0^{d_1} \overline{F}_1(x)dx + \int_0^{d_2} \overline{F}_2(x)dx = p > 0,\ d_1, d_2 \geq 0\right\}.$$

Moreover, we assume $p < \mathbb{E}[X_1] + \mathbb{E}[X_2]$. Otherwise, if $p \geq \mathbb{E}[X_1] + \mathbb{E}[X_2]$, then $L = \{(\infty, \infty)\}$ or $L = \emptyset$.

To derive the explicit solutions given in Theorems 4.4 and 4.5, we need the following Lemmas 4.1–4.3. The proofs of these lemmas are given in Appendix.

Lemma 4.1. *On the set L, the mapping from d_1 to d_2 is one-to-one. Denote the mapping as $d_2 = L(d_1)$. Then, $L(d_1)$ is continuous, differentiable and strictly decreasing in d_1, with $\frac{\partial d_2}{\partial d_1} = -\frac{\overline{F}_1(d_1)}{\overline{F}_2(d_2)}$.* □

Lemma 4.1 means that the set L is a continuous and strictly decreasing curve in the first quadrant and the inverse function L^{-1} of L is also continuous, differentiable and strictly decreasing.

To avoid tedious discussion, in the following, we further assume $\mathbb{E}[X_1] < p$ and $\mathbb{E}[X_2] < p$. Thus, both limits of $\lim_{d_2\to\infty} L^{-1}(d_2)$ and $\lim_{d_1\to\infty} L(d_1)$ exist on the set L. We denote by $\underline{d}_1 = \lim_{d_2\to\infty} L^{-1}(d_2)$ and $\underline{d}_2 = \lim_{d_1\to\infty} L(d_1)$. Therefore, $(\underline{d}_1, \infty)$ is the domain of the function $L(d_1)$ with $\lim_{d_1\downarrow \underline{d}_1} L(d_1) = \infty$ and $\underline{d}_2 = \lim_{d_1\to\infty} L(d_1)$. Furthermore, on the set L, $d_1 \downarrow \underline{d}_1 \Longleftrightarrow d_2 \to \infty$.

In the following, we denote

$$M(d_1, d_2) = \mathbb{E}[u(X_1 \wedge d_1 + X_2 \wedge d_2)],\ (d_1, d_2) \in L.$$

Note that $M(d_1, d_2) = M(d_1, L(d_1))$ is a univariate function of d_1 on the set L.

Lemma 4.2. *Let function u be continuous and monotonic such that $\mathbb{E}\big[|u(X_1+X_2)|\big] < \infty$. Then $M(d_1, d_2) = M(d_1, L(d_1))$ is continuous in $d_1 \in (\underline{d}_1, \infty)$ with*

$$\lim_{d_1\to\infty} M(d_1, L(d_1)) = M(\infty, \underline{d}_2) = \mathbb{E}[u(X_1 + X_2 \wedge \underline{d}_2)]$$

and

$$\begin{aligned}\lim_{d_1\downarrow \underline{d}_1} M(d_1, L(d_1)) &= \lim_{d_2\to\infty} M(L^{-1}(d_2), d_2) = M(\underline{d}_1, \infty) \\ &= \mathbb{E}[u(X_1 \wedge \underline{d}_1 + X_2)].\end{aligned}$$ □

Lemma 4.3. *Assume $u(x) \in C^1(\mathbb{R})$, i.e. $u'(x)$ is continuous on $\mathbb{R}$. Then $\frac{\partial^+}{\partial d_1} M(d_1, d_2)$ is right continuous in $d_1 \in (\underline{d}_1, \infty)$ and*

$$\begin{aligned}\frac{\partial^+}{\partial d_1} M(d_1, d_2) = \overline{F}_1(d_1)\Big(&\mathbb{E}[u'(d_1 + X_2 \wedge d_2) \mid X_1 > d_1] \\ &- \mathbb{E}[u'(d_2 + X_1 \wedge d_1) \mid X_2 > d_2]\Big).\end{aligned} \tag{4.2}$$ □

Now, applying the above preliminarily results, we can determine $(d_1^*, d_2^*) \in L$ such that

$$\mathbb{E}\Big[\big(X_1 \wedge d_1^* + X_2 \wedge d_2^*\big)^2\Big] = \inf_{(d_1,d_2)\in L} \mathbb{E}[(I^{d_1}(X_1) + I^{d_2}(X_2))^2], \tag{4.3}$$

$$\mathbb{E}\Big[\exp\{s(X_1 \wedge d_1^* + X_2 \wedge d_2^*)\}\Big] = \inf_{(d_1,d_2)\in L} \mathbb{E}[\exp\{s(I^{d_1}(X_1) + I^{d_2}(X_2))\}]. \tag{4.4}$$

Theorem 4.4. *Assume (X_1, X_2) is PDS and $\mathbb{E}\big[(X_1 + X_2)^2\big] < \infty$. For $d_1 \in (\underline{d}_1, \infty)$, define*

$$\begin{aligned}C_1(d_1) = \ &\mathbb{E}[(X_2 - L(d_1)) \wedge 0 \mid X_1 > d_1] \\ &- \mathbb{E}[(X_1 - d_1) \wedge 0 \mid X_2 > L(d_1)].\end{aligned}$$

Denote $r_1 = \sup\{d_1 \mid C_1(d_1) < 0\}$ and $r_2 = \inf\{d_1 \mid C_1(d_1) > 0\}$. Then $\underline{d}_1 < r_1 \le r_2 < \infty$ and for any $d_1^* \in [r_1, r_2]$, the retention vector $(d_1^*, L(d_1^*))$ is a solution to (4.3).

Proof. By setting $u(x) = x^2$ in (4.2) and noticing $d_2 = L(d_1)$, we have

$$\begin{aligned}\frac{\partial^+ M(d_1,d_2)}{\partial d_1} &= 2\,\overline{F}_1(d_1)\Big(\mathbb{E}[(d_1 + X_2\wedge d_2)\mid X_1 > d_1] \\ &\quad - \mathbb{E}[(d_2 + X_1 \wedge d_1)\mid X_2 > d_2]\Big)\\ &= 2\,\overline{F}_1(d_1)\Big(\mathbb{E}[(d_1 + X_2\wedge d_2) - (d_1+d_2)\mid X_1 > d_1] \\ &\quad - \mathbb{E}[(d_2 + X_1\wedge d_1) - (d_1+d_2)\mid X_2 > d_2]\Big)\\ &= 2\,\overline{F}_1(d_1)\Big(\mathbb{E}[(X_2 - d_2)\wedge 0 \mid X_1 > d_1] \\ &\quad - \mathbb{E}[(X_1 - d_1)\wedge 0\mid X_2 > d_2]\Big)\\ &= 2\,\overline{F}_1(d_1)\,C_1(d_1). \end{aligned} \tag{4.5}$$

Now we show that $C_1(d_1)$ is an increasing function of d_1 in $(\underline{d}_1, \infty)$. In doing so, let $d_1, d_1' \in (\underline{d}_1, \infty)$ and $d_1 < d_1'$. Since $X_2 \uparrow_{SI} X_1$, we have $X_2|(X_1 > d_1) \le_{\text{st}} X_2|(X_1 > d_1')$, see, for example, Barlow and Proschan (1981). Therefore, since $(x - L(d_1))\wedge 0$ is increasing in x and $L(d_1) > L(d_1')$, by the definition of $\le_{\text{st}}$, we have

$$\begin{aligned}&\mathbb{E}[(X_2 - L(d_1))\wedge 0 \mid X_1 > d_1]\\ &\quad\le \mathbb{E}[(X_2 - L(d_1))\wedge 0 \mid X_1 > d_1'] \\ &\quad\le \mathbb{E}[(X_2 - L(d_1'))\wedge 0 \mid X_1 > d_1'],\end{aligned} \tag{4.6}$$

which means that $\mathbb{E}[(X_2 - L(d_1))\wedge 0 \mid X_1 > d_1]$ is increasing in d_1. Similarly, since $(x - d_1)\wedge 0$ is increasing in x and $d_1' > d_1$, we have

$$\begin{aligned}\mathbb{E}[(X_1 - d_1)\wedge 0 \mid X_2 > L(d_1)] &\ge \mathbb{E}[(X_1 - d_1)\wedge 0 \mid X_2 > L(d_1')]\\ &\ge \mathbb{E}[(X_1 - d_1')\wedge 0 \mid X_2 > L(d_1')].\end{aligned}$$

Thus $\mathbb{E}[(X_1 - d_1)\wedge 0 \mid X_2 > L(d_1)]$ is decreasing in d_1. Therefore $C_1(d_1)$ is increasing in $d_1 \in (\underline{d}_1, \infty)$.

In the following, we examine the limits of $C_1(d_1)$ at two endpoints $\underline{d}_1$ and ∞ of the interval $(\underline{d}_1, \infty)$. For a fixed $d > d_1 > \underline{d}_1$, by (4.6), we have

$$\mathbb{E}[(X_2 - L(d_1))\wedge 0 \mid X_1 > d_1] \le \mathbb{E}[(X_2 - L(d_1))\wedge 0 \mid X_1 > d].$$

Then by the monotone convergence theorem, we have

$$\begin{aligned}&\lim_{d_1\downarrow \underline{d}_1} \mathbb{E}[(X_2 - L(d_1))\wedge 0 \mid X_1 > d_1]\\ &\quad\le \lim_{d_1\downarrow \underline{d}_1} \mathbb{E}[(X_2 - L(d_1))\wedge 0 \mid X_1 > d]\\ &\quad= \mathbb{E}[\lim_{d_1\downarrow \underline{d}_1}(X_2 - L(d_1))\wedge 0 \mid X_1 > d] = -\infty,\end{aligned} \tag{4.7}$$

where, the first limit exists because $\mathbb{E}[(X_2 - L(d_1))\wedge 0 \mid X_1 > d_1]$ is an increasing function of d_1 and the last equality follows from the fact that $\lim_{d_1\downarrow \underline{d}_1} L(d_1) = \infty$.

Since $X_1 \ge 0$, we have $\mathbb{E}[(X_1 - d_1)\wedge 0 \mid X_2 > L(d_1)] \ge \mathbb{E}[(-d_1)\wedge 0 \mid X_2 > L(d_1)] = -d_1$. Then, $\lim_{d_1\downarrow \underline{d}_1} \mathbb{E}[(X_1 - d_1)\wedge 0 \mid X_2 > L(d_1)] \ge \lim_{d_1\downarrow \underline{d}_1}(-d_1) = -\underline{d}_1$, which, together with (4.7) and the definition of $C_1(d_1)$, implies $\lim_{d_1\downarrow \underline{d}_1} C_1(d_1) = -\infty$. Thus, there exists $d_1 > \underline{d}_1$ such that $C(d_1) < 0$, which implies $\{d_1 \mid C_1(d_1) < 0\} \ne \emptyset$ and $r_1 = \sup\{d_1 \mid C_1(d_1) < 0\} > \underline{d}_1$.

Similarly, we have $\lim_{d_1\uparrow\infty} \mathbb{E}[(X_2 - L(d_1))\wedge 0 \mid X_1 > d_1] \ge -\underline{d}_2$ and $\lim_{d_1\uparrow\infty} \mathbb{E}[(X_1 - d_1)\wedge 0 \mid X_2 > L(d_1)] \le -\infty$. Therefore, $\lim_{d_1\uparrow\infty} C_1(d_1) = \infty$ and thus $\{d_1 \mid C_1(d_1) > 0\} \ne \emptyset$ and $r_2 = \inf\{d_1 \mid C_1(d_1) > 0\} < \infty$.

Since $C_1(d_1)$ is increasing in d_1, for any $x \in \{d_1 \mid C_1(d_1) < 0\}$, $y \in \{d_1 \mid C_1(d_1) > 0\}$, we have $x < y$, thus $r_1 = \sup\{d_1 \mid C_1(d_1) < 0\} \le \inf\{d_1 \mid C_1(d_1) > 0\} = r_2$. According to the definitions of r_1 and r_2, we have $C_1(d_1) < 0$ for all $d_1 \in (\underline{d}_1, r_1)$ and $C_1(d_1) > 0$ for all $d_1 \in (r_2, \infty)$. Moreover, if $d_1 > r_1$, then $C_1(d_1) \ge 0$; and if $d_1 < r_2$, then $C_1(d_1) \le 0$. Therefore, $C_1(d_1) = 0$ for all $d_1 \in (r_1, r_2)$.

By (4.5), we know that $\frac{\partial^+}{\partial d_1} M(d_1, L(d_1)) = 2\,\overline{F}_1(d_1)\,C(d_1)$ has the same sign as $C_1(d_1)$ on $(\underline{d}_1, \infty)$. Hence, $M(d_1, L(d_1))$ is strictly decreasing on $(\underline{d}_1, r_1)$, strictly increasing on (r_2, ∞), and a constant on (r_1, r_2) and thus a constant on $[r_1, r_2]$ since $M(d_1, L(d_1))$ is continuous. Therefore, $\inf_{d_1\in(\underline{d}_1,\infty)} M(d_1, L(d_1)) = M(d_1^*, L(d_1^*))$ for any $d_1^* \in [r_1, r_2]$. Notice that $M(d_1, L(d_1))$ is continuous in $d_1 \in (\underline{d}_1, \infty)$, strictly decreasing on $(\underline{d}_1, r_1)$, and strictly increasing on (r_2, ∞). Thus, according to Lemma 4.2, for any $d_1^* \in [r_1, r_2]$, $M(d_1^*, L(d_1^*)) < \lim_{d_1\to\infty} M(d_1, L(d_1)) = M(\infty, \underline{d}_2)$ and $M(d_1^*, L(d_1^*)) < \lim_{d_1\downarrow \underline{d}_1} M(d_1, L(d_1)) = M(\underline{d}_1, \infty)$. Hence, $\inf_{d_1\in[\underline{d}_1,\infty]} M(d_1, L(d_1)) = M(d_1^*, L(d_1^*))$ for any $d_1^* \in [r_1, r_2]$. It completes the proof of the theorem. □

Theorem 4.5. *Let $s > 0$ and assume (X_1, X_2) is PDS and $\mathbb{E}[\exp\{s(X_1 + X_2)\}] < \infty$. For $d_1 \in (\underline{d}_1, \infty)$, let*

$$\begin{aligned}C_2(d_1) &= \mathbb{E}[\exp\{s(X_2 - L(d_1))\wedge 0\} \mid X_1 > d_1]\\ &\quad - \mathbb{E}[\exp\{s(X_1 - d_1)\wedge 0\} \mid X_2 > L(d_1)].\end{aligned}$$

Denote $r_1 = \sup\{d_1 \mid C_2(d_1) < 0\}$ and $r_2 = \inf\{d_1 \mid C_2(d_1) > 0\}$. Then $\underline{d}_1 < r_1 \le r_2 < \infty$ and for any $d_1^* \in [r_1, r_2]$, the retention vector $(d_1^*, L(d_1^*))$ is a solution to (4.4).

Proof. By setting $u(x) = e^{sx}$ in (4.2) and noticing $d_2 = L(d_1)$, we have

$$\begin{aligned}\frac{\partial^+ M(d_1,d_2)}{\partial d_1} &= \overline{F}_1(d_1)\big(\mathbb{E}[s\,\exp\{s(X_2\wedge d_2)\}\mid X_1 > d_1]\\ &\quad - \mathbb{E}[s\,\exp\{s(X_1\wedge d_1)\}\mid X_2 > d_2]\big)\\ &= s\,e^{s(d_1+d_2)}\,\overline{F}_1(d_1)\Big(\mathbb{E}[\exp\{s(X_2 - d_2)\wedge 0\}\mid X_1 > d_1]\\ &\quad - \mathbb{E}[\exp\{s(X_1 - d_1)\wedge 0\}\mid X_2 > d_2]\Big)\\ &= s\,e^{s(d_1+d_2)}\,\overline{F}_1(d_1)\,C_2(d_1).\end{aligned}$$

Then, using the same arguments as in Theorem 4.4, we complete the proof. The same arguments are omitted. □

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Appendix

Proof of Lemma 4.1. To show that the mapping is one-to-one, it suffices to show that for any (d_1, d_2) and $(d_1', d_2') \in L$, $d_1 = d_1'$ if and only if $d_2 = d_2'$. First assume $d_1 = d_1'$, recall that

$$\begin{aligned}&\int_0^{d_1'} \overline{F}_1(x)dx + \int_0^{d_2'} \overline{F}_2(x)dx\\ &\quad= \int_0^{d_1} \overline{F}_1(x)dx + \int_0^{d_2} \overline{F}_2(x)dx = p,\end{aligned} \tag{A.1}$$

we have $\int_0^{d_2'} \overline{F}_2(x)dx = \int_0^{d_2} \overline{F}_2(x)dx$, or $\int_{d_2}^{d_2'} \overline{F}_2(x)dx = 0$, which implies $d_2 = d_2'$ since $\overline{F}_2(s) > 0$, $\forall x \in \mathbb{R}$. Similarly, $d_2 = d_2'$ implies $d_1 = d_1'$. Therefore, on L, there is a one-to-one mapping from d_1 to d_2.

Differentiating the second equation in (A.1) with respect to d_1 on both sides, we have $\overline{F}_1(d_1) + \overline{F}_2(d_2)\frac{\partial d_2}{\partial d_1} = 0$, which implies that $\frac{\partial d_2}{\partial d_1} = -\frac{\overline{F}_1(d_1)}{\overline{F}_2(d_2)} < 0$. Thus $L(d_1)$ is strictly decreasing. □

Proof of Lemma 4.2. Since $u(x)$ is monotonic, and $X_1, X_2 \geq 0$, then $|u(X_1 \wedge d_1 + X_2 \wedge d_2)|$ is bounded from above by either $|u(0)|$ or $|u(X_1 + X_2)|$, both of which are integrable. Therefore, according to Lebesgue dominated convergence theorem, for any $d_1 \in (\underline{d}_1, \infty)$, we have

$$\begin{aligned}\lim_{s\to d_1} M(s, L(s)) &= \mathbb{E}[\lim_{s\to d_1} u(X_1 \wedge s + X_2 \wedge L(s))] \\ &= \mathbb{E}[u(X_1 \wedge d_1 + X_2 \wedge L(d_1))] = M(d_1, L(d_1)),\end{aligned}$$

which means that $M(s, L(s))$ is continuous at d_1.

Similarly,

$$\begin{aligned}&\lim_{d_1\to\infty} M(d_1, L(d_1)) \\ &= \lim_{d_1\to\infty} \mathbb{E}[u(X_1 \wedge d_1 + X_2 \wedge L(d_1))] \\ &= \mathbb{E}[\lim_{d_1\to\infty} u(X_1 \wedge d_1 + X_2 \wedge L(d_1))] \\ &= \mathbb{E}[u(X_1 + X_2 \wedge L(\infty))] = \mathbb{E}[u(X_1 + X_2 \wedge \underline{d}_2)],\end{aligned}$$

and

$$\begin{aligned}\lim_{d_1\downarrow \underline{d}_1} M(d_1, L(d_1)) &= \lim_{d_2\to\infty} M(L^{-1}(d_2), d_2) \\ &= \lim_{d_2\to\infty} \mathbb{E}[u(X_1 \wedge L^{-1}(d_2) + X_2 \wedge d_2)] \\ &= \mathbb{E}[\lim_{d_2\to\infty} u(X_1 \wedge L^{-1}(d_2) + X_2 \wedge d_2)] \\ &= \mathbb{E}[u(X_1 \wedge L^{-1}(\infty) + X_2)] \\ &= \mathbb{E}[u(X_1 \wedge \underline{d}_1 + X_2)]. \quad \square\end{aligned}$$

Proof of Lemma 4.3. Denote $f(\omega, s) = u(X_1(\omega)\wedge s + X_2(\omega)\wedge L(s))$, then $M(d_1, d_2) = \mathbb{E}[f(\omega, d_1)] = \int_\Omega f(\omega, s)\mathbb{P}(d\omega)$. Notice that for any fixed $\omega \in \Omega$, the right derivative of $f(\omega, s)$ with respect to s exists for any $s \in (\underline{d}_1, \infty)$ and

$$\begin{aligned}\frac{\partial^+}{\partial s} f(\omega, s) &= u'(X_1 \wedge s + X_2 \wedge L(s)) \\ &\quad \times \left(\mathbb{I}\{X_1 > s\} + \mathbb{I}\{X_2 > L(s)\}\frac{\partial L(s)}{\partial s}\right).\end{aligned}$$

Let $[a, d_1] \subset (\underline{d}_1, \infty)$, then for any $(\omega, s) \in \Omega \times [a, d_1]$, we have $0 \leq X_1 \wedge s + X_2 \wedge L(s) \leq s + L(s) \leq d_1 + L(a) < \infty$, since $L(s)$ is decreasing. Therefore $u'(X_1\wedge s + X_2\wedge L(s))$ is bounded on $\Omega \times [a, d_1]$ since $u'(x)$ is continuous and thus bounded on the closed interval $[0,\ d_1 + L(a)]$. Also, by Lemma 4.1, we have

$$\begin{aligned}\left|\mathbb{I}\{X_1 > s\} + \mathbb{I}\{X_2 > L(s)\}\frac{\partial L(s)}{\partial s}\right| &\leq 1 + \left|\frac{\partial L(s)}{\partial s}\right| \\ &= 1 + \frac{\overline{F}_1(s)}{\overline{F}_2(L(s))} \\ &\leq 1 + \frac{\overline{F}_1(a)}{\overline{F}_2(L(a))} < \infty.\end{aligned}$$

Therefore, $\frac{\partial^+}{\partial s} f(\omega, s)$ is bounded on $\Omega \times [a, d_1]$. Denote the bound as A, then

$$\int_a^{d_1} \mathbb{E}\left[\left|\frac{\partial^+}{\partial s} f(\omega, s)\right|\right] ds \leq A\,(d_1 - a) < \infty.$$

According to Fubini's theorem, we could exchange the order of integration and expectation:

$$\int_a^{d_1} \mathbb{E}\left[\frac{\partial^+}{\partial s} f(\omega, s)\right] ds = \mathbb{E}\left[\int_a^{d_1} \frac{\partial^+}{\partial s} f(\omega, s) ds\right].$$

For any fixed $\omega \in \Omega$, it is easy to verify that $u(x)$ and $g(s) = X_1(\omega) \wedge s + X_2(\omega) \wedge L(s)$ satisfies Lipschitz condition on $[0,\ d_1 + L(a)]$ and on $[a,\ d_1]$ respectively. Therefore $f(\omega, s) = u \circ g(s)$ also satisfies Lipschitz condition on $[a, d_1]$, and thus is absolute continuous on $[a, d_1]$. Then $f(\omega, s)$ is differentiable with respect to s almost everywhere on $[a, d_1]$, and the derivative is equal to the right derivative. By Fundamental Theorem Π of Lebesgue integral, we have

$$\int_a^{d_1} \frac{\partial^+}{\partial s} f(\omega, s) ds = \int_a^{d_1} \frac{\partial}{\partial s} f(\omega, s) ds = f(\omega, d_1) - f(\omega, a).$$

Therefore,

$$\begin{aligned}&\int_a^{d_1} \mathbb{E}\left[\frac{\partial^+}{\partial s} f(\omega, s)\right] ds \\ &= \mathbb{E}\left[\int_a^{d_1} \frac{\partial^+}{\partial s} f(\omega, s) ds\right] \\ &= \mathbb{E}[f(\omega, d_1) - f(\omega, a)] = M(d_1, d_2) - \mathbb{E}[f(\omega, a)]. \qquad \text{(A.2)}\end{aligned}$$

Since $\frac{\partial^+}{\partial s} f(\omega, s)$ is right continuous in s and is bounded on $[a, d_1]$, according to Lebesgue dominated convergence theorem, we have $\mathbb{E}\left[\frac{\partial^+}{\partial s} f(\omega, s)\right]$ is right continuous in s.

It is easy to show that if $g(x)$ is right continuous and integrable on closed interval I and $G(x) = \int_a^x g(t)dt$, where $a \in I$, then $\frac{\partial^+}{\partial x} G(x) = g(x)$, $\forall x \in I$. Thus, taking right derivative on both sides of (A.2), we get

$$\begin{aligned}&\frac{\partial^+}{\partial d_1} M(d_1, d_2) \\ &= \frac{\partial^+}{\partial d_1} \int_a^{d_1} \mathbb{E}\left[\frac{\partial^+}{\partial s} f(\omega, s)\right] ds = \mathbb{E}\left[\frac{\partial^+}{\partial d_1} f(X, d_1)\right] \\ &= \mathbb{E}\left[u'(X_1 \wedge d_1 + X_2 \wedge d_2)\left(\mathbb{I}\{X_1 > d_1\} + \mathbb{I}\{X_2 > d_2\}\frac{\partial d_2}{\partial d_1}\right)\right] \\ &= \mathbb{E}[u'(d_1 + X_2 \wedge d_2)\,\mathbb{I}\{X_1 > d_1\}] \\ &\quad - \frac{\overline{F}_1(d_1)}{\overline{F}_2(d_2)}\,\mathbb{E}[u'(X_1 \wedge d_1 + d_2)\,\mathbb{I}\{X_2 > d_2\}] \\ &= \overline{F}_1(d_1)\Big(\mathbb{E}[u'(d_1 + X_2 \wedge d_2) \mid X_1 > d_1] \\ &\quad - \mathbb{E}[u'(d_2 + X_1 \wedge d_1) \mid X_2 > d_2]\Big). \qquad \text{(A.3)}\end{aligned}$$

The last equality follows from the fact that $\mathbb{E}[X\,\mathbb{I}\{Y \in B\}] = \mathbb{E}[X \mid Y \in B]\,\mathbb{P}\{Y \in B\}$ if $\mathbb{P}\{Y \in B\} > 0$. The right continuity of $\frac{\partial^+}{\partial d_1} M(d_1, d_2)$ is from (A.3). □

References

Aboudi, R., Thon, D., 1995. Second-degree stochastic dominance decisions and random initial wealth with application to the economics of insurance. The Journal of Risk and Insurance 62 (1), 30–49.

Barlow, R.E., Proschan, F., 1981. Statistical Theory of Reliability and Life Testing: Probability Models. To Begin With, Silver Spring, MD.

Block, H.W., Savits, T.H., Shaked, M., 1985. A concept of negative dependence using stochastic ordering. Statistics and Probability Letters 3 (2), 81–86.

Colangelo, A., Hu, T.Z., Shaked, M., 2008. Conditional orderings and positive dependence. Journal of Multivariate Analysis 99 (3), 358–371.

Colangelo, A., Scarsini, M., Shaked, M., 2005. Some notions of multivariate positive dependence. Insurance: Mathematics and Economics 37 (1), 13–26.

Denuit, M., Dhaene, J., Goovaerts, M., Kaas, R., 2005. Actuarial Theory for Dependent Risks: Measures, Orders and Models. John Wiley & Sons, Chichester.

Denuit, M., Vermandele, C., 1998. Optimal reinsurance and stopl-loss order. Insurance: Mathematics and Economics 22, 229–233.
Denuit, M., Vermandele, C., 1999. Lorenz and excess wealth orders, with applications in reinsurance theory. Scandinavian Actuarial Journal (2), 170–185.
Joe, H., 1997. Multivariate Models and Dependence Concepts. Chapman & Hall, New York.
Lehmann, E.L., 1966. Some concepts of dependence. Annals of Mathematical Statistics 37 (5), 1137–1153.
McNeil, A.J., Frey, R., Embrechts, P., 2005. Quantitative Risk Management: Concepts, Techniques and Tools. Princeton University Press, Princeton.
Müller, A., Stoyan, D., 2002. Comparison Methods for Stochastic Models and Risks. John Wiley & Sons, New York.
Ohlin, J., 1969. On a class of measures of dispersion with application to optimal reinsurance. ASTIN Bulletin 5 (2), 249–266.
Shaked, M., 1977. A family of concepts of dependence for bivariate distributions. Journal of American Statistical Association 72 (359), 642–650.
Shaked, M., Shanthikumar, J.G., 2007. Stochastic Orders. Springer, New York.
Van Heerwaarden, A.E., Kaas, R., Goovaerts, M.J., 1989. Optimal reinsurance in relation to ordering of risks. Insurance: Mathematics and Economics 8, 11–17.