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On the invariant properties of notions of positive dependence and copulas under increasing transformations

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ABSTRACT

Notions of positive dependence and copulas play important roles in modeling dependent risks. The invariant properties of notions of positive dependence and copulas under increasing transformations are often used in the studies of economics, finance, insurance and many other fields. In this paper, we examine the notions of the conditionally increasing (CI), the conditionally increasing in sequence (CIS), the positive dependence through the stochastic ordering (PDS), and the positive dependence through the upper orthant ordering (PDUO). We first use counterexamples to show that the statements in Theorem 3.10.19 of Müller and Stoyan (2002) about the invariant properties of CIS and CI under increasing transformations are not true. We then prove that the invariant properties of CIS and CI hold under strictly increasing transformations. Furthermore, we give rigorous proofs for the invariant properties of PDS and PDUO under increasing transformations. These invariant properties enable us to show that a continuous random vector is PDS (PDUO) if and only of its copula is PDS (PDUO). In addition, using the properties of generalized left-continuous and right-continuous inverse functions, we give a rigorous proof for the invariant property of copulas under increasing transformations on the components of any random vector. This result generalizes Proposition 4.7.4 of Denuit et al. (2005) and Proposition 5.6. of McNeil et al. (2005).

1. Introduction

Notions of positive dependence and copulas play important roles in modeling dependent risks. The invariant properties of notions of positive dependence and copulas under increasing transformations are often used in the studies of economics, finance, insurance and many other fields. Throughout this paper, 'increasing' means 'non-decreasing' and 'decreasing' means 'non-increasing'. In the literature, some of these invariant properties have been proved, while some were stated without proofs and have been assumed to be true.

In this paper, we examine the notions of the conditionally increasing (CI), the conditionally increasing in sequence (CIS), the positive dependence through the stochastic ordering (PDS), and the positive dependence through the upper orthant ordering (PDUO). The definitions of these notions will be defined later.

We first use two counterexamples to show that the statements in Theorem 3.10.19 of Müller and Stoyan (2002) about the invariant properties of CI and CIS under increasing transformations are not true. The counterexamples motivate us to verify the statements of Theorem 3.10.19 of Müller and Stoyan (2002) about the invariant properties of other notions of positive dependence under increasing transformations. Actually, it is easy to prove that most of the notions of positive dependence mentioned in Theorem 3.10.19 of Müller and Stoyan (2002) are preserved under increasing transformations. However, it is not easy to verify if those notions defined by using conditional expectations or conditional survival functions, such as CI, CIS, and PDS, are preserved under increasing transformations. It is straightforward to show that PDS is preserved under strictly increasing transformations. Indeed, Theorem 2.1 of Block et al. (1985) states that the negative dependence through

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the stochastic ordering (NDS), which is the counterpart of PDS, is preserved under strictly increasing transformations. To the best of our knowledge, the proof for the invariant property of PDS is not available. Indeed, the proof is not trivial and we need to prove several preliminary results.

As we know that there are connections between the notions of positive dependence of a random vector and its copula. A copula is the joint distribution function of a uniform random vector $(U_1, \dots, U_n)$ defined on $[0, 1]^n$. For any random vector $(X_1, \dots, X_n)$ with marginal distribution functions $F_i(x_i) = \mathbb{P}\{X_i \le x_i\}$, $i = 1, 2, \dots, n$, there exists a copula C such that the joint distribution function of $(X_1, \dots, X_n)$ can be expressed as the function of its marginal distributions through the copula C, namely, for any $(x_1, \dots, x_n) \in \mathbb{R}^n$,

$$\mathbb{P}\{X_1 \le x_1, \dots, X_n \le x_n\} = C(F_1(x_1), \dots, F_n(x_n)).$$

Such a copula C is called a copula of the random vector $(X_1, \dots, X_n)$ or its joint distribution function. In particular, if $(X_1, \dots, X_n)$ has continuous marginal distribution functions, then the joint distribution function of $(F_1(X_1), \dots, F_n(X_n))$ is its unique copula.

Under certain conditions, some notions of positive dependence of a random vector are the properties of its copula in the sense that a random vector has a notion of positive dependence if and only if its copula has the notion. In addition, some notions of positive dependence of a random vector can be characterized by its copula. Actually, we will show that a random vector $(X_1, \dots, X_n)$ is PDS (PUDO) if and only if $(F_1(X_1), \dots, F_n(X_n))$ is PDS (PDUO). Consequently, if $(X_1, \dots, X_n)$ has the continuous marginal distribution functions, then $(X_1, \dots, X_n)$ is PDS (PDUO) if and only if its copula is PDS (PDUO).

A very useful property of a copula is the invariance under strictly increasing transformations on the components of a continuous random vector or under increasing and continuous transformations on the components of any random vector. See, for example, Proposition 5.6. of McNeil et al. (2005), Proposition 4.7.4 of Denuit et al. (2005), Theorem 3.4.3 of Nelsen (2006), and Theorem 2.8 of Cherubini et al. (2004). Using the properties of generalized left-continuous and right-continuous inverse functions, we give rigorous proofs for the invariant properties of copulas under increasing transformations on the components of any random vector.

The invariant properties of notions of positive dependence and copulas under increasing transformations are often used in the studies of economics, finance, insurance and many other fields. It is necessary for one to give a detailed study of these invariant properties. In this paper, '$=_{st}$' means 'equal in distribution'.

The rest of the paper is organized as follows. In Section 2, we revisit several notions of positive dependence including the stochastically increasing (SI), CI, CIS, PDS, and PDUO. We use two counterexamples to show that the statements in Theorem 3.10.19 of Müller and Stoyan (2002) about the invariant properties of CI and CIS under increasing transformations are not true. We prove that CIS and CI are preserved under strictly increasing transformations. We give rigorous proofs for the invariant properties of SI, PDS, and PDUO under increasing transformations. These invariant properties enable us to show that a continuous random vector is PDS (PDUO) if and only if its copula is PDS (PDUO). In Section 3, using the properties of the generalized inverse functions, we also give a rigorous proof for the invariant property of copulas under increasing transformations on any random vector. This result generalizes Proposition 5.6. of McNeil et al. (2005) and Proposition 4.7.4 of Denuit et al. (2005). In Section 4, we give the characterization of PDUO in terms of survival copulas.

2. The invariant properties of the notions of positive dependence

In the literature, there are several notions of positive dependence, which describe positive dependence for two random variables or two random vectors. We refer to Müller and Stoyan (2002) and Shaked and Shanthikumar (2007) for a detailed treatment of these topics and to Denuit et al. (2005) for their applications in actuarial science and insurance. In this section, we will focus on the notions of SI, CI, CIS, PDS, PDUO. The notions of SI, CI, CIS, and PDS and their properties can be found in Block et al. (1985), Joe (1997), Shaked (1977), and references therein. The PDUO will be defined in this section. More notions of positive dependence can be found in Colangelo et al. (2005) and references therein. A new characterization of CIS is given in Fernández-Poncea et al. (2011).

We recall that for a random vector $\mathbf{Y} = (Y_1, \dots, Y_n)$, a support of $\mathbf{Y}$, denoted by $S(\mathbf{Y})$ or $S(Y_1, \dots, Y_n)$, is a Borel set of $\mathbb{R}^n$ such that $\mathbb{P}\{\mathbf{Y} \in S(\mathbf{Y})\} = 1$.

Definition 2.1. Let $(X_1, \dots, X_n)$ be a random vector and Y be a random variable.

(1) Y is said to be stochastically increasing (SI) in random vector $(X_1, \dots, X_n)$, denoted as $Y \uparrow_{SI} (X_1, \dots, X_n)$, if $\mathbb{P}\{Y > y \mid X_1 = x_1, \dots, X_n = x_n\}$ is increasing in $(x_1, \dots, x_n) \in S(X_1, \dots, X_n)$ for all $y \in \mathbb{R}$, or equivalently, $Y \uparrow_{SI} (X_1, \dots, X_n)$ if and only if $\mathbb{E}[u(Y) \mid X_1 = x_1, \dots, X_n = x_n]$ is increasing in $(x_1, \dots, x_n) \in S(X_1, \dots, X_n)$ for any increasing function u such that the conditional expectation exists.

(2) $(X_1, \dots, X_n)$ is said to be stochastically increasing (SI) in random variable Y, denoted as $(X_1, \dots, X_n) \uparrow_{SI} Y$, if $\mathbb{E}[u(X_1, \dots, X_n) \mid Y = y]$ is increasing in $y \in S(Y)$ for any increasing function $u : \mathbb{R}^n \to \mathbb{R}$ such that the conditional expectation exists.

(3) $(X_1, \dots, X_n)$ is said to be conditionally increasing in sequence (CIS) if $X_i \uparrow_{SI} (X_1, \dots, X_{i-1})$ for all $i = 2, \dots, n$.

(4) $(X_1, \dots, X_n)$ is said to be positively dependent through the stochastic ordering (PDS) if $(X_1, \dots, X_{i-1}, X_{i+1}, \dots, X_n) \uparrow_{SI} X_i$ for all $i = 1, \dots, n$.

(5) $(X_1, \dots, X_n)$ is said to be conditionally increasing (CI) if $(X_{\pi(1)}, \dots, X_{\pi(n)})$ is CIS for all permutations π of $(1, \dots, n)$. □

The natural extensions of $(X_1, \dots, X_n) \uparrow_{SI} Y$ and PDS are to define a notion of positive dependence by using the weaker condition of the conditional survival function $\Pr\{X_1 > x_1, \dots, X_n > x_n | Y = y\}$ instead of using the stronger condition of the condition expectation $\mathbb{E}[u(X_1, \dots, X_n) | Y = y]$. Thus, we can define the weaker notions of positive dependence than SI and PDS.

We first recall the definition of the upper orthant order.

Definition 2.2. An n-dimensional random vector $\mathbf{X} = (X_1, \dots, X_n)$ is said to be smaller than an n-dimensional random vector $\mathbf{Y} = (Y_1, \dots, Y_n)$ in the upper orthant order, denoted as $\mathbf{X} \le_{uo} \mathbf{Y}$, if $\mathbb{P}\{X_1 > x_1, \dots, X_n > x_n\} \le \mathbb{P}\{Y_1 > x_1, \dots, Y_n > x_n\}$ for any $(x_1, \dots, x_n) \in \mathbb{R}^n$. □

Definition 2.3. Let $\mathbf{X} = (X_1, \dots, X_n)$ be a random vector and Y be a random variable.

(1) $(X_1, \dots, X_n)$ is said to be weakly stochastically increasing (WSI) in Y, denoted as $(X_1, \dots, X_n) \uparrow_{WSI} Y$, if $\mathbb{P}\{X_1 > x_1, \dots, X_n > x_n \mid Y = y\}$ is increasing in $y \in S(Y)$ for any $(x_1, \dots, x_n) \in \mathbb{R}^n$.

(2) $(X_1, \dots, X_n)$ is said to be positively dependent through the upper orthant ordering (PDUO) if $(X_1, \dots, X_{i-1}, X_{i+1}, \dots, X_n) \uparrow_{WSI} X_i$ for all $i = 1, 2, \dots, n$. □

Note that $\mathbf{X}\uparrow_{WSI}Y$ is equivalent to $\mathbf{X} \mid Y = y_1 \leq_{uo} \mathbf{X} \mid Y = y_2$ for any $y_1, y_2 \in S(Y)$ with $y_1 < y_2$. It is clear that for two random variables X and Y, $X\uparrow_{WSI}Y$ is equivalent to $X\uparrow_{SI}Y$, and for a bivariate random vector (X_1, X_2), (X_1, X_2) is PDUO if and only if (X_1, X_2) is PDS. In general, we have SI $\Longrightarrow$ WSI and PDS $\Longrightarrow$ PDUO. In addition, we will see that PDUO can be characterized by the survival copulas for continuous random vectors. Hence, it is easy to construct a continuous PDUO random vector by copulas.

From the definitions, we know that CI, CIS, PDS, and PDUO describe the notions of positive dependence for a random vector and are defined by using conditional expectations or conditional survival functions. We summarize their implications as follows:

CI $\Longrightarrow$ CIS

and

CI $\Longrightarrow$ PDS $\Longrightarrow$ PDUO.

Theorem 3.10.19 of Müller and Stoyan (2002) states (without proofs) that several common notions of positive dependence including CIS, CI and PDS are preserved under increasing transformations. We first give two counterexamples, which show that the statements of Theorem 3.10.19 of Müller and Stoyan (2002) about CIS and CI are not true.

Example 2.4 (*CIS is Not Preserved Under General Increasing Transformations*)**.** Let X and Y be two independent random variables. Then it always holds that $X + Y\uparrow_{SI}X$. Now, assume that X and Y have the following probability functions: $\mathbb{P}(X = 0) = \mathbb{P}(X = 1) = 0.5$, $\mathbb{P}(Y = 0) = 0.4$, $\mathbb{P}(Y = 1) = 0.2$, and $\mathbb{P}(Y = 2) = 0.4$. Then it is easy to check that

$$\mathbb{P}(X > 0 \mid X + Y = 1) = \mathbb{P}(X = 1 \mid X + Y = 1) = 2/3,$$
$$\mathbb{P}(X > 0 \mid X + Y = 2) = \mathbb{P}(X = 1 \mid X + Y = 2) = 1/3.$$

Then $\mathbb{E}[X \mid X + Y = 1] = 2/3 > \mathbb{E}[X \mid X + Y = 2] = 1/3$, which means $\mathbb{E}[X \mid X + Y]$ is not increasing in $X + Y$.

Let $X_1 = X, X_2 = X + Y$, and $X_3 = X$. Then (X_1, X_2, X_3) is CIS since $X_2 = X + Y\uparrow_{SI}X_1 = X$ and $\mathbb{E}[u(X_3) \mid X_1 = x_1, X_2 = x_2] = \mathbb{E}[u(X) \mid X = x_1, X + Y = x_2] = u(x_1)$ is increasing in x_1 and x_2 for any increasing function u. Now, consider the increasing transformations of $f_1(x) = 1$ and $f_2(x) = f_3(x) = x$, then $(f_1(X_1), f_2(X_2), f_3(X_3)) = (1, X + Y, X)$ is not CIS since $\mathbb{E}[f_3(X_3) \mid f_1(X_1), f_2(X_2)] = \mathbb{E}[X \mid 1, X + Y] = \mathbb{E}[X \mid X + Y]$ is not increasing in $X+Y$. Therefore, CIS is not preserved under increasing transformations. □

Example 2.5 (*CI is Not Preserved Under General Increasing Transformations*)**.** Assume the conditional distribution of X, conditioning on Y, is given by the first table below. For instance, from the table, $\mathbb{P}\{X = 1 \mid Y = 2\} = 0.2$. Assume the marginal distribution of Y is $\mathbb{P}\{Y = i\} = 1/3, i = 0, 1, 2$. Thus, the conditional distribution of Y, conditioning on X, is given by the second table below. For example, from the table, $\mathbb{P}\{Y = 2 \mid X = 1\} = 2/7$.

$X \mid Y$	0	1	2
0	0.4	0.2	0.4
1	0.2	0.3	0.5
2	0.2	0.2	0.6

$Y \mid X$	0	1	2
0	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{4}$
1	$\frac{2}{7}$	$\frac{3}{7}$	$\frac{2}{7}$
2	$\frac{4}{15}$	$\frac{1}{3}$	$\frac{2}{5}$

It is easy to verify that $X\uparrow_{SI}Y$ and $Y\uparrow_{SI}X$. Consider the random vector $\mathbf{V} = (V_1, V_2, V_3) = (X, Y, X)$. Obviously, $\mathbf{V} = (V_1, V_2, V_3)$ is CI. Consider the increasing transformations of $f_1(x) = (x - 1)_+$, $f_2(x) = f_3(x) = x$. Now we examine the CI property of the random vector $(f_1(V_1), f_2(V_2), f_3(V_3)) = ((X - 1)_+, Y, X)$.

Note that

$$\mathbb{E}[X \mid ((X - 1)_+, Y) = (0, 1)] = \mathbb{E}[X \mid 0 \leq X \leq 1, Y = 1] = 0.6,$$
$$\mathbb{E}[X \mid ((X - 1)_+, Y) = (0, 2)] = \mathbb{E}[X \mid 0 \leq X \leq 1, Y = 2] = 0.5 < 0.6,$$

which means $\mathbb{E}[X|((X - 1)_+, Y) = (x, y)]$ is not increasing in x and y, and thus X is not stochastically increasing in $((X - 1)_+, Y)$. Therefore, $(f_1(V_1), f_2(V_2), f_3(V_3)) = ((X - 1)_+, Y, X)$ is not CI. □

However, we can show that the invariant properties of CIS and CI hold under strictly increasing transformations. Furthermore, we give rigorous proofs about the invariant properties of PDS and PDUO under increasing transformations.

Definition 2.6. For an increasing function $g : \mathbb{R} \to \mathbb{R}$, we denote the generalized left-continuous inverse function of g by $g^{-1} : \mathbb{R} \to [-\infty, \infty]$ and the generalized right-continuous inverse function of g by $g^{-1+} : \mathbb{R} \to [-\infty, \infty]$, which are defined as $g^{-1}(y) = \inf\{x \mid g(x) \geq y\}$ and $g^{-1+}(y) = \sup\{x \mid g(x) \leq y\}$ with the convention $\inf\{\emptyset\} = \infty$ and $\sup\{\emptyset\} = -\infty$. □

Proposition 2.7. *Assume random vector $(X_1, \ldots, X_n)$ is CIS. Then for any strictly increasing functions f_i, $i = 1, \ldots, n$, random vector $(f_1(X_1), \ldots, f_n(X_n))$ is also CIS.*

Proof. For any $k \in \{2, 3, \ldots, n\}$, we have $\sigma(f_1(X_1), \ldots, f_{k-1}(X_{k-1})) \subset \sigma(X_1, \ldots, X_{k-1})$. Thus,

$$\begin{aligned}&\mathbb{E}[f_k(X_k) \mid f_1(X_1), \ldots, f_{k-1}(X_{k-1})]\\ &\quad = \mathbb{E}[\mathbb{E}[f_k(X_k) \mid X_1, \ldots, X_{k-1}] \mid f_1(X_1), \ldots, f_{k-1}(X_{k-1})]\\ &\quad = \mathbb{E}[h_k(X_1, \ldots, X_{k-1}) \mid f_1(X_1), \ldots, f_{k-1}(X_{k-1})],\end{aligned} \tag{2.1}$$

where $h_k(x_1, \ldots, x_{k-1}) = \mathbb{E}[f_k(X_k) \mid X_1 = x_1, \ldots, X_{k-1} = x_{k-1}]$. Since $X_k\uparrow_{SI}(X_1, \ldots, X_{k-1})$, by the definition of $\uparrow_{SI}$, we know that h_k is increasing in each argument. Recall that f_i is strictly increasing. Thus, f_i^{-1} is increasing and $f_i^{-1}(f_i(x)) = x$. Therefore, by (2.1), we have

$$\begin{aligned}&\mathbb{E}[f_k(X_k) \mid f_1(X_1), \ldots, f_{k-1}(X_{k-1})]\\ &\quad = \mathbb{E}[h_k(f_1^{-1}(f_1(X_1)), \ldots, f_{k-1}^{-1}(f_{k-1}(X_{k-1})))\\ &\qquad \mid f_1(X_1), \ldots, f_{k-1}(X_{k-1})]\\ &\quad = h_k(f_1^{-1}(f_1(X_1)), \ldots, f_{k-1}^{-1}(f_{k-1}(X_{k-1})))\\ &\quad = g(f_1(X_1), \ldots, f_{k-1}(X_{k-1})),\end{aligned} \tag{2.2}$$

where $g(x_1, \ldots, x_{k-1}) = h_k(f_1^{-1}(x_1), \ldots, f_{k-1}^{-1}(x_{k-1}))$ is increasing in each argument, which means $f_k(X_k)\uparrow_{SI}(f_1(X_1), \ldots, f_{k-1}(X_{k-1}))$. It is interesting to note that the step (2.2) requires strict increasingness and rules out increasingness. □

Corollary 2.8. *Assume random vector $(X_1, \ldots, X_n)$ is CI. Then for any strictly increasing functions $f_i, i = 1, \ldots, n$, random vector $(f_1(X_1), \ldots, f_n(X_n))$ is also CI.*

Proof. It is straightforward from Proposition 2.7 and the definition of CI. □

For a set $A \subseteq \mathbb{R}$, we denote the inverse image of the set A under function $g : \mathbb{R} \to \mathbb{R}$ by $g^{-1}(A) = \{x \in \mathbb{R} \mid g(x) \in A\}$. Thus, for any $y \in \mathbb{R}$, $g^{-1}(\{y\}) = \{x \in \mathbb{R} \mid g(x) = y\}$.

For an increasing function g, we define the following three sets:

$$F_0 = \{y \in \mathbb{R} \mid g^{-1}(\{y\}) = \emptyset\} = \{y \in \mathbb{R} \mid \text{ a point does not exist } x \in \mathbb{R} \text{ such that } g(x) = y\},$$
$$F_1 = \{y \in \mathbb{R} \mid g^{-1}(\{y\}) \text{ contains exactly one element}\} = \{y \in \mathbb{R} \mid \text{ there exists exactly one point } x \in \mathbb{R} \text{ such that } g(x) = y\},$$
$$F_2 = \{y \in \mathbb{R} \mid g^{-1}(\{y\}) \text{ contains more than one element}\} = \{y \in \mathbb{R} \mid \text{ there exist more than one point } x \in \mathbb{R} \text{ such that } g(x) = y\}.$$

Moreover, for an increasing function g, we recall that g has at most countably many points of discontinuities and that if g is discontinuous at x, then the left and right limits of g at x exist with $g(x-) < g(x+)$. Furthermore, since g is increasing, the sets F_0, F_1, F_2 are mutually disjoint and $F_0 \cup F_1 \cup F_2 = \mathbb{R}$. Note that if $g(x) \in F_1$, then $g^{-1}(g(x)) = x$.

Lemma 2.9. *If g is an increasing function, then the set F_2 is countable.*

Proof. For any $y \in F_2$, there exist two points $x_1(y) < x_2(y)$ in $\mathbb{R}$ such that $x_1(y), x_2(y) \in g^{-1}(\{y\})$, then $g(x_1(y)) = g(x_2(y)) = y$. Thus $g(x) = y$ for any $x \in (x_1(y), x_2(y))$ since g is increasing. Note that for any $y_1 \neq y_2 \in F_2$, the open intervals $(x_1(y_1), x_2(y_1))$ and $(x_1(y_2), x_2(y_2))$ are disjoint. Therefore, there is a one-to-one mapping between F_2 and the set of the mutually disjoint open intervals of $\mathbb{R}$ and thus F_2 is countable. □

For increasing function g and random variables X and $Y = g(X)$, denote $F_3 = \{y \in F_2 \mid \mathbb{P}\{Y = y\} > 0\}$ and $F_4 = \{y \in F_2 \mid \mathbb{P}\{Y = y\} = 0\}$, then F_3 and F_4 are disjoint and $F_3 \cup F_4 = F_2$. From Lemma 2.9, we know that F_4 is countable. Thus, $\mathbb{P}\{Y \in F_4\} = 0$. Note that $\mathbb{P}\{Y \in F_0\} = \mathbb{P}\{X \in \emptyset\} = 0$ and $F_0 \cup F_1 \cup F_3 \cup F_4 = \mathbb{R}$. Hence, $\mathbb{P}\{Y \in F_1 \cup F_3\} = 1 - \mathbb{P}\{Y \in F_4\} - \mathbb{P}\{Y \in F_0\} = 1$.

Furthermore, for any function u such that $\mathbb{E}[|u(X)|] < \infty$ and for $y \in F_3$, we define

$$q_u(y) = \frac{\mathbb{E}[u(X)\,\mathbb{I}\{Y = y\}]}{\mathbb{P}\{Y = y\}}.$$

Proposition 2.10. *Let $\mathbb{E}[|u(X)|] < \infty$ and g be an increasing function. Then*

$$\mathbb{E}[u(X) \mid g(X)] = u(X)\,\mathbb{I}\{g(X) \in F_1\} + q_u(g(X))\,\mathbb{I}\{g(X) \in F_3\}. \tag{2.3}$$

Proof. Let X be defined on the probability space $(\Omega, \mathfrak{F}, \mathbb{P})$. Note that if $g(X) \in F_1$, then $g^{-1}(g(X)) = X$. Denote $Y = g(X)$ and

$$m_u(Y) = u(g^{-1}(Y))\,\mathbb{I}\{Y \in F_1\} + q_u(Y)\,\mathbb{I}\{Y \in F_3\}. \tag{2.4}$$

Thus, according to the definition of the conditional expectation, to prove the expression (2.3), it is sufficient to show $\mathbb{E}[u(X)\,\mathbb{I}_A] = \mathbb{E}[m_u(Y)\,\mathbb{I}_A]$ for all $A \in \sigma(Y)$. For any $A \in \sigma(Y)$, there exists $B \in \mathcal{B}(\mathbb{R})$ such that $A = \{Y \in B\} = \{\omega \in \Omega \mid Y(\omega) \in B\}$. Recall that $g^{-1}(Y) = g^{-1}(g(X)) = X$ if $Y = g(X) \in F_1$ and $\mathbb{P}\{Y \in F_1 \cup F_3\} = 1$, we have

$$\begin{aligned}
\mathbb{E}[u(X)\,\mathbb{I}_A] &= \mathbb{E}[u(X)\,\mathbb{I}\{Y \in B\}] \\
&= \mathbb{E}[u(X)\,\mathbb{I}\{Y \in B \cap (F_1 \cup F_3)\}] \\
&= \mathbb{E}[u(X)\,\mathbb{I}\{Y \in B \cap F_1\}] + \mathbb{E}[u(X)\,\mathbb{I}\{Y \in B \cap F_3\}] \\
&= \mathbb{E}[u \circ g^{-1}(Y)\,\mathbb{I}\{Y \in B \cap F_1\}] + \mathbb{E}\left(\sum_{y_0 \in B \cap F_3} u(X)\,\mathbb{I}\{Y = y_0\}\right) \\
&= \mathbb{E}[u \circ g^{-1}(Y)\,\mathbb{I}\{Y \in B \cap F_1\}] + \sum_{y_0 \in B \cap F_3} \mathbb{E}[u(X)\,\mathbb{I}\{Y = y_0\}] \quad (2.5) \\
&= \mathbb{E}[u \circ g^{-1}(Y)\,\mathbb{I}\{Y \in B \cap F_1\}] + \sum_{y_0 \in B \cap F_3} q_u(y_0)\,\mathbb{P}\{Y = y_0\} \quad (2.6) \\
&= \mathbb{E}[u \circ g^{-1}(Y)\,\mathbb{I}\{Y \in B \cap F_1\}] + \mathbb{E}[q_u(Y)\,\mathbb{I}\{Y \in B \cap F_3\}] \\
&= \mathbb{E}[m_u(Y)\,\mathbb{I}\{Y \in B\}] = \mathbb{E}[m_u(Y)\,\mathbb{I}_A],
\end{aligned}$$

where (2.5) holds by the Lebesgue convergence theorem and (2.6) holds by the definition of $q_u(y)$. □

Corollary 2.11. *For increasing function g and random variable X, it holds that $X \uparrow_{SI} g(X)$.*

Proof. By Proposition 2.10, we have $\mathbb{E}[u(X) \mid g(X)] = m_u(Y)$, where $m_u(Y)$ is defined by (2.4). In order to prove $X \uparrow_{SI} g(X)$, it is sufficient to show that, for any increasing function u, $m_u(y) = u \circ g^{-1}(y)\,\mathbb{I}\{y \in F_1\} + q_u(y)\,\mathbb{I}\{y \in F_3\}$ is increasing in $y \in F_1 \cup F_3$, which is a support of Y since $\mathbb{P}(Y \in F_1 \cup F_3) = 1$.

For any set $A \subseteq \mathbb{R}$ and the function u, we denote $u(A) = \{u(x) \mid x \in A\}$, $\sup\{A\} = \sup\{x \mid x \in A\}$ and $\inf\{A\} = \inf\{x \mid x \in A\}$. Let $B(y) = g^{-1}(\{y\})$, then $B(y) \neq \emptyset$ for any $y \in F_1 \cup F_3$. For $y \in F_3$, we have

$$q_u(y) = \frac{\mathbb{E}[u(X)\,\mathbb{I}\{Y = y\}]}{\mathbb{P}\{Y = y\}} = \frac{\mathbb{E}[u(X)\,\mathbb{I}\{X \in B(y)\}]}{\mathbb{P}\{Y = y\}} \leq \frac{\mathbb{E}[\sup\{u(B(y))\}\,\mathbb{I}\{X \in B(y)\}]}{\mathbb{P}\{Y = y\}} = \sup\{u(B(y))\}.$$

Similarly, $q_u(y) \geq \inf\{u(B(y))\}$. If $y \in F_1$, we have $\inf\{u(B(y))\} = u(g^{-1}(y)) = \sup\{u(B(y))\}$ since $B(y) = \{g^{-1}(y)\}$ is a single point set in this case. Since for any fixed $y \in F_1 \cup F_3$, $m_u(y)$ is of the form either $u(g^{-1}(y))$ or $q_u(y)$, we have $\inf\{u(B(y))\} \leq m_u(y) \leq \sup\{u(B(y))\}$.

Consider $y_1 < y_2 \in F_1 \cup F_3$. For any $x_1 \in B(y_1)$, $x_2 \in B(y_2)$, we have $g(x_1) = y_1 < y_2 = g(x_2)$, then $x_1 < x_2$ since g is increasing. Thus, $u(x_1) \leq u(x_2)$ and then $\sup\{u(B(y_1))\} \leq \inf\{u(B(y_2))\}$. Therefore $m_u(y_1) \leq \sup\{u(B(y_1))\} \leq \inf\{u(B(y_2))\} \leq m_u(y_2)$. □

Proposition 2.12. *Let $\mathbf{X} = (X_1, \ldots, X_n)$ be an n-dimensional random vector and Y be a random variable. If $\mathbf{X} \uparrow_{SI} Y$, then $f(\mathbf{X}) \uparrow_{SI} g(Y)$ for any increasing functions $f : \mathbb{R}^n \to \mathbb{R}^k$ and $g : \mathbb{R} \to \mathbb{R}$, where $k \in \mathbb{N}$.*

Proof. First, it is easy to show that $\mathbf{X} \uparrow_{SI} Y \implies f(\mathbf{X}) \uparrow_{SI} Y$. Indeed, for any increasing function $h : \mathbb{R}^k \to \mathbb{R}$, $h \circ f : \mathbb{R}^n \to \mathbb{R}$ is also increasing. By the definition of $\uparrow_{SI}$, we know that $\mathbb{E}[h \circ f(X_1, \ldots, X_n) \mid Y = y]$ is increasing in $y \in S(Y)$, which means $f(X_1, \ldots, X_n) \uparrow_{SI} Y$.

Then, to complete the proof, it is sufficient to show that $\mathbf{X} \uparrow_{SI} Y \implies \mathbf{X} \uparrow_{SI} g(Y)$. Denote $Z = g(Y)$, note that $\sigma(Z) \subset \sigma(Y)$. Thus, for any increasing function $u : \mathbb{R}^n \to \mathbb{R}$, we have

$$\mathbb{E}[u(\mathbf{X}) \mid Z] = \mathbb{E}[\mathbb{E}[u(\mathbf{X}) \mid Y] \mid Z] = \mathbb{E}[h_u(Y) \mid Z], \tag{2.7}$$

where $h_u(Y) = \mathbb{E}[u(\mathbf{X}) \mid Y]$ is increasing in Y since $\mathbf{X} \uparrow_{SI} Y$. By the properties of conditional expectations, we know that (2.7) implies $\mathbb{E}[u(\mathbf{X}) \mid Z = z] = \mathbb{E}[h_u(Y) \mid Z = z]$ for all $z \in S(Z)$, where $S(Z)$ is a support of Z. By Corollary 2.11, we have $Y \uparrow_{SI} Z$ and thus $h_u(Y) \uparrow_{SI} Z$. Therefore $\mathbb{E}[u(\mathbf{X}) \mid Z = z] = \mathbb{E}[h_u(Y) \mid Z = z]$ is increasing in $z \in S(Z)$, which implies that $\mathbf{X} \uparrow_{SI} Z$. □

From Proposition 2.12, we immediately get the following property.

Proposition 2.13. *If random vector $(X_1, \ldots, X_n)$ is PDS, then $(f_1(X_1), \ldots, f_n(X_n))$ is PDS for any increasing functions f_i, $i = 1, \ldots, n$.* □

Corollary 2.14. *Random vector $(X_1, \ldots, X_n)$ is PDS if and only if $(F_1(X_1), \ldots, F_n(X_n))$ is PDS, where F_i is the distribution function of $X_i, i = 1, \ldots, n$.*

Proof. Since $F_i(x), i = 1, \ldots, n$ are increasing, according to Proposition 2.13, we have $(X_1, \ldots, X_n)$ PDS implies $(F_1(X_1), \ldots, F_n(X_n))$ PDS. On the other hand, from Proposition A.4 of McNeil et al. (2005), we know that $X_i = F_i^{-1} \circ F_i(X_1)$ holds with probability 1 for all $i = 1, \ldots, n$. By Proposition A.3(i) of McNeil et al. (2005), $F_i^{-1}, i = 1, \ldots, n$ are increasing. Thus, if $(F_1(X_1), \ldots, F_n(X_n))$ is PDS, by Proposition 2.13, we have $(F_1^{-1} \circ F_1(X_1), \ldots, F_n^{-1} \circ F_n(X_n))$ is PDS, and hence $(X_1, \ldots, X_n)$ is PDS. □

Proposition 2.15. *Assume $g(x)$ and $g_i(x), i = 1, 2, \ldots, n$, are increasing functions. For random vector $\mathbf{X} = (X_1, \ldots, X_n)$ and random variable Y, if $\mathbf{X} = (X_1, \ldots, X_n) \uparrow_{WSI} Y$, then $(g_1(X_1), \ldots, g_n(X_n)) \uparrow_{WSI} g(Y)$.*

Proof. Since $\mathbf{X} \uparrow_{WSI} Y$, we have $\mathbf{X} \mid Y = y_1 \le_{uo} \mathbf{X} \mid Y = y_2$ for any $y_1, y_2 \in S(Y)$ with $y_1 < y_2$. Thus, by Theorem 6.G.3 of Shaked and Shanthikumar (2007), we know that the upper orthant order $\le_{uo}$ is preserved under componentwise increasing transformations. Thus we have $(g_1(X_1), \ldots, g_n(X_n)) \mid Y = y_1 \le_{uo} (g_1(X_1), \ldots, g_n(X_n)) \mid Y = y_2$ for any $y_1, y_2 \in S(Y)$ with $y_1 < y_2$, which means $h(y) = \mathbb{P}\{g_1(X_1) > x_1, \ldots, g_n(X_n) > x_n \mid Y = y\}$ is increasing in $y \in S(Y)$ for any $(x_1, \ldots, x_n) \in \mathbb{R}^n$.

On the other hand, since $\sigma(g(Y)) \subseteq \sigma(Y)$, we have

$$\begin{aligned}
&\mathbb{E}[\mathbb{I}\{g_1(X_1) > x_1, \ldots, g_n(X_n) > x_n\} \mid g(Y)] \\
&= \mathbb{E}[\mathbb{E}[\mathbb{I}\{g_1(X_1) > x_1, \ldots, g_n(X_n) > x_n\} \mid Y] \mid g(Y)] \\
&= \mathbb{E}[h(Y) \mid g(Y)].
\end{aligned}$$

According to Corollary 2.11, we have $Y \uparrow_{SI} g(Y)$, thus $\mathbb{E}[h(Y) \mid g(Y) = y]$ is increasing in $y \in S(g(Y))$, which implies $\mathbb{P}\{g_1(X_1) > x_1, \ldots, g_n(X_n) > x_n \mid g(Y) = y\}$ is increasing in $y \in S(g(Y))$ for any $(x_1, \ldots, x_n) \in \mathbb{R}^n$. □

Corollary 2.16. *Assume $g_i(x), i = 1, 2, \ldots, n$ are increasing functions. If random vector $\mathbf{X} = (X_1, \ldots, X_n)$ is PDUO, then $(g_1(X_1), \ldots, g_n(X_n))$ is PDUO.* □

Proof. The proof follows immediately from the definition of PDUO and Proposition 2.15. □

Corollary 2.17. *Let F_i be the distribution function of X_i for $i = 1, \ldots, n$. Then, $(X_1, \ldots, X_n)$ is PDUO if and only if $(F_1(X_1), \ldots, F_n(X_n))$ is PDUO.*

Proof. The proof is similar to that for Corollary 2.14 and is omitted. □

Corollaries 2.14 and 2.17 imply that if $(X_1, \ldots, X_n)$ has continuous marginal distributions $F_i, i = 1, 2, \ldots, n$, then $(X_1, \ldots, X_n)$ is PDS (PDUO) if and only if its copula is PDS (PDUO).

3. Generalized inverse functions and the copula invariance

For the inverse functions g^{-1} and g^{-1+} defined in Definition 2.6, it is easy to check that g^{-1} is left-continuous while g^{-1+} is right-continuous. The generalized inverse functions of increasing functions appear in many studies. Below, we prove a property of the generalized inverse functions, which will be used to derive the invariant property of copulas under increasing transformations.

Proposition 3.1. *Let $g : \mathbb{R} \to \mathbb{R}$ be an increasing function and $x, z \in \mathbb{R}$.*

(i) *If g is left continuous, then $g(x) \le z$ if and only if $x \le g^{-1+}(z)$.*
(ii) *If g is right continuous, then $g(x) \ge z$ if and only if $x \ge g^{-1}(z)$.*
(iii) *The following implications hold: $x < g^{-1+}(z) \Longrightarrow g(x) \le z \Longrightarrow x \le g^{-1+}(z)$.*

Proof. (i) If $g(x) \le z$, then $x \in \{y \mid g(y) \le z\}$ and thus $x \le \sup\{y \mid g(y) \le z\} = g^{-1+}(z)$. Conversely, if $x \le g^{-1+}(z)$, then $g(x) \le g(g^{-1+}(z))$ since g is increasing. Because $g^{-1+}(z)$ is the supremum of the set $\{y \mid g(y) \le z\}$, there exists a series $\{x_n\}_{n=1}^{\infty}$ in the set such that $g(x_n) \le z$ and $x_n \uparrow g^{-1+}(z)$ as $n \to \infty$. Since g is left-continuous, then $g(g^{-1+}(z)) = \lim_{n\to\infty} g(x_n) \le z$. Thus, $g(x) \le g(g^{-1+}(z)) \le z$.

(ii) The statement is from Proposition A.3(iv) in McNeil et al. (2005).

(iii) Assume $x < g^{-1+}(z)$. If $g(x) > z$, then $g(x) > g(y)$ for all $y \in \{t : g(t) \le z\}$. Hence, $x > y$ for all $y \in \{t \mid g(t) \le z\}$ since g is increasing. Thus, $x \ge \sup\{t \mid g(t) \le z\} = g^{-1+}(z)$, which contradicts the assumption of $x < g^{-1+}(z)$. Therefore, $x < g^{-1+}(z) \Longrightarrow g(x) \le z$. Furthermore, assume $g(x) \le z$, then $x \in \{y \mid g(y) \le z\}$ and thus $x \le \sup\{y \mid g(y) \le z\} = g^{-1+}(z)$. □

Lemma 3.2. *Assume random variable X_i has continuous marginal distribution function F_i for $i = 1, \ldots, n$ and $(X_1, \ldots, X_n)$ has copula C. If $f_1, \ldots, f_n$ are increasing functions, then $(f_1(X_1), \ldots, f_n(X_n))$ also has the copula C.*

Proof. Note that $\mathbb{P}\{X_1 \le x_1, \ldots, X_n \le x_n\} = C(F_1(x_1), \ldots, F_n(x_n))$. For any $i = 1, \ldots, n$, we have $\mathbb{P}\{X_i = f_i^{-1+}(z_i)\} = 0$ since X_i has a continuous distribution function. According to Proposition 3.1 (iii), we have for any $i = 1, \ldots, n$, $\{X_i < f_i^{-1+}(z_i)\} \subseteq \{f_i(X_i) \le z_i\} \subseteq \{X_i \le f_i^{-1+}(z_i)\}$, which, together with $\mathbb{P}\{X_i = f_i^{-1+}(z_i)\} = 0$, implies that $F_{f_i(X_i)}(z_i) = \mathbb{P}\{f_i(X_i) \le z_i\} = \mathbb{P}\{X_i \le f_i^{-1+}(z_i)\} = F_i \circ f_i^{-1+}(z_i)$. Therefore,

$$\begin{aligned}
&\mathbb{P}\{f_1(X_1) \le z_1, \ldots, f_n(X_n) \le z_n\} \\
&= \mathbb{P}\{X_1 \le f_1^{-1+}(z_1), \ldots, X_n \le f_n^{-1+}(z_n)\} \\
&= C(F_1 \circ f_1^{-1+}(z_1), \ldots, F_n \circ f_n^{-1+}(z_n)) \\
&= C(F_{f_1(X_1)}(z_1), \ldots, F_{f_n(X_n)}(z_n)),
\end{aligned}$$

which means that C is also a copula of $(f_1(X_1), \ldots, f(X_n))$. □

Theorem 3.3. *Assume $f_1, \ldots, f_n$ are increasing functions. If random vector $(X_1, \ldots, X_n)$ has copula C, then $(f_1(X_1), \ldots, f_n(X_n))$ also has the copula C.*

Proof. Since random vector $(X_1, \ldots, X_n)$ has copula C, by the last paragraph of the proof for Theorem 5.3 of McNeil et al. (2005), we know that there exists a uniform random vector $(U_1, \ldots, U_n)$ defined on $[0, 1]^n$ such that $(U_1, \ldots, U_n)$ has distribution function $C(u_1, \ldots, u_n)$ and $(F_1^{-1}(U_1), \ldots, F_n^{-1}(U_n)) =_{st} (X_1, \ldots, X_n)$.

Thus, $(f_1(X_1), \ldots, f_n(X_n)) =_{st} (f_1(F_1^{-1}(U_1)), \ldots, f_n(F_n^{-1}(U_n)))$ or $(f_1(X_1), \ldots, f_n(X_n))$ and $(f_1(F_1^{-1}(U_1)), \ldots, f_n(F_n^{-1}(U_n)))$ have the same joint distribution function and hence they have the same copula. On the other hand, since $f_i \circ F_i^{-1}$ is increasing and U_i has the continuous marginal distribution function, by Lemma 3.2, we know that $(f_1(F_1^{-1}(U_1)), \ldots, f_n(F_n^{-1}(U_n)))$ and $(U_1, \ldots, U_n)$ have the same copula C. Therefore, $(f_1(X_1), \ldots, f_n(X_n))$ has the copula C as well. □

Theorem 3.3 generalizes Proposition 4.7.4 of Denuit et al. (2005) and Proposition 5.6. of McNeil et al. (2005).

4. The characterization of PDUO in terms of survival copulas

If the distribution function F_i of X_i is continuous for $i = 1, \ldots, n$, then $F_i(X_i)$ has the uniform distribution over $[0, 1]$ and thus the joint distribution function of $(F_1(X_1), \ldots, F_n(X_n))$ is the unique copula of $(X_1, \ldots, X_n)$, which links the marginal distributions of $X_1, \ldots, X_n$. This means that the PDS and PDUO properties

$$\begin{aligned}&\mathbb{P}\{U_1 > u_1, \dots, U_{k-1} > u_{k-1}, U_{k+1} > u_{k+1}, \dots, U_n > u_n \mid U_k = u_k\}\\&= \lim_{\Delta u \searrow 0} \frac{\mathbb{P}\{U_1 > u_1, \dots, U_k > u_k, \dots, U_n > u_n\} - \mathbb{P}\{U_1 > u_1, \dots, U_k > u_k + \Delta u, \dots, U_n > u_n\}}{\mathbb{P}\{U_k \in (u_k, u_k + \Delta u]\}}\\&= \lim_{\Delta u \searrow 0} \frac{\bar{C}(u_1, \dots, u_k, \dots, u_n) - \bar{C}(u_1, \dots, u_k + \Delta u, \dots, u_n)}{\Delta u}\\&= -\frac{\partial}{\partial u_k}\bar{C}(u_1, \dots, u_k, \dots, u_n)\end{aligned} \tag{4.1}$$

Box I.

of a continuous random vector can be characterized by their copulas. Actually, if the joint distribution function of a continuous random vector $(X_1, \dots, X_n)$ is linked by a Gaussian copula, then $(X_1, \dots, X_n)$ is PDS if and only if all off-diagonal elements of the covariance matrix in the Gaussian copula are non-negative. See, for example, Joe (1997). Now, using Corollary 2.17, we could develop a sufficient and necessary condition for PDUO in terms of survival copulas.

Let $(X_1, \dots, X_n)$ be a random vector with marginal distribution functions $F_1, \dots, F_n$ and marginal survival functions $\bar{F}_1(x_1) = 1 - F_1(x_1), \dots, \bar{F}_n(x_n) = 1 - F_n(x_n)$. The joint survival function of the random vector is denoted by $\bar{F}(x_1, \dots, x_n) = \mathbb{P}\{X_1 > x_1, \dots, X_n > x_n\}$, which is linked by a copula $\widehat{C}$ as

$$\bar{F}(x_1, \dots, x_n) = \widehat{C}(\bar{F}_1(x_1), \dots, \bar{F}_n(x_n)).$$

Such a copula $\widehat{C}$ is called a survival copula of the random vector $(X_1, \dots, X_n)$. See, for example, McNeil et al. (2005).

For a random vector $(X_1, \dots, X_n)$ with continuous marginal distribution functions $F_i(x_i) = \mathbb{P}\{X_i \le x_i\}$ for $i = 1, \dots, n$, the unique copula of $(X_1, \dots, X_n)$, denoted by C, is the joint distribution of the uniform random vector $(U_1, \dots, U_n) = (F_1(X_1), \dots, F_n(X_n))$ valued on $[0, 1]^n$. We denote the survival function of the copula C by $\bar{C}$, which is the joint survival function of the uniform random vector $(U_1, \dots, U_n)$ or $\bar{C}(x_1, \dots, x_n) = \mathbb{P}\{U_1 > x_1, \dots, U_n > x_n\}$. In this case, let $\widehat{C}$ be the joint distribution of $(1-U_1, \dots, 1-U_n)$, then, $\widehat{C}$ is the survival copula of $(X_1, \dots, X_n)$ and for $u_i \in [0, 1], i = 1, 2, \dots, n$,

$$\bar{C}(u_1, \dots, u_n) = \widehat{C}(1 - u_1, \dots, 1 - u_n).$$

See McNeil et al. (2005) for detailed discussions on survival copulas.

Proposition 4.1. *Assume $(X_1, \dots, X_n)$ has continuous marginal distribution functions $F_1, \dots, F_n$ with survival copula $\widehat{C}$. Then $(X_1, \dots, X_n)$ is PDUO if and only if $\widehat{C}$ is concave in each argument.*

Proof. Denote $U_i = F_i(X_i), i = 1, 2, \dots, n$, then U_i has a uniform distribution over $[0, 1]$. According to Corollary 2.17, it is sufficient to show that $(U_1, \dots, U_n)$ is PDUO if and only if $\widehat{C}$ is concave in each argument.

Let $\bar{C}$ be the survival function of $(U_1, \dots, U_n)$. Note that the survival function $\bar{C}$ is decreasing in each argument and thus differentiable with respect with each argument almost everywhere. Thus, for any $u_k \in [0, 1]$ and $k \in \{1, 2, \dots, n\}$, Eq. (4.1) given in Box I holds. Hence, $(X_1, \dots, X_{k-1}, X_{k+1}, \dots, X_n) \uparrow_{SI} X_k \iff -\frac{\partial}{\partial u_k}\bar{C}(u_1, u_2, \dots, u_n)$ is increasing in $u_k \in [0, 1]$ (almost everywhere) for any fixed $u_1, \dots, u_{k-1}, u_{k+1}, \dots, u_n \iff \bar{C}$ is concave in each argument. On the other hand, $\bar{C}(u_1, u_2, \dots, u_n) = \widehat{C}(1 - u_1, 1 - u_2, \dots, 1 - u_n)$. Hence, $\bar{C}(u_1, \dots, u_k, \dots, u_n)$ is concave in u_k if and only if $\widehat{C}(u_1, \dots, u_k, \dots, u_n)$ is concave in u_k. Therefore, we conclude that continuous random vector $(X_1, \dots, X_n)$ is PDUO if and only if their survival copula $\widehat{C}$ is concave in each argument. $\square$

Proposition 4.1 enables one to easily construct a PDUO random vector by choosing a copula such that the copula is concave in each argument. We give such an example below.

Example 4.2. Assume that the joint survival function of a continuous random vector $(X_1, \dots, X_n)$ is linked by an Archimedean copula with

$$\widehat{C}(u_1, u_2, \dots, u_n) = \Psi^{-1}\left(\sum_{k=1}^{n} \Psi(u_k)\right),\quad u_k \in [0, 1],\ k = 1, 2, \dots, n, \tag{4.2}$$

where the generator $\Psi^{-1}(x)$ is completely monotonic and the function Ψ satisfies $\Psi(1) = 0, \lim_{x\to 0}\Psi(x) = \infty, \Psi'(x) < 0$, and $\Psi''(x) > 0$.

Rewriting (4.2), we get $\Psi(\widehat{C}) = \sum_{k=1}^{n}\Psi(u_k)$. Then, differentiating with respect to u_k on both sides of the equation, we have $\Psi'(\widehat{C}) \times \frac{\partial \widehat{C}}{\partial u_k} = \Psi'(u_k)$. Therefore, $\frac{\partial \widehat{C}}{\partial u_k} = \frac{\Psi'(u_k)}{\Psi'(\widehat{C})}$ and

$$\begin{aligned}\frac{\partial^2 \widehat{C}}{\partial u_k^2} &= \frac{\Psi'(\widehat{C})\Psi''(u_k) - \Psi''(\widehat{C})\frac{\partial \widehat{C}}{\partial u_k}\Psi'(u_k)}{[\Psi'(\widehat{C})]^2}\\&= \frac{[\Psi'(u_k)]^2}{\Psi'(\widehat{C})} \times \left[\frac{\Psi''(u_k)}{[\Psi'(u_k)]^2} - \frac{\Psi''(\widehat{C})}{[\Psi'(\widehat{C})]^2}\right].\end{aligned}$$

Recall that the survival copula $\widehat{C} = \widehat{C}(u_1, u_2, \dots, u_n)$ is a copula. By the Fréchet bounds for copulas, we have $\widehat{C} = \widehat{C}(u_1, u_2, \dots, u_n) \le \min\{u_1, \dots, u_n\} \le u_k$. Thus, $\widehat{C}(u_1, \dots, u_n)$ is concave in each argument, or $\frac{\partial^2 \widehat{C}}{\partial u_k^2} \le 0$ for each $k \in \{1, \dots, n\}$, if

$$\frac{\Psi''(x)}{[\Psi'(x)]^2} \text{ is increasing in } x \in [0, 1]. \tag{4.3}$$

Hence, if the joint survival function of a continuous random vector $(X_1, \dots, X_n)$ is linked by the Archimedean copula (4.2), then $(X_1, \dots, X_n)$ is PDUO if (4.3) holds.

Two examples of the Archimedean copula satisfying (4.3) are the multivariate Gumbel copula with $\Psi(x) = (-\ln x)^\theta, \theta \ge 1$ and multivariate the Clayton copula with $\Psi(x) = x^{-\theta} - 1,\ \theta > 0$. In both cases, the condition (4.3) holds. We refer to Müller and Scarsini (2005) for a detailed study of the relationships between Archimedean copulas and other notions of positive dependence. $\square$

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