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Quantile estimation and the statistical relative efficiency curve

Summary - In this article, we introduce a new practical tool—*relative efficiency curve* (REC)—for comparison of two competing statistical procedures. While in other scientific areas the term of REC has been around for some time, in statistics it seems to be new. In estimation, the curve is constructed by employing asymptotic properties of quantile estimators. Suppose two *consistent* and *asymptotically normal* estimators of a fixed quantile of the underlying distribution are available. Plotting of the ratio of their variances versus quantiles at various probability levels yields an REC. Such a curve provides information about the accuracy of one estimator relative to another when both are designed to estimate the same (fixed but arbitrary) quantile of the distribution. Thus, depending on the objective of application, the REC can help one choose between parametric, robust parametric, empirical nonparametric or other method of estimation for the measure of interest. Further, other possibilities for defining (statistical) RECs are also discussed, and illustrative examples for (equivalent) Pareto and exponential, and lognormal and normal distributions are provided. Specifically, graphs of RECs of maximum likelihood, method of trimmed moments, and empirical nonparametric estimators of distribution quantiles are presented.

Key Words - Parametric distributions; Quantiles; Relative efficiency; Robust estimation.

1. Introduction

In many statistical estimation problems, "good" estimators are both consistent and asymptotically normal. If for particular problem there are two (or more) such estimators, then we are interested in determining which one is better (or best among all). Let us first focus on situations involving estimation of a single parameter θ. Suppose that two *consistent* and *asymptotically normal* estimators of the parameter θ, say $\widehat{\theta}_1$ and $\widehat{\theta}_2$, are available and let their respective asymptotic variances be denoted $\sigma_1^2(\theta)/n$ and $\sigma_2^2(\theta)/n$, where n represents the sample size; n is large. If comparison of estimator performances is based upon the variance criterion, then the ratio $\sigma_1^2(\theta)/\sigma_2^2(\theta)$ is used to identify which

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estimator is better and it is called the asymptotic relative efficiency (ARE) of estimator $\widehat{\theta}_2$ relative to estimator $\widehat{\theta}_1$. In simulations, where sample sizes are small or moderate and bias has to be accounted for, the variance is replaced with the mean-square error; also ARE becomes simply RE. Interpretations of such measures are usually given in terms of the ratio of sample sizes needed for the two estimators to perform equivalently with respect to the adopted criterion (see Serfling, 1980, Section 1.15.4).

Extensions of ARE to the multi-parameter case also exist. Consider estimation of a k-dimensional parameter $\boldsymbol{\theta} = (\theta_1, \ldots, \theta_k)$ by $\widehat{\boldsymbol{\theta}}^{(i)} = (\widehat{\theta}_1^{(i)}, \ldots, \widehat{\theta}_k^{(i)})$, where $\widehat{\boldsymbol{\theta}}^{(i)}$ is asymptotically k-variate normal with mean $\boldsymbol{\theta}$ and (nonsingular) covariance matrix $n^{-1}\boldsymbol{\Sigma}_{\boldsymbol{\theta}}^{(i)}$, for $i = 1, 2$. A general form of comparison of the two estimators $\widehat{\boldsymbol{\theta}}^{(1)}$ and $\widehat{\boldsymbol{\theta}}^{(2)}$ is based on the condition

$$\boldsymbol{\Sigma}_{\boldsymbol{\theta}}^{(2)} - \boldsymbol{\Sigma}_{\boldsymbol{\theta}}^{(1)} \qquad \text{nonnegative definite.} \tag{1.1}$$

Condition (1.1) is quite natural since it requires the asymptotic distribution of $\widehat{\boldsymbol{\theta}}^{(1)}$ to possess a concentration ellipsoid contained entirely within that of the asymptotic distribution of $\widehat{\boldsymbol{\theta}}^{(2)}$. For computational purposes, however, one uses a less general but numerically convenient measure based on the concentration ellipsoid volumes $\left|\boldsymbol{\Sigma}_{\boldsymbol{\theta}}^{(1)}\right|$ and $\left|\boldsymbol{\Sigma}_{\boldsymbol{\theta}}^{(2)}\right|$:

$$\text{ARE} \quad \left(\widehat{\boldsymbol{\theta}}^{(2)}, \widehat{\boldsymbol{\theta}}^{(1)}\right) = \left(\left|\boldsymbol{\Sigma}_{\boldsymbol{\theta}}^{(1)}\right| / \left|\boldsymbol{\Sigma}_{\boldsymbol{\theta}}^{(2)}\right|\right)^{k}, \tag{1.2}$$

where $|\cdot|$ denotes the determinant of matrix. For further discussion on asymptotic efficiency and optimality in estimation, see, e.g., Lehmann *et al.* (1998, Chapter 6) and Serfling (1980, Section 4.1).

Note that (1.2) is a scalar measure which, for example, for location-scale families yields a number (*i.e.*, it is parameter-free). In addition, this measure includes, and thus naturally generalizes, the univariate ARE, and is quite an effective method for comparison and ranking of estimators. In applications, however, a fitted distribution is not the ultimate goal, rather one is interested in certain aspects/segments of the underlying distribution. For instance: in actuarial science (or, more broadly, in quantitative risk management), various upper-tail risk measures, such as *conditional tail expectation* or *value-at-risk*, are used to measure the riskiness of portfolio; in economics, the *Lorenz curve* is a standard tool for measuring income inequality in a population of people; and, in engineering, estimation of *survival probabilities* is essential for assessing the reliability of engineering systems. Clearly, while the upper tail is the main focus of the first and third area of application, dispersion of the entire population is central to the economics example. Thus, in view of this discussion, it certainly is desirable to have an efficiency measure that not only allows for comparison of estimators/procedures but also supplies the researcher with information about the estimator's relative performance across all segments of the underlying distribution.

Furthermore, there exist practical situations for which methodological options are limited. For example, one cannot make reliable inference beyond the observed data using the empirical approach. In other cases, however, there is always the question of what type of methodology (e.g., parametric, robust parametric, empirical nonparametric or other) is most appropriate for the problem at hand. This issue is usually "solved" by choosing the method that the researcher prefers. For example, in Jones and Zitikis (2003), the empirical nonparametric approach was introduced for estimation of, and testing based on, actuarial risk measures; and, Cowell and Victoria-Feser (2006) promote robust parametric methodology for measuring income inequality. In the actuarial literature, there were attempts to address this problem through simulations (see Brazauskas and Kaiser, 2004, and Kaiser and Brazauskas, 2006) but a quicker and more rigorous approach is desirable. Also notice that such questions cannot be answered by employing a measure such as (1.2) because of its purely parametric nature.

To address the issues raised above, in this article we introduce a new practical tool—relative efficiency curve (REC)—for comparison of two competing statistical procedures. Mathematical treatment of efficiency as a function (curve) is certainly not new and fundamental results can be found in the classic text of Shorack and Wellner (1986). Development of practical tools based on the existing theory, however, has not been addressed in the statistical literature. Note that the term of REC does appear in the educational measurement literature (Lord, 1974), in physics (Hayashi *et al.*, 2000), and in other scientific areas as well. Note also that while the motivation for introduction of the statistical and other RECs is similar—we want to compare tests/estimators/procedures/methods—each field relies on its own interpretation of efficiency and employs different theoretical tools to construct RECs.

In statistical estimation, the curve is constructed by using asymptotic properties of quantile estimators. Suppose two consistent and asymptotically normal estimators of a fixed quantile of the underlying distribution are available. Plotting the ratio of their variances versus quantiles at various probability levels yields an REC. Such a curve provides information about the accuracy of one estimator relative to another when both are designed to estimate the same (fixed but arbitrary) quantile of the distribution. Thus, depending on the objective of application, the REC can help one choose between parametric, robust parametric, empirical nonparametric or other method of estimation for the measure of interest. Finally, since testing problems can usually be recast in the context of estimation (and vice versa), the REC can also be applied to hypotheses testing problems.

The rest of the article is organized as follows. In Section 2, necessary asymptotic distributions are specified and the efficiency formulas used in construction of RECs are derived. The subsequent section provides two illustrative examples for (equivalent) Pareto and exponential, and lognormal and normal distributions. Specifically, RECs of maximum likelihood, method of trimmed moments, and empirical nonparametric estimators of quantiles of these distributions are plotted. Some final remarks are presented in Section 4.

2. Relative efficiency curves

Consider a sample of n independent and identically distributed (i.i.d.) continuous random variables, $X_1, \dots, X_n$, whose distribution (cdf), density (pdf), and quantile (qf) functions we denote by F, f, and Q, respectively. Assume that the cdf, pdf, and qf are given in a parametric form, and suppose that they are indexed by a k-dimensional parameter $\boldsymbol{\theta} = (\theta_1, \dots, \theta_k)$. Let us denote $X_{1:n} \leq \cdots \leq X_{n:n}$ the order statistics of $X_1, \dots, X_n$.

For a fixed probability level p, $0 < p < 1$, consider empirical nonparametric and parametric estimators of the population quantile $Q(p)$. As is well-known (see, e.g., Shorack and Wellner, 1986, Section 18.1), the empirical estimator $\widehat{Q}_{\mathrm{EMP}}(p) = X_{\lceil np \rceil:n}$, where $\lceil \cdot \rceil$ denotes "greatest integer part", is asymptotically normal ($\mathcal{AN}$) with mean $Q(p)$ and variance $p(1-p)/[nf^2(Q(p))]$. In short,

$$\widehat{Q}_{\mathrm{EMP}}(p) \quad \text{is} \quad \mathcal{AN}\left(Q(p),\, n^{-1}\frac{p(1-p)}{f^2(Q(p))}\right). \tag{2.1}$$

For the parametric approach, suppose that a k-dimensional parameter $\boldsymbol{\theta}$ is estimated by $\widehat{\boldsymbol{\theta}}$, where $\widehat{\boldsymbol{\theta}}$ is asymptotically k-variate normal with mean $\boldsymbol{\theta}$ and covariance matrix $n^{-1}\boldsymbol{\Sigma}_{\boldsymbol{\theta}}$. That is,

$$\widehat{\boldsymbol{\theta}} \quad \text{is} \quad \mathcal{AN}\left(\boldsymbol{\theta},\, n^{-1}\boldsymbol{\Sigma}_{\boldsymbol{\theta}}\right).$$

Suppose $Q(p)$ is a sufficiently smooth function of $\boldsymbol{\theta}$, denoted $g_p(\boldsymbol{\theta})$. Then, an application of the delta method (cf., e.g., Serfling, 1980, Section 3.3) implies that the parametric estimator $\widehat{Q}_{\mathrm{PAR}}(p) = g_p(\widehat{\boldsymbol{\theta}})$ is asymptotically normal with mean $Q(p)$ and variance $n^{-1}\mathbf{d}_p\boldsymbol{\Sigma}_{\boldsymbol{\theta}}\mathbf{d}_p'$, that is,

$$\widehat{Q}_{\mathrm{AR}}(p) \quad \text{is} \quad \mathcal{AN}\left(Q(p),\, n^{-1}\mathbf{d}_p\boldsymbol{\Sigma}_{\boldsymbol{\theta}}\mathbf{d}_p'\right), \tag{2.2}$$

where the vector $\mathbf{d}_p$ is $(\partial g_p/\partial\widehat{\theta}_1, \dots, \partial g_p/\partial\widehat{\theta}_k)$ evaluated at $\boldsymbol{\theta} = (\theta_1, \dots, \theta_k)$.

Now a definition of the asymptotic relative efficiency, ARE of $\widehat{Q}_{\mathrm{EMP}}(p)$ relative to $\widehat{Q}_{\mathrm{PAR}}(p)$, naturally emerges from the conditions (2.1) and (2.2)—we take the ratio of the asymptotic variances:

$$\begin{aligned} \mathrm{ARE}_p : &= \mathrm{ARE}\left(\widehat{Q}_{\mathrm{EMP}}(p),\ \widehat{Q}_{\mathrm{PAR}}(p)\right) \\ &= \frac{f^2(Q(p))}{p(1-p)} \mathbf{d}_p \boldsymbol{\Sigma}_{\boldsymbol{\theta}} \mathbf{d}_p' \quad \text{ for } 0 < p < 1. \end{aligned} \tag{2.3}$$

If one is interested in comparison of two parametric estimators, $\widehat{Q}^{(1)}_{\mathrm{PAR}}(p)$ and $\widehat{Q}^{(2)}_{\mathrm{PAR}}(p)$, then

$$\mathrm{ARE}_p := \mathrm{ARE}\left(\widehat{Q}^{(2)}_{\mathrm{PAR}}(p),\ \widehat{Q}^{(1)}_{\mathrm{PAR}}(p)\right) = \frac{\mathbf{d}_p \boldsymbol{\Sigma}^{(1)}_{\boldsymbol{\theta}} \mathbf{d}_p'}{\mathbf{d}_p \boldsymbol{\Sigma}^{(2)}_{\boldsymbol{\theta}} \mathbf{d}_p'} \quad \text{ for } 0 < p < 1. \tag{2.4}$$

Plotting the points $(p,\ \mathrm{ARE}_p)$, where ARE_p is defined by (2.3) or (2.4), yields corresponding RECs.

Note 2.1 We shall emphasize here that RECs given by (2.3) and (2.4) should not be interpreted as continuous curves, rather they provide visualization for finite number of points. However, since in practice data are measured and observed discretely, the current version of REC suffice. Notice also that for the continuous treatment, different mathematical tools would have to be employed. In particular, asymptotic theory of empirical quantile processes should be consulted (see Shorack and Wellner, 1986, Chapter 18). □

Note 2.2 Similar efficiency measures, and hence curves, can be constructed using other multiparameter "suppression" methods. For example, cdf or pdf based point estimators can be used instead of the qf estimators. Indeed, for the cdf approach, asymptotic normality results for the empirical estimator $\widehat{F}_{\mathrm{EMP}}(x) = n^{-1}\sum_{j=1}^{n} \mathbf{1}\left\{X_j \le x\right\}$ and a parametric estimator $\widehat{F}_{\mathrm{PAR}}(x)$ are available, and thus a simple repetition of the above steps leads to

$$\mathrm{ARE}_x := \mathrm{ARE}\left(\widehat{F}_{\mathrm{EMP}}(x),\ \widehat{F}_{\mathrm{PAR}}(x)\right) = \frac{\mathbf{b}_x \boldsymbol{\Sigma}_{\boldsymbol{\theta}} \mathbf{b}_x'}{F(x)(1-F(x))} \quad \text{for} x \in \mathcal{X}, \tag{2.5}$$

where $\mathcal{X} = \{x \mid -\infty < x < \infty,\ 0 < F(x) < 1\}$, the vector $\mathbf{b}_x$ is $(\partial h_x/\partial\widehat{\theta}_1, \ldots, = \partial h_x/\partial\widehat{\theta}_k)$ evaluated at $\boldsymbol{\theta} = (\theta_1, \ldots, \theta_k)$, and $h_x(\boldsymbol{\theta}) := F(x)$. Which measure, (2.3) or (2.5), is "better"? This, of course, depends on the definition of "better". We find it easier to work with ARE_p, given by (2.3), because it is defined on a finite interval, *i.e.*, (0, 1), and can be interpreted in terms of probabilities or percentiles. Also, the pdf approach in this context seems to be even less appealing since empirical estimation of pdf (at a fixed point) is not as straightforward as that of cdf or qf. □

3. ILLUSTRATIVE EXAMPLES

In this section, we provide examples of RECs for two pairs of equivalent distributions: Pareto and exponential, and lognormal and normal. (The distributions are equivalent in the following sense: after the logarithmic transformation, Pareto and lognormal become shifted-exponential (with known location), and normal, respectively.) In each example, we choose the corresponding maximum likelihood estimator (MLE) as the benchmark parametric estimator. Then, using formulas (2.3) and (2.4), we evaluate respective ARE_p's for the empirical estimator (EMP) and for various method of trimmed moments (MTM) estimators. (The MTM estimators were introduced by Brazauskas *et al.* (2009); derivations of the asymptotic properties of these estimators and small-sample investigations can also be found in the same paper.) For all distributions, ten MTM estimators were chosen with the following trimming proportions (a, b):

Estimator (a, b)	MTM-1 (.05, .05)	MTM-2 (.10, .10)	MTM-3 (.15, .15)	MTM-4 (.25, .25)	MTM-5 (.49, .49)

Estimator (a, b)	MTM-6 (.00, .10)	MTM-7 (.00, .30)	MTM-8 (.05, .00)	MTM-9 (.25, .00)	MTM-10 (.50, .00)

3.1. *Pareto and Exponential Models*

Let $X_1, \ldots, X_n$ be i.i.d. random variables, each with the same Pareto distribution

$$\text{Pareto}(x_0, \alpha): \quad F(x) = 1 - \left(\frac{x}{x_0}\right)^{-\alpha}, \quad x > x_0 \tag{3.1}$$

where $\alpha > 0$ is an unknown parameter, and $x_0 > 0$ is assumed to be known.

The qf and pdf are: $Q(p) = x_0(1-p)^{-1/\alpha}$ and $f(x) = (\alpha/x_0)(x/x_0)^{-\alpha-1}$, respectively. As is well-known (see, e.g., Arnold, 1983), the maximum likelihood estimator of α is given by $\widehat{\alpha}_{\text{MLE}} = \left[n^{-1}\sum_{i=1}^{n}\log(X_i/x_0)\right]^{-1}$, and

$$\widehat{\alpha}_{\text{MLE}} \text{ is } \mathcal{AN}\left(\alpha,\ \frac{\alpha^2}{n}\right). \tag{3.2}$$

The MTM estimator of α is given by $\widehat{\alpha}_{\text{MTM}} = c(a,b)\Big[(n-m_n-m_n^*)^{-1}\sum_{i=m_n+1}^{n-m_n^*} \log(X_{i:n}/x_0)\Big]^{-1}$, where $c(a,b) = -(1-a-b)^{-1}\int_a^{1-b}\log(1-u)\,\mathrm{d}u$ with $m_n/n \to a$ and $m_n^*/n \to b$, and

$$\widehat{\alpha}_{\text{MTM}} \text{ is } \mathcal{AN}\left(\alpha,\ \frac{\alpha^2}{n}C\right) \quad \text{with} \quad C = \frac{\int_a^{1-b}\int_a^{1-b}\frac{\min\{u,v\}-uv}{(1-u)(1-v)}\,\mathrm{d}v\,\mathrm{d}u}{\left(\int_a^{1-b}\log(1-u)\,\mathrm{d}u\right)^2}. \tag{3.3}$$

Notice that when $m_n = m_n^* = 0$, then $\widehat{\alpha}_{\text{MTM}} \to \widehat{\alpha}_{\text{MLE}}$ because $c(a,b) \to 1$ as $a = b \to 0$; also, since $C \to 1$ as $a = b \to 0$, the MLE's asymptotic distribution (3.2) follows from (3.3).

Now we have everything that is needed for computation of (2.3) and (2.4). That is: for the MLE, $\mathbf{d}_p\boldsymbol{\Sigma}_{\boldsymbol{\theta}}\mathbf{d}_p' = (x_0/\alpha)^2(1-p)^{-2/\alpha}\log^2(1-p)$; for the EMP estimator, $f^2(Q(p)) = (\alpha/x_0)^2(1-p)^{2+(2/\alpha)}$; and, for the MTM estimator, $\mathbf{d}_p\boldsymbol{\Sigma}_{\boldsymbol{\theta}}\mathbf{d}_p' = C(x_0/\alpha)^2(1-p)^{-2/\alpha}\log^2(1-p)$. These imply that:

$$\text{ARE}_p\,(\text{EMP, MLE}) = \frac{1-p}{p}\log^2(1-p) \quad \text{and} \quad \text{ARE}_p\,(\text{MTM, MLE}) = C^{-1}. \tag{3.4}$$

Note 3.1 As was mentioned above, the Pareto(x_0, α) is equivalent—through the logarithmic transformation of the variable—to a shifted-exponential variable. That is, if X is Pareto(x_0, α) with the cdf defined by equation (3.1), then $Z = \log X$ is a shifted-exponential variable with the cdf $F_Z(z) = 1 - e^{-(z-z_0)/\theta}$, $z > z_0$, where $z_0 = \log x_0$ and $\theta = \alpha^{-1}$. The corresponding qf and pdf are: $Q_Z(p) = z_0 - \theta\log(1-p)$ and $f_Z(z) = (1/\theta)e^{-(z-z_0)/\theta}$, respectively. Then, we have that $\widehat{\theta}_{\text{MLE}} = n^{-1}\sum_{i=1}^{n}(Z_i - z_0)$ is $\mathcal{AN}\left(\theta,\ \theta^2/n\right)$ and $\widehat{\theta}_{\text{MTM}} = c(a,b)(n - m_n - m_n^*)^{-1}\sum_{i=m_n+1}^{n-m_n^*}(Z_{i:n} - z_0)$ is $\mathcal{AN}\left(\theta,\ C\theta^2/n\right)$ with the same constants as above. And while intermediate results for the exponential distribution differ from those for the Pareto model, the relative efficiency formulas are identical to those in (3.4). Hence, RECs for these two models are the same. □

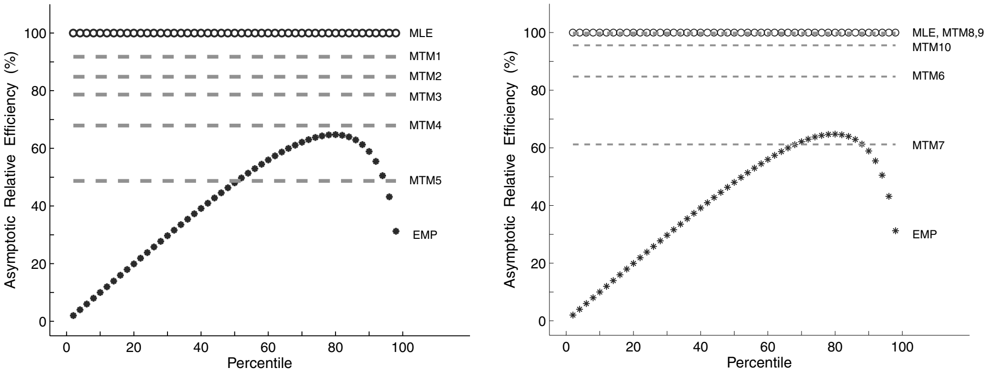

Figure 1. RECs of MLE, MTM, and empirical estimators of Pareto and exponential quantiles. Left panel: symmetrically trimmed MTMs. Right panel: asymmetrically trimmed MTMs.

In Figure 1, RECs of MLE, MTM and EMP estimators of Pareto and exponential quantiles are plotted. The plots summarize and confirm some well-known facts. First, estimation of lower/upper tail quantiles with EMP estimator is inefficient (due to a scarcity of sample data in the tails). Second, approximately 80th empirical percentile contains most information about its theoretical counterpart. The latter fact has been reported and rediscovered by multiple authors (see discussion by Arnold, 1983, Section 5.2.8) who considered quantile and/or order statistics estimation for Pareto and exponential models. For example, one of the earliest references on the topic (Sarhan *et al.*, 1963) reports the optimal level of a single quantile of 0.7968. Later, for the problem of Pareto tail estimation, Koutrouvelis (1981) derived optimal quantile estimators based on k ($k \geq 2$) quantiles. Predictably, as k grows larger, most of those quantiles cluster around the 80% level. For instance: for $k = 3$, they are 0.53, 0.83, 0.97; and, for $k = 5$, they are 0.39, 0.67, 0.84, 0.94, 0.99. This suggests that most information about Pareto α (or, equivalently, Exponential θ) is contained in the upper tail of the distribution. One can also arrive at similar conclusions by studying various trimming schemes of the MTM estimator. Indeed, larger proportions of trimming in symmetrically trimmed MTMs lead to a gradual and unavoidable decline of efficiency. That is, for $a = b = 0.05, 0.10, 0.15, 0.25, 0.49$, the respective ARE$_p$'s are: 92%, 85%, 79%, 68%, 48%. Among these, the first four estimators are uniformly more efficient than their empirical counterparts. But the most extreme trimming case of $a = b = 0.49$ (*i.e.*, it essentially represents a median-based estimator) is not as competitive, being only about half of the time more efficient than the empirical approach. On the other hand, asymmetric trimming may yield 100% efficient estimators (e.g., MTM-8 with $a = 0.05, b = 0.00$, and MTM-9 with $a = 0.25, b = 0.00$). Even more dramatically, the MTM-10 estimator, with $a = 0.50, b = 0.00$, uses only half of the actual data and still maintains 96%

efficiency. However, if we trim upper part of the sample, the corresponding estimator efficiencies drop significantly: MTM-6 with $a = 0.00, b = 0.10$, has $\text{ARE}_p = 85\%$; MTM-7 with $a = 0.00, b = 0.30$, has $\text{ARE}_p = 61\%$.

3.2. Lognormal and Normal Models

Let $X_1, \ldots, X_n$ be i.i.d. random variables, each with the same lognormal distribution

$$\text{LN}(\mu, \sigma): \quad F(x) = \Phi\left(\frac{\log(x) - \mu}{\sigma}\right), \quad x > 0, \tag{3.5}$$

where $-\infty < \mu < \infty$ and $\sigma > 0$ are unknown parameters, and Φ is the standard normal cdf. The qf and pdf are: $Q(p) = e^{\mu + \sigma\Phi^{-1}(p)}$ and $f(x) = (\sigma x)^{-1}\phi((\log(x) - \mu)/\sigma)$, where Φ^{-1} and ϕ are the standard normal qf and pdf, respectively.

Let us start with the maximum likelihood estimation of μ and σ. It is known (see, e.g., Serfling, 2002) that $\widehat{\mu}_{\text{MLE}} = n^{-1}\sum_{i=1}^{n}\log(X_i)$ and $\widehat{\sigma}_{\text{MLE}} = \sqrt{n^{-1}\sum_{i=1}^{n}\left(\log(X_i) - \widehat{\mu}_{\text{MLE}}\right)^2}$, and that

$$\left(\widehat{\mu}_{\text{MLE}}, \widehat{\sigma}_{\text{MLE}}\right) \quad \text{is} \quad \mathcal{AN}\left((\mu, \sigma), \frac{\sigma^2}{n}\mathbf{S_0}\right) \quad \text{with} \quad \mathbf{S_0} = \begin{bmatrix} 1 & 0 \\ 0 & 1/2 \end{bmatrix}. \tag{3.6}$$

The MTM estimators of μ and σ are given by $\widehat{\mu}_{\text{MTM}} = \widehat{t}_1 - c_1$ and $\widehat{\sigma}_{\text{MTM}} = \sqrt{(\widehat{t}_2 - \widehat{t}_1^2)/(c_2 - c_1^2)}$ where, for $k = 1, 2$, the constant $c_k = (1 - a - b)^{-1}\int_a^{1-b}\left[\Phi^{-1}(u)\right]^k \mathrm{d}u$ and the trimmed sample moment $\widehat{t}_k = (n - m_n - m_n^*)^{-1}\sum_{i=m_n+1}^{n-m_n^*}\left[\log(X_{i:n})\right]^k$ with $m_n/n \to a$ and $m_n^*/n \to b$. Further,

$$\left(\widehat{\mu}_{\text{MTM}}, \widehat{\sigma}_{\text{MTM}}\right) \quad \text{is} \quad \mathcal{AN}\left((\mu, \sigma), \frac{\sigma^2}{n}\mathbf{S}\right) \quad \text{with} \quad \mathbf{S} = \begin{bmatrix} s_{11} & s_{12} \\ s_{12} & s_{22} \end{bmatrix}, \tag{3.7}$$

where the parameter-free entries s_{ij} are as follows: $s_{11} = (c_{11}^* c_2^2 - 2c_1c_2c_{12}^* + c_1^2c_{22}^*)/(c_2 - c_1^2)^2$, $s_{12} = (-c_{11}^*c_1c_2 + c_2c_{12}^* + c_1^2c_{12}^* - c_1c_{22}^*)/(c_2 - c_1^2)^2$, and $s_{22} = (c_{11}^*c_1^2 - 2c_1c_{12}^* + c_{22}^*)/(c_2 - c_1^2)^2$. Here the constants c_1 and c_2 are defined as above and $c_{ij}^* = (1-a-b)^{-2}\int_a^{1-b}\int_a^{1-b}\left(\min\{u, v\} - uv\right)\left[\Phi^{-1}(u)\right]^{i-1}\left[\Phi^{-1}(v)\right]^{j-1} \mathrm{d}\Phi^{-1}(v)\,\mathrm{d}\Phi^{-1}(u)$, for $i, j = 1, 2$.

Now we have everything that is needed for computation of (2.3) and (2.4). That is: for the MLE, $\mathbf{d}_p\mathbf{\Sigma_\theta}\mathbf{d}_p' = \sigma^2\left[1 + 0.5(\Phi^{-1}(p))^2\right]$; for the EMP estimator,

$f^2(Q(p)) = \sigma^{-2}\phi^2(\Phi^{-1}(p))$ and, for the MTM estimator, $\mathbf{d}_p \boldsymbol{\Sigma}_{\boldsymbol{\theta}} \mathbf{d}_p' = \sigma^2\big[s_{11} + 2s_{12}\Phi^{-1}(p) + s_{22}(\Phi^{-1}(p))^2\big]$. These imply that:

$$\mathrm{ARE}_p\,(\mathrm{EMP},\ \mathrm{MLE}) = \frac{1 + 0.5(\Phi^{-1}(p))^2}{p(1-p)}\phi^2(\Phi^{-1}(p)),$$
$$\mathrm{ARE}_p\,(\mathrm{MTM},\ \mathrm{MLE}) = \frac{1 + 0.5(\Phi^{-1}(p))^2}{s_{11} + 2s_{12}\Phi^{-1}(p) + s_{22}(\Phi^{-1}(p))^2}. \tag{3.8}$$

Note 3.2 As was mentioned above, the LN$(\mu,\ \sigma)$ is equivalent—through the logarithmic transformation of the variable—to a normal variable. That is, if X is LN$(\mu,\ \sigma)$ with the cdf defined by equation (3.5), then $Z = \log X$ is a normal variable with the cdf $F_Z(z) = \Phi((z-\mu)/\sigma)$, $-\infty < z < \infty$. The corresponding qf and pdf are: $Q_Z(p) = \mu + \sigma\Phi^{-1}(p)$ and $f_Z(z) = \sigma^{-1}\phi((z - \mu)/\sigma)$, respectively. The formulas of MLE and MTM estimators of μ and σ do not change (except that $\log X_i$'s are now replaced with Z_i's) and their asymptotic distributions agree with (3.6) and (3.7), respectively. Similar to the first example (see Section 3.1), intermediate results for the normal distribution differ from those for the lognormal but the relative efficiency formulas are identical to those in (3.8). Hence, RECs for these two models are the same. □

In Figure 2, RECs of MLE, MTM and EMP estimators of lognormal and normal quantiles are plotted. The plots reveal several interesting facts. First of all, estimation of lower/upper tail quantiles with EMP estimator is inefficient. Secondly, as expected, the empirical REC is symmetric. Thirdly, a somewhat surprising finding is that approximately 20th and 80th percentiles—not the median—can be estimated most efficiently via the empirical estimator, with $\mathrm{ARE}_{0.20} = \mathrm{ARE}_{0.80} = 66\%$ and $\mathrm{ARE}_{0.50} = 64\%$. (Of course, the difference is not substantial.) Further, unlike in the Pareto and exponential examples, ARE_p of the MTM estimator is not constant, and thus RECs of MTMs are now indeed curves. For symmetrically trimmed MTMs, the corresponding RECs are symmetric and, when compared to the empirical REC, are: uniformly above it (MTM-1, -2, -3), uniformly below it (MTM-5), and crossing it at several points (MTM-4). The inefficiency of the MTM-5 estimator can be attributed to the fact that, in general, median-based estimators have poor efficiency properties for estimating the scale parameter which is needed for the estimation of quantiles. While among asymmetrically trimmed MTMs there are only two estimators (MTM-6, -8) that are uniformly more efficient than EMP, every asymmetric MTM also has a percentile region of excellent performance (*i.e.*, $\mathrm{ARE}_p \geq 95\%$). For example: MTM-6 at $0.08 \leq p \leq 0.54$, MTM-7 at $0.14 \leq p \leq 0.34$, MTM-8 at $0.32 \leq p \leq 0.98$, MTM-9 at $0.62 \leq p \leq 0.86$, MTM-10 at $0.76 \leq p \leq 0.86$.

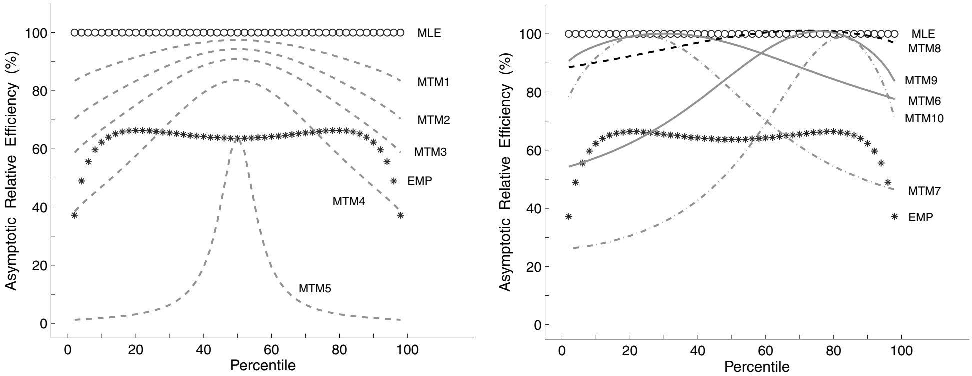

Figure 2. RECs of MLE, MTM, and empirical estimators of lognormal and normal quantiles. Left panel: symmetrically trimmed MTMs. Right panel: asymmetrically trimmed MTMs.

4. Final remarks

In this paper, we have introduced a *practical* tool for comparison of two competing statistical procedures, which we call the relative efficiency curve, or REC for short. Such a curve is motivated by situations (applications) where the researcher is interested in certain aspects of the underlying distribution and has several methodological options at his/her disposal. Our illustrations have been focused on parametric, robust parametric, and empirical nonparametric approaches for quantile estimation, though similar curves can be constructed using other (e.g., semiparametric) techniques as well.

Further, for regular parametric families, MLE is the most efficient method of estimation and thus we used it as a benchmark procedure for construction of RECs. In such setting, the REC of the empirical approach can be interpreted as a *parametric signature* of empirical quantiles. Then, such signatures can be employed, for example, for choosing trimming proportions of robust estimators or for selection of most informative probability levels for quantile-based estimators. Furthermore, the approach might also be useful for comparing inefficient but perhaps more flexible and/or easily computable estimators. This is especially true for nonregular parametric families, for which either MLE properties are difficult to establish or it simply does not exist (e.g., for the generalized Pareto distribution with the tail parameter $\gamma > 1$). In these cases, however, there is no obvious benchmark estimator and thus RECs will not be capped at 100%.

In summary, no matter what statistical methodology is used or what benchmark estimator is chosen, RECs are fairly easy to construct and can form a better picture about the model-fitting process. If properly interpreted, they not only help us to know *what* happens with various model-fitting procedures but also provide an insight of *why* it happens.

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