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A Note on Sunspot Equilibria in Search Models of Fiat Money*

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Search models can generate an endogenous role for fiat money, in the sense that there exist equilibria where intrisically useless, unbacked, paper currency is valued due to its function as a medium of exchange. In this note, I ask if there exist sunspot equilibria in these models, where the value or acceptability of money fluctuates along with extrinsic random events even though the fundamentals of the economy are deterministic and time invariant. The answer is yes. *Journal of Economic Literature* Classification Numbers: C70, E40.

I. INTRODUCTION

Search models of the exchange process can generate an endogenous role for fiat money, in the sense that there exist equilibria where intrinsically useless, unbacked, paper currency is valued due to its function as a medium of exchange; see, for example Kiyotaki and Wright [3, 4]. In this note, I ask if there exist equilibria in these models where the value or the acceptability of money fluctuates along with extrinsic random events, called *sunspots*, even though the fundamentals of the economy are deterministic and time invariant.

In terms of its relation to the literature, this continues the research program that attempts to characterize the set environments in which

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sunspots matter.[1] There has been almost no work on the possibility of sunspot equilibria in search models. An exception is Howitt and McAfee [2], but their model has nothing to do with money. It is especially interesting to consider the effects of extrinsic uncertainty in models with fiat currency, since there is a venerable notion that monetary economies are particularly susceptible to randomness or indeterminacy (see Mill [6]). Further, many standard monetary models can generate indeterminacy or sunspot equilibria, such as overlapping generations models (Azariadis [1]), cash-in-advance models (Woodford [9]), and models with money in the utility function (Matsuyama [5]). It therefore seems natural to inquire as to whether search models have similar properties. This note answers in the affirmative.

II. The Model

The basic framework is the one in Kiyotaki and Wright [3,4], to which the reader can refer for additional motivation. There is a continuum of infinitely lived symetric agents with a total mass of unity and a set C of indivisible commodities. Each agent i is characterized by a set $C_i \subseteq C$, such that he derives utility $u > 0$ from consuming one unit of good j if $j \in C_i$ and no utility if $j \notin C_i$. It is assumed that for a randomly selected agent i and good j, $\text{prob}(j \in C_i) = x$, where $x \in [0, 1]$. Each agent i can produce one commodity $j \notin C_i$, and it is assumed that the distribution of production goods is uniform. Production occurs instantaneously after consumption of a commodity in C_i, at a cost normalized to zero. (Note that production cannot occur except after consumption.) The rate of time preference is $r > 0$.

At the initial date, agents in a set of measure M are each endowed with one unit of an indivisible object that has no utility value to any consumer and cannot be produced by any producer, called *fiat money*. The rest of the agents are endowed with their production goods. Agents meet bilaterally and at random in continuous time, according to a Poisson process with an arrival rate $\beta > 0$. When two agents meet, they swap inventories one-for-one if and only if mutually agreeable. Call agents with money *buyers* and agents with consumption goods *sellers*. Buyers must trade money for goods in order to consume; sellers may trade goods for money and then use the money to buy goods, or they may barter goods directly.

There is a transaction cost for accepting a consumption good in trade, denoted $\varepsilon \in (0, u)$. We consider only *symmetric* outcomes here, where no

[1] It is known that in finite, convex exonomies with complete, unrestricted markets and competitive behavior, there cannot exist equilibria where sunspots matter, and that in an economy that does not satisfy all of these qualifications, there can exist such equilibria. See the discussion and references in Shell and Wright [7].

agents or commodities are treated as special. In this case, the transaction cost implies that agent i will not accept any good in exchange that is not in C_i. (If he did, he would only have to trade it again, and since all goods are equally tradable he is better off not accepting it and thus saving the transaction cost.) This means that agent i accepts consumption good j if and only if $j \in C_i$. The interesting issue is to determine if agents accept money. For simplicity, it is assumed that there is no transaction cost to accepting money, but this can be relaxed (see Kiyotaki and Wright [4]).

Let $s \in S$ represent an extrinsic random variable, called a *sunspot*, where $S = \{1, 2, ..., n\}$. It evolves according to a Poisson process: when $s = i$ it switches to $s = j$ at rate λ_{ij} (by convention, $\lambda_{ii} = 0$). The *hazard rate* is the rate at which s switches from state i to any other state: $H_i = \sum_j \lambda_{ij}$. Let V_j^s denote the expected discounted utility (the *payoff* or *value* function) for an agent with object j in state s, where $j = \mathrm{c}$ if he has a commodity and $j = \mathrm{m}$ if he has money. Let Π_s denote the probability that agents accept money in state s (mixed strategies are allowed). It is possible in principle for payoffs and strategies to depend on s, even though preferences, technology, and other fundamentals in the economy do not. However, given s, payoffs and strategies are independent of calendar time (that is, only steady-state equilibria will be considered).

A convenient recursive representation for the continuous time payoff functions can be derived as follows. Suppose first that time proceeds in discrete units of length δ. The value of holding a commodity in state s is given by

$$V_{\mathrm{c}}^s = (1 + r\delta)^{-1} \Big\{ \beta\delta(1 - M)\, x^2(u - \varepsilon + V_{\mathrm{c}}^s) + \beta\delta M x \pi_s V_{\mathrm{m}}^s + \sum_t \lambda_{st}\, \delta V_{\mathrm{c}}^t$$
$$+ [1 - \beta\delta(1 - M)\, x^2 - \beta\delta M x \pi_s - \delta H_s]\, V_{\mathrm{c}}^s + o(\delta) \Big\}, \qquad (1)$$

where $o(\delta)$ represents the payoff in the event that two or more Poisson arrivals occur in the interval δ and therefore satisfies $o(\delta)/\delta \to 0$ as $\delta \to 0$. The variable π_s is the probability with which the individual agent accepts money in state s and is chosen to maximize V_{c}^s, taking as given the probability with which other agents are accepting money in each state, Π_s for all $s \in S$.[2]

[2] The first term inside the braces in (1) is the probability that a seller meets another seller and both want to trade, $\beta\delta(1 - M)\, x^2$, times the value of trading. The second term is the probability that a seller meets a buyer who wants to trade, $\beta\delta M x$, times the gain from choosing whether to accept money. The third terms sums the probabilities that the state changes times the value functions in the new states. The fourth term is the probability that none of these events occurs. Actually, each probability is an approximation when $\delta > 0$, given the Poisson assumption, which is why $o(\delta)$ appears.

Rearranging (1) and letting $\delta \to 0$ yields

$$rV_c^s = \beta(1-M)\,x^2(u-\varepsilon) + \beta M x \pi_s (V_m^s - V_c^s) + \sum_t \lambda_{st}(V_c^t - V_c^s). \quad (2)$$

A similar derivation yields a similar representation for the value of holding money:

$$rV_m^s = \beta(1-M)\,x\Pi_s(u-\varepsilon + V_c^s - V_m^s) + \sum_t \lambda_{st}(V_m^t - V_m^s). \quad (3)$$

Let $\Delta_s = V_m^s - V_s$. Then (2) and (3) imply that

$$[r + (1-M)\,\Pi_s + M\pi_s]\,\Delta_s = (1-M)(u-\varepsilon)(\Pi_s - x) + \sum_t \lambda_{st}(\Delta_t - \Delta_s), \quad (4)$$

if we normalize time (with no loss in generality) so that $\beta x = 1$. For any s, (4) implies that the best response condition for an individual in state s is described by

$$\Delta_s > 0 \Rightarrow \pi_s = 1; \qquad \Delta_s < 0 \Rightarrow \pi_s = 0; \qquad \Delta_s = 0 \Rightarrow \pi_s = [0, 1]. \quad (5)$$

For any exogenously specified process for sunspots, as summarized by the transition rates λ_{st} for all $s, t \in S$, an *equilibrium* is defined as a value for Π_s for all $s \in S$ that satisfies (5), where Δ_s is given by (2)–(4) with $\pi_s = \Pi_s$. One possibility is that agents choose to ignore sunspots, so that $\Pi_s = \Pi$, $V_j^s = V_j$, and $\Delta_s = \Delta$, for all s. In this case, (5) immediately implies that there are exactly three equilibrium values of Π: $\Pi = 0$, x, and 1. In a *proper* sunspot equilibrium, it is not the case that $\Pi_s = \Pi$ for all s. It is assumed below that Π_s is different in each of the n states, and states will be rebelled if necessary so that $\Pi_1 < \Pi_2 < \cdots < \Pi_n$.

III. The Case of Two States

For now, consider the special case where $n = 2$, so that $H_1 = \lambda_{12}$ and $H_2 = \lambda_{21}$ (we return to the general n-state case below). After inserting the equilibrium condition $\pi_s = \Pi_s$, (4) implies that

$$(r + \Pi_1)\,\Delta_1 = (1-M)(u-\varepsilon)(\Pi_1 - x) + H_1(\Delta_2 - \Delta_1) \quad (6)$$

$$(r + \Pi_2)\,\Delta_2 = (1-M)(u-\varepsilon)(\Pi_2 - x) + H_2(\Delta_1 - \Delta_2). \quad (7)$$

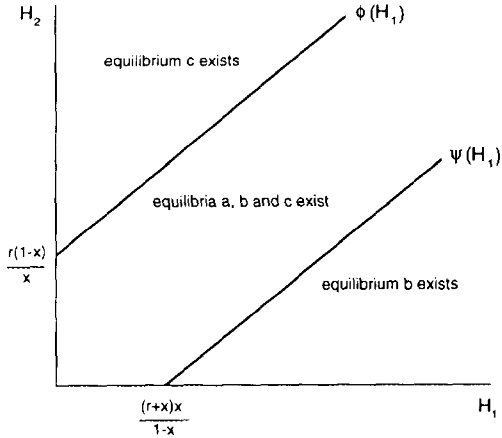

FIG. 1. Existence of equilibria.

Solving (6) and (7) yields

$$D\Delta_1 = (r + \Pi_2 + H_2)(\Pi_1 - x) + H_1(\Pi_2 - x) \tag{8}$$

$$D\Delta_2 = (r + \Pi_1 + H_1)(\Pi_2 - x) + H_2(\Pi_1 - x), \tag{9}$$

where $D \equiv [(r + \Pi_1 + H_1)(r + \Pi_2 + H_2) - H_1 H_2]/(1 - M)(u - \varepsilon) > 0$.

We will construct equilibria where $\Pi_1 < \Pi_2$. There are exactly four possibilities: (a) $\Pi_1 = 0$, $\Pi_2 = 1$; (b) $\Pi_1 = 0 < \Pi_2 < 1$; (c) $0 < \Pi_1 < 1 = \Pi_2$;

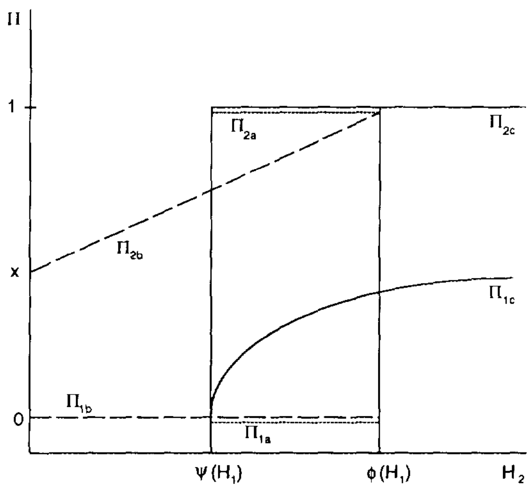

FIG. 2. Equilibrium acceptance probabilities.

and (d) $0 < \Pi_1 < \Pi_2 < 1$. Whether these possibilities actually constitute equilibria depends on H_1 and H_2. It will be useful to define the functions

$$\varphi(H_1) \equiv (r + H_1)(1 - x)/x \tag{10}$$

$$\psi(H_1) \equiv H_1(1 - x)/x - (r + x). \tag{11}$$

Then I claim (a) $\Pi_1 = 0$, $\Pi_2 = 1$ is an equilibrium iff $\psi(H_1) \leqslant H_2 \leqslant \varphi(H_1)$; (b) $\Pi_1 = 0 < \Pi_2 < 1$ is an equilibium iff $H_2 < \varphi(H_1)$; (c) $0 < \Pi_1 < 1 = \Pi_2$ is an equilibrium iff $H_2 > \psi(H_1)$; and $0 < \Pi_1 < \Pi_2 < 1$ is never an equilibrium.

Consider case (a). By virtue of (5), this is an equilibrium iff $\Delta_1 \leqslant 0 \leqslant \Delta_2$. If we insert $\Pi_1 = 0$ and $\Pi_2 = 1$ into (8) and (9), it can be seen after some minor algebra that that $\Delta_1 \leqslant 0 \leqslant \Delta_2$ iff $\varphi(H_1) \geqslant H_2 \geqslant \psi(H_1)$, as claimed. Now consider case (b), which is an equilibrium iff $\Delta_1 \leqslant 0 = \Delta_2$. If we set $\Pi_1 = 0$ and $\Delta_2 = 0$, (9) implies that $\Pi_2 = x + xH_2/(r + H_1)$. This is consistent with $\Pi_2 \in (0, 1)$ iff $H_2 < \varphi(H_1)$. Similarly, case (c) is an equilibrium iff $\Pi_1 = x - (1 - x) H_1/(r + x + H_2)$, which is consistent with $\Pi_1 \in (0, 1)$ iff $H_2 > \psi(H_1)$. Finally, case (d) is an equilibrium iff $\Delta_1 = \Delta_2 = 0$; but $\Delta_1 = \Delta_2 = 0$ implies that $\Pi_1 = \Pi_2 = x$, and hence there exists no equilibrium with $0 < \Pi_1 < \Pi_2 < 1$. This establishes the claim.

Figure 1 shows the regions in the (H_1, H_2) plane where the various equilibria exist. Figure 2 shows the equilibrium values of Π_1 and Π_2 as correspondences of H_2, given a fixed (but arbitrary) value of H_1; the branch labeled Π_{sj} depicts the value of Π_s in an equilibrium of type j, where $j =$ a, b, or c. Consider, for example, equilibrium a, where $\Pi_1 = 0$ and $\Pi_2 = 1$. For sellers to refuse money in state 1, the probability of switching to state 2 cannot be too great relative to the probability of switching back, which is the restriction $\psi(H_1) \leqslant H_2$. Similarly, for sellers to accept money in state 2, the probability of switching to state 1 cannot be too great, which is the restriction $H_2 \leqslant \varphi(H_1)$. If these restrictions both hold, then the equilibrium with $\Pi_1 = 0$ and $\Pi_2 = 1$ exists. Note that when Π_s is strictly between 0 and 1 it is increasing in H_2; hence, the acceptability of money rises with the probability that it will fall in value. Of course, it is the fact that Π_s goes up when H_2 falls that keeps sellers indifferent between accepting and rejecting money and thereby allows them to use a mixed strategy.

IV. The General Case

We now return to the general case. Since there are $n(n-1)$ different transition rates λ_{st}, the results cannot be summarized as easily as in the previous section, but the idea is basically the same. As in the $n = 2$ case, there are potentially four types of (proper n-state sunspot) equilibria with

$\Pi_1 < \Pi_2 < \cdots < \Pi_n$: (a) $\Pi_1 = 0$, $\Pi_n = 1$; (b) $\Pi_1 = 0 < \Pi_n < 1$; (c) $0 < \Pi_1 < 1 = \Pi_n$; and (d) $0 < \Pi_1$, $\Pi_n < 1$. As in the two-state model, it is easy to show that case (d) cannot arise, since $0 < \Pi_s < 1$ for all s implies that $\Pi_s = x$ for all s. This leaves cases (a), (b), and (c).

Consider case (a). It we set $\Delta_s = 0$ for $s = 2, 3, ..., n-1$, then (4) can be solved for $s = 1$ and $s = n$,

$$D\Delta_1 = -x(r+1+H_n) + \lambda_{1n}(1-x) \tag{12}$$

$$D\Delta_n = (r+H_1)(1-x) - \lambda_{n1}x,$$

where $D \equiv [(r+H_1)(r+1+H_n) - \lambda_{1n}\lambda_{n1}]/(1-M)(u-\varepsilon) > 0$. For $\Pi_1 = 0$ and $\Pi_n = 1$ to be an equilibrium, we require $\Delta_1 \leqslant 0 \leqslant \Delta_n$. This holds iff

$$\lambda_{1n} \leqslant \frac{x}{1-x}(r+1+H_n) \tag{14}$$

$$\lambda_{n1} \leqslant \frac{1-x}{x}(r+H_1). \tag{15}$$

Now, given Δ_1 and Δ_n from (12) and (13), we can rearrange $\Delta_s = 0$ for $s = 2, 3, ..., n-1$ to yield

$$\Pi_s = x - (\lambda_{s1}\Delta_1 + \lambda_{sn}\Delta_n)/(1-M)(u-\varepsilon). \tag{16}$$

We could substitute Δ_1 and Δ_n into (16) to get the equilibrium value of Π_s explicitly, although the result is not particularly enlightening. What remains to be checked in order to guarantee this is an equilibrium is $0 < \Pi_s < 1$, $s = 2, 3, ..., n-1$. Clearly, these conditions hold as long as λ_{s1} and λ_{sn} are not too large. Moreover, Π_s can be greater or less than x, and $\Pi_s > \Pi_t$ iff $\lambda_{s1}\Delta_1 + \lambda_{sn}\Delta_n < \lambda_{t1}\Delta_1 + \lambda_{tn}\Delta_n$. This shows that Π_s depends on λ_{s1} and λ_{sn}, and it depends on λ_{1n}, λ_{n1}, H_1, and H_n because Δ_1 and Δ_n do; however, perhaps surprisingly, the other transition rates λ_{ij} do not affect Π_s at all. In any event, this shows that case (a) equilibria can exist.

Now consider case (b), $\Pi_1 = 0 < \Pi_2 < \cdots < \Pi_n < 1$. For this to be an equilibrium we require that $\Delta_1 \leqslant 0$ and $\Delta_s = 0$ for $s = 2, 3, ..., n$. Then (4) yields

$$(r+H_1)\Delta_1 = -(1-M)(u-\varepsilon)x < 0. \tag{17}$$

Now we can solve $\Delta_s = 0$ for $s = 2, 3, ..., n$ to yield

$$\Pi_s = x - \lambda_{s1}\Delta_1/(1-M)(u-\varepsilon) = x + \lambda_{s1}x/(r+H_1). \tag{18}$$

Note that $\Pi_s > x$, and $\Pi_s < 1$ iff $\lambda_{s1} < \varphi(H_1)$, where φ is defined in (10). Also, Π_s depends only on λ_{s1} and H_1 and is increasing in s provided that we label states so that λ_{s1} is increasing in s.

Finally, consider case (c), $0 < \Pi_1 < \cdots < \Pi_{n-1} < 1 = \Pi_n$. For this to be an equilibrium we require that $\Delta_s = 0$ for $s = 1, 2, ..., n-1$ and $\Delta_n \geqslant 0$. Solving for Δ_n, we have

$$(r + 1 + H_n)\,\Delta_n = (1 - M)(u - \varepsilon)(1 - x) > 0. \tag{19}$$

Now we can solve $\Delta_s = 0$ for $s = 1, 2, ..., n-1$ to yield

$$\Pi_s = x - \lambda_{sn}\,\Delta_n/(1 - M)(u - \varepsilon) = x - \lambda_{sn}(1 - x)/(r + 1 + H_n). \tag{20}$$

Note that $\Pi_s < x$, and $\Pi_s > 0$ iff $\psi(\lambda_{sn}) < H_n$, where ψ is defined in (11). Also, Π_s depends only on λ_{sn} and H_n and is increasing in s provided that we label states so that λ_{sn} is decreasing in s. In any event, all three types of equilibria exist for some parameter values.

V. Conclusion

It has been shown that search models of fiat money, like some other models of fiat money, and like some other models where the welfare theorems do not apply, are susceptible to sunspot equilibria. In these equilibria there are fluctuations over time in the value of money, as measured by the probability that a random seller accepts it. An interesting extension would be to look for sunspot equilibria in search based models of fiat money where currency is always accepted, but the price at which it is accepted varies over time. Progress along these lines has been made recently by Shi [8].

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