

Search, Bargaining, Money, and Prices

Alberto Trejos

Northwestern University and Instituto de Análisis Económico de Barcelona

Randall Wright

University of Pennsylvania and Federal Reserve Bank of Minneapolis

The goal of this paper is to extend existing search-theoretic models of fiat money, which until now have assumed that the price level is exogenous, by explicitly incorporating bilateral bargaining. This allows us to determine the price level endogenously and leads to additional insights concerning the role of money. For example, we find that monetary equilibria are generally inefficient in the sense that output and prices differ from the solution to a social planner's problem, although the difference can become small as the discount rate or search friction vanishes. We also find that there exist nonstationary inflationary equilibria.

I. Introduction

Search-theoretic models can be used to formalize the role of money as a medium of exchange and thereby provide a theoretical foundation for monetary economics. The framework allows one to be precise about the frictions that can make monetary exchange an equilibrium or an efficient arrangement, to determine endogenously which objects serve as media of exchange, and to address many other substan-

We thank Melvyn Coles, Nobuhiro Kiyotaki, George Mailath, Akihiko Matsui, Shouyong Shi, the referees and editors of this *Journal*, and participants in many workshops and conferences for their very helpful input. We also thank the National Science Foundation and the University of Pennsylvania Research Foundation for financial support. The views expressed here are those of the authors, not necessarily those of the Federal Reserve Bank of Minneapolis or the Federal Reserve System.

[*Journal of Political Economy*, 1995, vol. 103, no. 1]

© 1995 by The University of Chicago. All rights reserved. 0022-3808/95/0301-0006\$01.50

tive issues in monetary economics. The first generation of models in this literature is incomplete, however, in the sense that while the papers analyze in detail the *exchange process*, they neglect the determination of *exchange rates*. That is, either it is assumed that prices are fixed or it is assumed that all objects are indivisible and therefore must trade one for one, or some other more or less ad hoc restriction is imposed on the relative prices of goods and money.¹

The goal of this project is to extend these models by endogenizing prices. In particular, into a search-based model of fiat currency with divisible commodities, we introduce a standard strategic bargaining game (which also has a limiting form that is equivalent to a simple axiomatic bargaining model). This not only allows us to determine the price level endogenously but also leads to additional insights into the role of money in the exchange process. Moreover, bargaining is natural in the context of bilateral exchange and fits into the framework in such a way that we are able to maintain both the spirit and the simplicity of the search-theoretic approach to monetary economics.

In terms of results, we characterize both steady-state and dynamic equilibria in several versions of the model. For some specifications there is a unique steady-state monetary equilibrium; for others there are multiple monetary steady states. Additionally, there are nonstationary equilibria that exhibit inflation. We also study the relationship between prices and the amount of money in the system. If the amount of money is small (i.e., if few agents are endowed with fiat currency), then an increase in the money supply could actually cause the price level to fall because of a liquidity effect; but as the money supply increases further, prices necessarily begin to rise. In any event, for a fixed endowment of money, equilibria are generally inefficient in the sense that output and the price level differ from the solution to a social planner's problem. Under certain conditions, however, the difference becomes small as the discount rate or the search friction vanishes.

We present the analysis in the following steps. In Section II, after describing the basic assumptions, we first consider the "monetary theory" part of the model ignoring the "price theory" part; that is, we show how a fixed-price search model of fiat currency emerges as a special case. In Section III, we then consider the "price theory" part ignoring the "monetary theory" part; that is, we study the bargaining

¹ Some examples from the literature include Jones (1976), Diamond (1984), Kiyotaki and Wright (1989, 1991, 1993), Oh (1989), Marimon, McGrattan, and Sargent (1990), Aiyagari and Wallace (1991, 1992), Matsuyama, Kiyotaki, and Matsui (1993), Trejos (1993), Zhou (1993), Cuadras-Morato (1994), Williamson and Wright (1994), Li (in press), and Wright (in press). See Ostroy and Starr (1990) for an extensive survey of related models of monetary exchange.

problem taking as given that money has value. In Section IV, we integrate these two parts of the model in order to characterize the set of steady-state equilibria and derive our substantive results concerning efficiency and the relationship between money and prices. In Section V, we consider some alternative assumptions relating to the search and bargaining processes and show how certain results can change. In Section VI, we study dynamics. In Section VII, we present some concluding remarks.²

II. Monetary Theory

The economy is populated by a continuum of infinitely lived agents, with the total population normalized to unity. Agents produce and consume *services* (or, equivalently, nonstorable goods) at discrete points in continuous time. These services are specialized in the following sense: at any point in time, each agent has a demand for a particular variety of services that can be produced by only a fraction x of the population; symmetrically, each agent has the ability to produce a variety of services that is demanded by only a fraction x of the population. The smaller x is, other things being equal, the more specialized the economy and the more difficult exchange is. We assume that agents never produce services for their own consumption (although all that is really necessary is to assume that they do not always do so). This means that they need to trade in order to consume.

Initially, a fraction $M \in [0, 1]$ of the population are each endowed with one (normalized) unit of fiat money, which by definition is an object that cannot be consumed or produced by any private individual and has potential use only as a medium of exchange. For simplicity, we want it to be the case that when agents with money spend their money, they always spend *all* of it, since this implies that it is trivial to solve for the equilibrium distribution of money holdings: at every date the population partitions into a group of M buyers with one unit of currency and a group of $1 - M$ sellers with no currency.³ The

² In independent research, Shi (in press) developed a model that is very similar to ours in many respects. He also describes some extensions and applications other than those considered here, including the analysis of economies with multiple currencies and of equilibria with extrinsic uncertainty. Earlier, Diamond (1984) discussed a way of endogenizing the price level in a search model of money that was motivated by a bargaining story, but that approach does not actually model bargaining explicitly.

³ We are also implicitly assuming that buyers do not sell to other buyers; i.e., they never acquire a second unit of money. This can be motivated in a variety of ways; e.g., we can suppose that agents must consume in order to produce (see Aiyagari and Wallace 1991, 1992), or we can suppose that buyers never meet each other in the search process (see Burdett et al. 1993). As discussed in Sec. V, if agents are allowed to accumulate more than one unit or to spend fractional parts of their cash balances, then the equilibrium distribution of money will be complicated and has to be deter-

most straightforward way to guarantee that buyers always spend all their cash is simply to assume that the monetary object is indivisible, and this is what we do for most of the analysis here. However, we demonstrate in Section V that some key results are robust to relaxing these assumptions and allowing agents to hold any amount of money.

Consumption goods or services, in contrast to money, are assumed to be perfectly divisible, and if one unit of fiat currency exchanges for q units of consumption, the implied price level is $p = 1/q$. When a seller produces q units for a buyer, the buyer enjoys utility $u(q)$ and the seller suffers disutility $c(q)$. We assume that $u'(q) > 0$ and $c'(q) > 0$, for all $q > 0$. We also assume that $u''(q) \leq 0$ and $c''(q) \geq 0$, with at least one strict inequality, for all $q > 0$. We also assume that $u(0) = c(0) = 0$, that $u'(0) > c'(0) = 0$, and that there exists a $\hat{q} > 0$ such that $u(\hat{q}) = c(\hat{q})$. As additional notation to describe preferences, let q^* be the value of q in $(0, \hat{q})$ that satisfies $u'(q^*) = c'(q^*)$, and let $r > 0$ be the rate of time preference.

For now, we want to consider a version of the model in which there is no direct barter (but see Sec. V). One way to motivate this is to assume that buyers search while sellers remain at distinct physical locations, say, because they must be in a particular place in order to produce. Then no two sellers ever meet, and in all trades buyers travel to sellers' locations and pay in cash.⁴ There are *frictions* involved in travel, which are modeled here by assuming that the amount of time it takes to get to a seller's location is an exponentially distributed random variable. This is formally equivalent to the standard assumption in search theory that buyers sample locations according to a Poisson process. We assume that the rate at which a buyer locates sellers in this search process is proportional to the number of sellers, $\beta(1 - M)$, and the rate at which a seller locates buyers is proportional to the number of buyers, βM .

When a buyer and a random seller meet, the probability that they can trade is given by x , since this is the proportion of sellers who can produce the appropriate service.⁵ Hence, the arrival rate of appro-

mined endogenously. See Diamond and Yellin (1987, 1990) for analyses of the problem in related models.

⁴ Alternatively, one can rule out direct barter by assuming that because of specialization, there is never a double coincidence; i.e., the probability that one agent can produce the service another agent desires *and* vice versa is zero. In any case, we shall consider a version of the model with barter in addition to monetary exchange in Sec. V.

⁵ This assumes that buyers sample sellers randomly. Alternatively, we could assume that buyers always go directly to the appropriate seller (i.e., one that can produce the desired service); all that matters is that it takes time to get there. In the version of the model with no barter, this would be merely a renormalization of the arrival rate, or the special case in which $x = 1$. In the generalized version of the model presented in

appropriate buyers for a seller is βMx , and the arrival rate of appropriate sellers for a buyer is $\beta(1 - M)x$. For most of what follows, we normalize $\beta x = 1$ (with no loss in generality). When an appropriate buyer and seller meet, they bargain over q , the amount of the service to be provided in exchange for the money. If and when they come to an agreement, they trade, the buyer becomes a seller, and the seller becomes a buyer. The nature of the bargaining problem will be described in detail below. First we consider the model in which the time path for the value of money is given exogenously by $q_t = Q_t$, say, where Q_t is a right-differentiable function of t .

Let V_{st} and V_{bt} denote expected lifetime utilities for a seller and a buyer at date t (the *value functions*), given a time path for Q_t . Consider first V_{st} , the value function of a seller. Under the Poisson assumption and our normalization $\beta x = 1$, in a short discrete time interval of length $\Delta > 0$ the probability that exactly one appropriate buyer will arrive is approximately $M\Delta$, the probability that no appropriate buyer will arrive is approximately $1 - M\Delta$, and the probability that more than one will arrive is proportional to $o(\Delta)$, where $o(\Delta)/\Delta \rightarrow 0$ as $\Delta \rightarrow 0$. Therefore,

$$V_{st} = \frac{1}{1 + r\Delta} \{M\Delta[V_{bt+\Delta} - c(Q_{t+\Delta})] + (1 - M\Delta)V_{st+\Delta} + o(\Delta)\}.$$

Simplification yields

$$r\Delta V_{st} = M\Delta[V_{bt+\Delta} - V_{st+\Delta} - c(Q_{t+\Delta})] + V_{st+\Delta} - V_{st} + o(\Delta).$$

If we divide by Δ and take the limit as $\Delta \rightarrow 0$, we find

$$rV_{st} = M[V_{bt} - V_{st} - c(Q_t)] + \dot{V}_{st}. \quad (1)$$

A very similar derivation yields

$$rV_{bt} = (1 - M)[u(Q_t) + V_{st} - V_{bt}] + \dot{V}_{bt}. \quad (2)$$

In the special case in which $Q_t = Q$ is constant with respect to time ($0 < Q < \hat{q}$), (1) and (2) can be solved explicitly for the steady-state value functions

$$V_s = V_s(Q) = \frac{M}{r(1 + r)} [(1 - M)u(Q) - (r + 1 - M)c(Q)] \quad (3)$$

and

$$V_b = V_b(Q) = \frac{1 - M}{r(1 + r)} [(r + M)u(Q) - Mc(Q)]. \quad (4)$$

Sec. V, which allows barter, we shall want to assume that buyers sample sellers more or less at random and that $x < 1$, since this makes direct barter difficult and thereby generates an interesting role for a medium of exchange.

If Q is exogenous, (3) and (4) constitute a version of the standard fixed-price search model of fiat money. In particular, observe that $V_b - c(Q) > V_s > 0$, which means that sellers voluntarily accept money, if and only if $(1 - M)u(Q) > (r + 1 - M)c(Q)$. The result $V_b > 0$ implies that money has value or yields indirect utility, even though it yields no direct utility from consumption.

III. Price Theory

In this section we introduce bargaining over q , taking as given that money has some value $V_b > 0$. For now, we focus on steady states, where all endogenous variables are constant with respect to time. The basic bargaining solution can be thought of either as the axiomatic model of Nash (1950) or as the strategic model of Rubinstein (1982), since the latter has a reduced form that approaches the former as the time between rounds in the bargaining game goes to zero. However, as is well known, the exact Nash representation can depend critically on details of the underlying strategic formulation (see Binmore, Rubinstein, and Wolinsky 1986; Osborne and Rubinstein 1990); hence, it is important to be careful in specifying this part of the model.

When an appropriate buyer and seller meet, one of them is chosen at random to propose a value of q , to which the other can respond by either accepting or rejecting. If he accepts, the money changes hands and services are rendered. If he rejects, he can choose to walk away from the bargaining table to search for a new partner. The other agent can also choose to walk away from the bargaining table to search for a new partner. If either walks away, the agents cannot reconvene. If neither walks away, they wait a length of time Δ for another round, at which point someone is again chosen at random to make a proposal, and so the process continues. As is standard in this type of model, in equilibrium no one ever terminates the bargaining process voluntarily, all offers are made so that they are accepted, and negotiations are completed in the first round at a quantity that depends on whether it is the seller or the buyer who gets to go first.

Although offers are never rejected in equilibrium, it is the threat of rejecting and delaying settlement that drives the solution. The extent to which this threat matters depends on what happens during the interval Δ between a rejection and the next offer. In this section, we adopt the assumption that agents never meet other potential trading partners during this interval; in Section V we consider an alternative assumption and show how it makes a difference. However, under either assumption, there will be a unique subgame perfect equilib-

rium to the bargaining game (when V_s and V_b are taken as given), and this equilibrium has the property that a seller always proposes $q_s = q_s(\Delta)$, a buyer always proposes $q_b = q_b(\Delta)$, and these proposals are always accepted. Furthermore, as Δ becomes small, q_s and q_b converge to the same limit, q , which has a convenient representation in terms of a Nash bargaining solution.

To see how this works, note that the offers q_s and q_b are constructed so that the proposer gets as much utility as possible, subject to the responder's acceptance of his offer. Therefore, they satisfy the following relations:

$$V_s + u(q_s) = \frac{1}{1 + r\Delta} [V_s + \frac{1}{2}u(q_s) + \frac{1}{2}u(q_b)],$$

$$V_b - c(q_b) = \frac{1}{1 + r\Delta} [V_b - \frac{1}{2}c(q_s) - \frac{1}{2}c(q_b)].$$

The left-hand side of the first equation is the value to the buyer of accepting the seller's offer q_s , which is the utility of consuming q_s and becoming a seller. The right-hand side is the expected discounted value of rejecting: with equal probability each of the agents gets to make the next offer; in either case the offer is accepted (in equilibrium), and so the expected payoff next period is $V_s + \frac{1}{2}u(q_s) + \frac{1}{2}u(q_b)$. The second equation has a similar interpretation for a seller evaluating a buyer's proposal.

Rearranging these relations, we get

$$2\Delta r[V_s + u(q_s)] = u(q_b) - u(q_s),$$

$$2\Delta r[V_b - c(q_b)] = c(q_b) - c(q_s).$$

As $\Delta \rightarrow 0$, the left-hand sides vanish. This implies that in the limit $q_s = q_b = q$. Furthermore, if we take the ratio of the previous two conditions and then let $\Delta \rightarrow 0$, we get

$$\frac{V_s + u(q)}{V_b - c(q)} = \frac{u'(q)}{c'(q)}.$$

Hence, the limit q can be represented as

$$q = \operatorname{argmax}[V_s + u(q)][V_b - c(q)], \quad (5)$$

which means that it is the solution to a Nash bargaining problem.⁶

⁶ In general, a bilateral Nash bargaining problem takes the form $\max(U_1 - \bar{U}_1)(U_2 - \bar{U}_2)$, where U_j is the payoff and \bar{U}_j is the threat point of agent j . Here the payoffs are $V_s + u(q)$ for the buyer and $V_b - c(q)$ for the seller, and the threat points are zero. The threat points are nonzero under the alternative assumption made in Sec. V that agents can meet other potential trading partners during the bargaining

In bargaining over q , the individuals take V_s and V_b as given. One way to interpret this is to observe that V_s and V_b are both functions of the value of Q prevailing in the market, as given in (3) and (4), and in any bilateral meeting the agents negotiate over q taking Q as given. In equilibrium, of course, $q = Q$. Additionally, as is standard, we have to be sure that it is individually rational for both sellers and buyers to accept the q generated by (5). In the present context, this imposes the following constraints:

$$V_b(Q) - c(q) \geq V_s(Q) \quad (6)$$

and

$$V_s(Q) + u(q) \geq V_b(Q). \quad (7)$$

We take as our bargaining solution the q that satisfies (5) subject to (6) and (7).

IV. Equilibrium: Integrating Monetary and Price Theory

A *steady-state equilibrium* is a list (Q, V_s, V_b) satisfying the following conditions: (i) $q = Q$ solves the bargaining problem (5) subject to constraints (6) and (7), taking $V_b = V_b(Q)$ and $V_s = V_s(Q)$ as given; and (ii) V_s and V_b satisfy (3) and (4), taking Q as given. In principle, equilibria can be of two types. An *unconstrained equilibrium* occurs when $q = Q$ solves the unconstrained Nash problem (5) and does not violate (6) or (7). A *constrained equilibrium* occurs when Q satisfies constraints (6) and (7) and one of them is binding, and the unconstrained solution to the Nash problem violates the binding constraint. We also distinguish between *nonmonetary* steady-state equilibria with $Q = 0$ and *monetary* steady-state equilibria with $Q > 0$.

In equilibrium, $q = Q$, and we can substitute $V_s(q)$ and $V_b(q)$ into the constraints and simplify to yield the following results: (6) holds if and only if

$$\varphi(q) \equiv (1 - M)u(q) - (r + 1 - M)c(q) \geq 0; \quad (8)$$

and (7) holds if and only if

$$\psi(q) \equiv (r + M)u(q) - Mc(q) \geq 0. \quad (9)$$

Any q that satisfies $\varphi(q) \geq 0$ also satisfies $\psi(q) \geq 0$; hence, we need to check only that the former is satisfied in order to verify both constraints. Further, our assumptions imply $\varphi(0) = 0$, $\varphi'(0) > 0$, $\varphi''(q) \leq$

game, as is consistent with standard results in bargaining theory (see, e.g., Osborne and Rubinstein 1990).

0 for all q , and $\varphi(q) < 0$ for large q . Therefore, the constraints are satisfied if and only if q is below some critical value $\bar{q} = \bar{q}(r, M)$.⁷

The discussion above leads to the following method for constructing steady-state equilibria. To construct an unconstrained equilibrium, find a solution q to the unconstrained Nash problem, and show that it is less than the critical value \bar{q} . To construct a constrained equilibrium, show that when $Q = \bar{q}$ is taken as given the unconstrained solution to the Nash problem exceeds \bar{q} , which means that \bar{q} constitutes a constrained equilibrium.

PROPOSITION 1. For any $r > 0$ and $M \in (0, 1)$, there exists a non-monetary steady-state equilibrium and a unique monetary steady-state equilibrium. The monetary equilibrium is unconstrained, and it satisfies $u'(q) > c'(q)$.

Proof. The existence of the nonmonetary equilibrium is immediate, since $Q = V_b = V_s = 0$ satisfies all the equilibrium conditions. Consider unconstrained monetary equilibria. The necessary and sufficient condition for (5) can be written $[V_b - c(q)]u'(q) - [V_s + u(q)]c'(q) = 0$. If we insert V_s and V_b and set $Q = q$, this reduces to $T(q) = 0$, where

$$T(q) \equiv [(r + M)(1 - M)u(q) - \Phi c(q)]u'(q) - [\Phi u(q) - M(r + 1 - M)c(q)]c'(q) \quad (10)$$

and $\Phi \equiv r(1 + r) + M(1 - M)$. One can show $T(0) = 0$, $T'(0) > 0$, and $T(\bar{q}) < 0$. By continuity, there exists a $q \in (0, \bar{q})$ such that $T(q) = 0$.

This establishes the existence of an unconstrained equilibrium. To show that the unconstrained equilibrium is unique, rewrite $T(q) = 0$ as

$$\frac{c'(q)}{u'(q)} = \frac{(r + M)(1 - M)u(q) - \Phi c(q)}{\Phi u(q) - M(r + 1 - M)c(q)},$$

and note that the left-hand side is strictly increasing and the right-hand side is strictly decreasing in the interval $(0, \bar{q})$. To show that there does not exist a constrained equilibrium, note that $T(\bar{q}) < 0$ implies that, when $Q = \bar{q}$ is taken as given, the unconstrained solution to the Nash bargaining problem does not violate the constraint. Finally, to show $u'(q) > c'(q)$ in equilibrium, let q^* solve $u'(q^*) = c'(q^*)$ and observe that $T(q^*) < 0$. Q.E.D.

This establishes the existence of equilibrium where our intrinsically

⁷ By comparing (8) and (9) with (3) and (4), we see that the constraints on the bargaining problem are equivalent to the conditions that V_s and V_b are nonnegative in the steady state. Hence, they guarantee that agents do not want to drop out of the economy.

worthless asset has value determined by bargaining. The proposition also establishes the uniqueness of monetary equilibrium, although this depends critically on the assumption that there is no barter: in the generalized model with barter in addition to monetary exchange discussed in Section V, monetary equilibria generically come in pairs.

Perhaps the key result is $u'(q) > c'(q)$. That is, q is less than the q^* that equates marginal utility and marginal cost. Define *welfare* by $W = MV_b + (1 - M)V_s$, which is interpretable as average utility in the steady state or as the ex ante expected utility of all agents before the initial endowment of money is randomly distributed. Equations (3) and (4) imply

$$rW = M(1 - M)[u(q) - c(q)].$$

Hence, q^* is the value of q that a social planner would dictate, if he could, in order to maximize welfare. Since $q < q^*$, the monetary equilibrium is socially inefficient. This is so despite the fact that q is bilaterally efficient, in the sense that it is on the frontier of the payoff space in the bargaining game, with V_s and V_b taken as given.

Indeed, a stronger result is actually true. Not only is the equilibrium q too low according to the criterion W , it is also inefficiently low according to the ex post Pareto criterion. Let q_s^* and q_b^* denote the values of q that, if imposed on all agents, maximize V_s and V_b . It is not hard to verify that $q_s^* < q^* < q_b^*$. Given the distribution of money, any $q \in [q_s^*, q_b^*]$ is Pareto efficient, because if we either increase or decrease such a q , we make either buyers or sellers worse off. Simple algebra implies that $q < q_s^*$ in equilibrium. Hence, even sellers would prefer a lower price, if that price could be imposed on all transactions.

It might be suspected that the result $q < q^*$ is an artifact of our assumption that buyers always spend all their money. We show in Section V that it is not; that is, the result continues to hold in a generalized version of the model in which agents can store and trade any amount of money. The economic reason that q is inefficiently low is that sellers are receiving for their services cash that can be spent only in the future, and since they discount the future, they are willing to provide less than if they could turn the proceeds into immediate consumption (see Townsend [1980] for a similar finding). To confirm that it is discounting that drives the result, note from (10) that $q \rightarrow q^*$ as $r \rightarrow 0$. If we do not normalize the arrival rate β , an equivalent statement is that $q \rightarrow q^*$ as $\beta \rightarrow \infty$ (since it is only the ratio r/β that matters in the model). Further, it is shown in the Appendix that $\partial q/\partial r < 0$ and $\partial q/\partial \beta > 0$; hence, the greater the discount rate or the search friction, the greater the distance between q^* and q .

We close this section by considering the effect of changes in M , where M equals the money supply and also the number of buyers,

since by assumption each buyer has one unit of money. In the Appendix it is verified that there exists a critical value $\mu = \mu(r)$, which is strictly positive for small r , such that $\partial q/\partial M \leq 0$ if and only if $M \geq \mu$. Hence, when M is small, an increase in M may actually increase q and decrease p . However, $\mu < 1/2$ for all $r > 0$; so as M increases, the price level eventually begins to rise. The heuristic explanation for the result that p could fall with M is this: when M is small, an increase in M will increase the number of meetings between buyers and sellers—a “liquidity effect”—which can increase the amount of production and exchange. It is possible for this to dominate the tendency for increases in M to reduce the value of money.

We emphasize that the experiment above involves giving the indivisible monetary object to a larger number of agents.⁸ When we change the number of agents with money, we affect the number of meetings between buyers and sellers, and this affects real economic activity. In particular, differentiating welfare with respect to M , we get

$$r \frac{\partial W}{\partial M} = (u - c)(1 - 2M) + M(1 - M)(u' - c') \frac{\partial q}{\partial M}.$$

The first term is the liquidity effect. It implies that if q is fixed exogenously, then W is maximized at $M = 1/2$. In equilibrium, however, the second term is negative at $M = 1/2$ (since $\partial q/\partial M < 0$ at $M = 1/2$ and $q < q^*$). Hence, in a model with divisible commodities, the welfare-maximizing value of M balances the trade-off between providing liquidity and raising the price level.⁹

V. Alternative Assumptions

In this section we present some extensions of the basic model. One is to consider an alternative bargaining environment in which there is the possibility of meeting other trading partners during the period of delay after a rejected offer. Another is to allow direct barter in addition to monetary exchange. We also sketch a version of the model with divisible money; although we cannot solve this model completely, we show how the key result $q < q^*$ will continue to hold. We also discuss in this section the relation between our results and those of some previous literature.

To describe the alternative bargaining environment, first note that

⁸ An alternative experiment is to give each of the agents with money a larger-denomination object, which is obviously neutral.

⁹ One can also study the effect of M on other variables, such as nominal output, real output, and velocity (see Trejos and Wright 1993b).

we continue to assume that the proposer is chosen at random and the responder can accept or reject; if he rejects, they can either walk away or wait an interval of time Δ for another round. However, now buyers and sellers continue to meet new partners at rates $1 - M$ and M between rounds. We assume that traders continue to meet new partners irrespective of any actions they take; also, when a new partner who can demand or supply the appropriate service arrives, a trader automatically enters a bargaining game with him and abandons his old partner. There are no choices along these dimensions. The essential distinction between this model and the one employed earlier is that now there is a chance of being left without a partner if delay occurs; following the literature, we call this bargaining with the possibility of *exogenous breakdowns*.¹⁰

It is still the case that there is a unique subgame perfect equilibrium when V_s and V_b are taken as given, with the property that a seller always proposes $q_s = q_s(\Delta)$, a buyer always proposes $q_b = q_b(\Delta)$, and these proposals are always accepted. And as $\Delta \rightarrow 0$, q_s and q_b converge to the same limit q . To derive the properties of q in this case, let λ_s and λ_b denote the probabilities that sellers and buyers are left without partners if there is delay. Then $\lambda_s = \Delta(1 - M)$ and $\lambda_b = \Delta M$ if we ignore higher-order terms that become irrelevant as $\Delta \rightarrow 0$.¹¹ Now q_s and q_b satisfy

$$V_s + u(q_s) = \frac{1}{1 + r\Delta} \{ \lambda_b V_b + (1 - \lambda_b) [V_s + \frac{1}{2}u(q_s) + \frac{1}{2}u(q_b)] \}$$

and

$$V_b - c(q_b) = \frac{1}{1 + r\Delta} \{ \lambda_s V_s + (1 - \lambda_s) [V_b - \frac{1}{2}c(q_s) - \frac{1}{2}c(q_b)] \},$$

which have interpretations very similar to those of the analogous equations in the model with no exogenous breakdowns.

Rearranging these conditions, letting $\Delta \rightarrow 0$, and simplifying, one can show that

$$\frac{V_s + u(q) - V_b}{V_b - c(q) - V_s} = \frac{u'(q)}{c'(q)}.$$

¹⁰ Other possibilities also could be considered. For example, if we take seriously the notion that sellers remain at fixed locations and buyers travel to them, it might be natural to assume that sellers can meet new buyers but not vice versa. One can also endogenize the search intensity decisions and hence the arrival rates, as in Wolinsky (1987).

¹¹ Since agents will never terminate bargaining voluntarily, one agent ends up with no partner only when his current partner meets a new trader and he does not. This probability is $\Delta(1 - M)(1 - \Delta M) + o(\Delta)$ for a seller and $\Delta M[1 - \Delta(1 - M)] + o(\Delta)$ for a buyer; but we ignore second- and higher-order terms since we are going to let $\Delta \rightarrow 0$.

Hence, the limiting value of q solves

$$\max[V_s + u(q) - V_b][V_b - c(q) - V_s], \quad (11)$$

which is a Nash bargaining problem similar to (5), except that now the threat points are V_b for the buyer and V_s for the seller. To make sure the q that solves the Nash problem is acceptable, we continue to impose constraints (6) and (7), which continue to hold if and only if q is below some \bar{q} .

PROPOSITION 2. Consider the model with exogenous breakdowns. For any $r > 0$ and $M \in (0, 1)$, there exists a nonmonetary steady-state equilibrium and a unique monetary steady-state equilibrium. The monetary equilibrium is unconstrained but does not necessarily satisfy $u'(q) > c'(q)$.

Proof. See the Appendix.

The substantive difference that results from exogenous breakdowns is that we do not necessarily have $u'(q) > c'(q)$. The intuition developed in the previous section concerning the effect of discounting is still relevant, but the outcome is complicated here by the effect of exogenous breakdowns on relative bargaining power. For instance, when M becomes small and there are relatively few buyers, the threat of being abandoned between bargaining rounds impinges more severely on sellers than on buyers; this tilts settlement in favor of the buyer, and q goes up. If this effect is large, we can have $q > q^*$. In the proof of proposition 2, it is shown that in the limit, as $r \rightarrow 0$, we have $(1 - M)u'(q) = Mc'(q)$. Therefore, when $M = 1/2$ and the threat of a breakdown is the same for both parties, $q \rightarrow q^*$ as $r \rightarrow 0$, exactly as in the previous section.¹²

The next step is to allow barter, which means that sellers must meet other sellers in addition to buyers.¹³ The rate at which a seller meets other sellers is $\beta(1 - M)$. When they meet, the probability that they can trade is x^2 , since each has to be able to provide the service desired by the other. Given our normalization $\beta x = 1$, the rate at which appropriate barter partners arrive is therefore $(1 - M)x$. When barter occurs, it is easy to show that the outcome of the bargaining game is that each seller provides the other with q^* , the quantity that satisfies $u'(q^*) = c'(q^*)$. What needs to be determined is, What quantity does a seller provide in exchange for cash, and how is this affected by the possibility of barter?

¹² It is still the case that $\partial q/\partial r < 0$ and $\partial q/\partial \beta > 0$ here, as in the model with no exogenous breakdowns. However, now it can be shown that $\partial q/\partial M < 0$ and $\partial p/\partial M > 0$ for all M and r .

¹³ The only trades that we allow are those in which a buyer and seller exchange cash for services and those in which two sellers barter services directly; we do not allow buyers to provide services (but see the next extension in this section).

The generalizations of (1) and (2) (in the steady state) are

$$rV_s = M[V_b - V_s - c(Q)] + \Omega \quad (12)$$

and

$$rV_b = (1 - M)[u(Q) + V_s - V_b], \quad (13)$$

where $\Omega \equiv (1 - M)x[u(q^*) - c(q^*)]$. The generalization of the relevant constraint is

$$\varphi(q) \geq \Omega, \quad (14)$$

where φ is defined in (8). This is satisfied for all q in the interval $[q, \bar{q}]$, where both q and \bar{q} depend on M and x , and $q < \bar{q}$ as long as r is not too big. The model with no barter corresponds to setting $\Omega = 0$, in which case $q = 0$ and (14) is satisfied as long as $q \leq \bar{q}$.

We can consider the barter model either with or without exogenous breakdowns. In either case, it turns out that whether or not a monetary equilibrium exists depends on r , M , and x ; but whenever one exists, multiple monetary equilibria exist.

PROPOSITION 3. Consider the model with barter and no exogenous breakdowns. There is a critical value $\bar{r} \equiv \bar{r}(M, x)$, $\bar{r} > 0$, with the following properties: if $r > \bar{r}$, there are no monetary equilibria; if $r < \bar{r}$, there are two monetary equilibria, one constrained and one unconstrained.

Proof. See the Appendix.

PROPOSITION 4. Consider the model with barter and exogenous breakdowns. There is a critical value $\hat{r} \equiv \hat{r}(M, x)$, where $\hat{r} > 0$ at least if x is sufficiently small, with the following properties: if $r > \hat{r}$, there are no monetary equilibria; if $r < \hat{r}$, there generically exists an even number of monetary equilibria, all of which are unconstrained.

Proof. See the Appendix.

In contrast to the model without barter, where the monetary equilibrium is unique, now there exist (generically) multiple monetary equilibria. Also, without barter, monetary equilibria exist for all r , but now they exist only if r is sufficiently low. The models without barter can be interpreted formally as limiting cases of the models with barter in which $x \rightarrow 0$ (and $\beta \rightarrow \infty$, to keep our normalization $\beta x = 1$). In the limit, all but the monetary equilibrium with the highest q either disappear or coalesce with the nonmonetary equilibrium, which explains the uniqueness of monetary equilibrium when barter is not possible.

These results bring out an analogy between the set of equilibria in fixed-price search models of money and the models in this paper. For example, in the model of Kiyotaki and Wright (1993), with q exogenous, there are three equilibria: a nonmonetary equilibrium in

which no one accepts money, a pure monetary equilibrium in which money is accepted with probability one, and a mixed-strategy equilibrium in which money is accepted with probability x . In the limit as $x \rightarrow 0$, the mixed-strategy monetary equilibrium and nonmonetary equilibrium coalesce. This corresponds to the results in proposition 3, with pure and mixed-strategy monetary equilibria replaced by unconstrained and constrained monetary equilibria. Further, in the fixed-price model, sellers are indifferent about accepting or rejecting money in a mixed-strategy equilibrium, and here sellers are indifferent about accepting or rejecting money in a constrained equilibrium. Further still, in the fixed-price model, equilibria can be welfare-ranked (under assumptions discussed in Kiyotaki and Wright [1993]), and the same is true here.¹⁴

There is also a relationship between our results and those that emerge from models such as the one in Rubinstein and Wolinsky (1985). Consider a model with an indivisible consumption good and a divisible object that Rubinstein and Wolinsky call "money," although it has little in common with the fiat money in our model (see below). There are M buyers with one unit of money and $1 - M$ sellers with one unit of the good. Meetings occur randomly, with arrival rates for sellers and buyers given by M and $1 - M$. When a buyer and seller meet, they bargain over the money price of the good, p . Rubinstein and Wolinsky assume that the seller and buyer derive utilities $u_s(p) = p$ and $u_b(p) = 1 - p$ from a completed transaction, after which they exit the market and are replaced by a new pair. As in our model, the distribution of money holdings across the population is fixed. In contrast to our model, the utility of money is exogenous and the total surplus from trade is fixed at $u_s + u_b = 1$.

The value functions now satisfy

$$rV_s = M[u_s(p) - V_s]$$

and

$$rV_b = (1 - M)[u_b(p) - V_b].$$

Depending on whether or not we allow exogenous breakdowns, an underlying strategic model yields one of the following Nash bargaining solutions as a reduced form when $\Delta \rightarrow 0$:

¹⁴ Note that there are no mixed-strategy monetary equilibria in the model in this paper, since if a seller is indifferent about accepting or rejecting fiat currency, a buyer can always entice him to strictly prefer accepting by reducing q by an arbitrarily small amount. The mixed-strategy equilibrium in the fixed-price search model is an artifact of indivisibilities.

$$p = \operatorname{argmax} u_b(p) u_s(p),$$

$$p = \operatorname{argmax} [u_b(p) - V_b][u_s(p) - V_s].$$

The former implies $p = 1/2$. The latter implies $p = 1/2(1 + V_s - V_b)$, which simplifies to $p = (r + M)/(1 + 2r)$ after the value functions are eliminated.

The most important distinction between the Rubinstein-Wolinsky model and our model is that we assume that agents do not exit the economy after trade. Rather, the buyer becomes a seller, the seller becomes a buyer, and the money stays within the system. This is how fiat currency can have value without being an argument of anyone's utility function: money is valued today for what it will buy in the future. The basic search and bargaining setup is the same in the two models, and therefore some results are similar (e.g., the importance of exogenous breakdowns). However, the fact that the value of money is endogenous here makes a crucial difference. For instance, while the Rubinstein-Wolinsky model has a unique equilibrium, our model generates multiplicity. In some sense, the framework here seems to have more in common with earlier search models of money, despite the fact that they impose exogenous prices.

To close this section, we briefly consider a version of the model in which money is divisible and agents can hold any amount $m \in \mathbb{R}_+$. Let $F(m)$ be the distribution of money holdings across agents and $V(m)$ the value function of an agent with m dollars. For simplicity, assume that there is no barter, that meetings occur at Poisson rate β , and that in any meeting an agent will have an opportunity to buy with probability x and also to sell with probability x (which is why V does not have a subscript for buyer or seller). Hence, the arrival rate of both potential sales and potential purchases is βx , which we normalize to one. Let $d(m_b, m_s)$ and $q(m_b, m_s)$ be the amount of dollars and services that are traded between an agent holding m_b units of money and an agent holding m_s units of money when the former desires the services of the latter (i.e., when the former turns out to be a buyer and the latter a seller in a particular meeting).

Generalizing the arguments leading to (1) and (2), we get

$$rV(m) = \int \{u[q(m, m_s)] + V[m - d(m, m_s)] - V(m)\} dF(m_s) \\ + \int \{-c[q(m_b, m)] + V[m + d(m_b, m)] - V(m)\} dF(m_b). \quad (15)$$

The first term is the expected gain from buying from someone with m_s dollars, and the second term is the expected gain from selling to someone with m_b dollars, where m_s and m_b are random draws from $F(m)$. Also, if we assume no exogenous breakdowns, $d(m_b, m_s)$ and

$q(m_b, m_s)$ solve the Nash problem

$$\max_{(d, q)} [u(q) + V(m_b - d)][-c(q) + V(m_s + d)] \quad (16)$$

subject to $u(q) + V(m_b - d) \geq V(m_b)$ and $-c(q) + V(m_s + d) \geq V(m_s)$. Given the total stock of money, $M = \int m dF(m)$, a steady-state equilibrium is now defined to be a list of four functions (V, d, q, F) such that (i) given F and (d, q) , V satisfies (15); (ii) given V , (d, q) solves the bargaining problem; and (iii) given (d, q) , F is a stationary distribution for money holdings.

We have not yet been able either to prove existence or to completely characterize monetary equilibria in this model. However, at least as long as V is concave, it is possible to say some things about the properties of equilibrium using only the conditions that follow directly from bargaining. In particular, we claim that $q(m_b, m_s) < q^*$ for all (m_b, m_s) . To see this, first note that the first-order conditions from (16) imply

$$\frac{V'(m_b - d)}{V'(m_s + d)} = \frac{u'(q)}{c'(q)} = \frac{u(q) + V(m_b - d)}{V(m_s + d) - c(q)}$$

Suppose by way of contradiction that $u'(q) \leq c'(q)$. Then $m_b - d \geq m_s + d$ by the first equality, given that V is concave, and, a fortiori, $V(m_s + d) - c(q) < u(q) + V(m_b - d)$. Hence, $u'(q) > c'(q)$ by the second equality.

This establishes that the result $q < q^*$ is *not* an artifact of the indivisible money assumption, but is actually quite general. Some other properties of monetary equilibria (if they exist and if V is concave) are the following. When trade occurs, the seller leaves with more money and the buyer leaves with more utility: $m_s + d > m_b - d$ and $u(q) + V(m_b - d) > -c(q) + V(m_s + d)$. Also, when trade occurs, the quantities q , $m_s + d$, and $m_b - d$ depend only on $\bar{m} = m_b + m_s$, not on m_b or m_s individually. One can show $\partial(m_s + d)/\partial\bar{m} > 0$ and $\partial(m_b - d)/\partial\bar{m} > 0$. The sign of $\partial q/\partial\bar{m}$ is indeterminate, in general, although both $u(q) + V(m_b - d)$ and $-c(q) + V(m_s + d)$ are increasing in \bar{m} . The quantity d depends on m_s and m_b , not just on \bar{m} , and satisfies $\partial d/\partial m_s < 0$ and $\partial d/\partial m_b > 0$; that is, ceteris paribus, richer agents get less money when they sell and pay more money when they buy.

VI. Dynamics

In this section we return to the simple version of the model in which agents always hold either zero or one unit of money, and we consider non-steady-state equilibria. We do not analyze the underlying strategic bargaining game; rather, we adopt as a primitive assumption that q_t solves a Nash bargaining problem at each point in time. The equi-

librium path for q_t can still vary over time if the value functions V_{st} and V_{bt} do, and the value functions can vary over time if agents expect that Q_t will.¹⁵

Suppose that there are no exogenous breakdowns (the other case is similar; see Trejos and Wright [1993a]). Also, we rule out barter. For given values of V_{st} and V_{bt} , let the unique solution to the unconstrained Nash bargaining problem (5) be $q_t = q(V_{st}, V_{bt})$. If we substitute this into (1) and (2), we get

$$\dot{V}_s = (r + M)V_s - MV_b + Mc[q(V_s, V_b)] \quad (17)$$

and

$$\dot{V}_b = (r + 1 - M)V_b - (1 - M)V_s - (1 - M)u[q(V_s, V_b)], \quad (18)$$

where time subscripts are ignored from now on when there is no risk of confusion. This defines a dynamical system $\dot{V} = f(V)$, where $V \equiv (V_s, V_b)$. Further, if we substitute $q = q(V_s, V_b)$ into constraints (6) and (7), they can be written as

$$c[q(V_s, V_b)] \leq V_b - V_s \quad (19)$$

and

$$u[q(V_s, V_b)] \geq V_b - V_s. \quad (20)$$

Inequalities (19) and (20) define a region of the (V_s, V_b) plane called the *admissible region* and denoted by \mathcal{A} .

A *rational expectations equilibrium* is defined to be a solution to $\dot{V} = f(V)$ that remains in \mathcal{A} for all time. Any solution that remains in \mathcal{A} must be bounded, since it must satisfy $u(q) \geq V_b - V_s \geq c(q)$ for all t and hence $q_t \leq \hat{q}$ for all t , where $u(\hat{q}) = c(\hat{q})$. This implies that V_s and V_b are bounded above by $u(\hat{q})/r$, which is the value of consuming \hat{q} at the arrival rate of trading partners for the rest of time. Note that there are no initial conditions in the definition of equilibrium because V is not predetermined at $t = 0$; V_s and V_b can assume any starting values, as long as the implied path from this point stays in \mathcal{A} .

The locus of points in (V_s, V_b) space satisfying $\dot{V}_s = 0$ is called the V_s curve, and the locus of points satisfying $\dot{V}_b = 0$ is called the V_b curve. Intersections of the two curves constitute steady states of the system and correspond, of course, to the steady-state equilibria studied in Section IV. Both curves go through the origin, which is the nonmonetary steady state. Given our assumptions on $u(q)$ and $c(q)$, the V_s curve has a flatter slope than the V_b curve at the origin. As

¹⁵ In recent work, Coles and Wright (1994) solve the strategic bargaining game explicitly and discuss how this compares to imposing the Nash solution outside of the steady state.

shown in figure 1, the V_s curve is upward sloping, and the V_b curve rises to the northwest from the origin, then turns and heads northeast. We know from proposition 1 that the curves intersect exactly once in the positive quadrant, since there is a unique monetary steady state. We also know that at the monetary steady state the constraints do not bind, and so this intersection is in the interior of \mathcal{A} .

Figure 1 also shows the flow of the dynamical system. Given our assumptions on $u(q)$ and $c(q)$, the interior steady state is a source and the origin is a saddle point. Hence, in addition to the steady states, there exist only two other types of orbits: the saddle path that originates at the interior steady state and converges to the origin, and all other paths, which eventually become unbounded and leave the constraint set. The latter class of solutions cannot be equilibria. Paths beginning on the saddle path are equilibria if we can guarantee that the saddle path is contained in \mathcal{A} . Now the saddle path is contained in the region between the V_s curve and the V_b curve, in which $\dot{V}_s < 0$ and $\dot{V}_b < 0$. Whenever $\dot{V}_s < 0$ and $\dot{V}_b < 0$ in the positive quadrant, (17) and (18) indicate that $c[q(V_s, V_b)] \leq V_b - V_s \leq u[q(V_s, V_b)]$, and so both constraints are satisfied. Hence, the saddle path lies entirely within \mathcal{A} .

Summarizing the discussion above, we have the following result.

PROPOSITION 5. The set of equilibria consists of a nonmonetary steady state, a monetary steady state, and a one-dimensional continuum that is described as follows. For any initial V_s in the open interval between the two steady-state values, there is a unique V_b such that

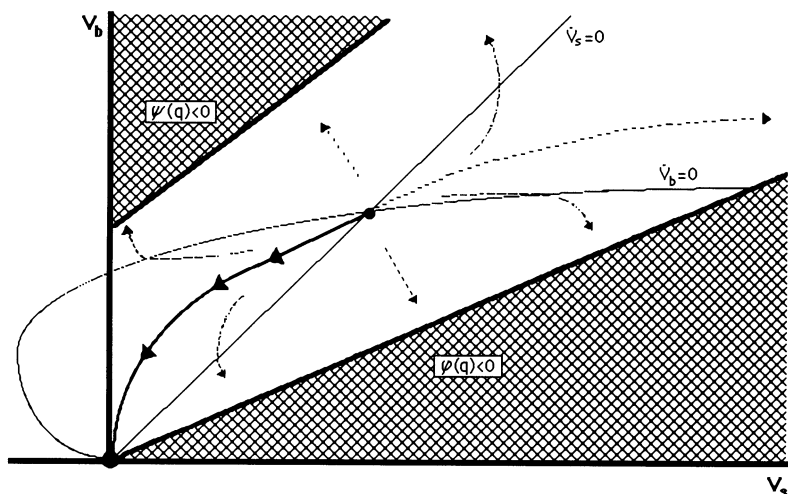


FIG. 1.—Dynamic equilibria

the path beginning at these initial conditions and converging to the origin is an equilibrium.

The equilibria starting on the saddle path and converging to the origin imply that $q_t \rightarrow 0$ and therefore $p_t \rightarrow \infty$. Such an outcome is simply due to self-fulfilling expectations. If the initial value of q is too low and therefore the initial value of p is too high, as compared to the monetary steady state, the economy sets off on an inflationary spiral. Expectations that prices are going to rise lead agents to agree to fewer and fewer services per dollar over time, which rationalizes the expectation of rising prices, and so on. The analogy between the set of equilibria here and the set of equilibria in the standard overlapping generations model of fiat money (see, e.g., Azariadis 1993) should be apparent.¹⁶

VII. Conclusion

This paper has introduced bargaining into a simple model of monetary exchange in order to determine the nominal price level endogenously. This also yields some new insights into the role of money. For example, we discovered a tendency for the price level to be too high in equilibrium, in the sense that the quantity produced and sold for a dollar is less than the amount a social planner would like. As the rate of time preference or the frictions in the exchange process vanish, the monetary equilibrium does converge to the planner's solution in at least some versions of the model. In other versions, the result has to be qualified on the basis of considerations associated with relative bargaining power.

We also found that if liquidity is sufficiently scarce, the value of money can actually rise with an increase in the money supply. However, as the money supply increases further, the value of money eventually begins to fall. This effect implies that the welfare-maximizing amount of money in the model is less than that predicted by fixed-price search models. Some other findings, such as the multiplicity of equilibria, are very similar to those obtained in earlier formulations. Still others, such as the dynamic inflationary equilibria, could not even be discussed without endogenous prices. Future research in this area may concentrate on analysis of the version of the model in which the distribution of money holdings is generalized and determined endogenously, or may use simple versions of the model to address some more applied issues in monetary economics.

¹⁶ If we allow barter, there are two monetary steady states, one of which is a saddle and the other a source. This is similar to overlapping generations models when government spending is added (see Azariadis 1993).

Appendix

CLAIM 1. With or without exogenous breakdowns, $\partial q/\partial r < 0$ and $\partial q/\partial \beta > 0$.

Proof. Consider the model with no exogenous breakdowns. Differentiating condition (10), we see that $\partial q/\partial r$ has the same sign as

$$\alpha = \alpha(M, r) \equiv (1 - M)uu' + Mcc' - (1 + 2r)(uc' + cu').$$

For any M , $r = 0$ implies $\alpha < 0$. If $\alpha \geq 0$ for some r , then $\partial \alpha/\partial r \geq 0$ at the lowest value of r such that $\alpha = 0$. But it is easy to see that $\partial \alpha/\partial r < 0$ whenever $\alpha = 0$. We conclude that $\alpha < 0$, and thus $\partial q/\partial r < 0$, for all r and M . The proof for the model with exogenous breakdowns is similar. Finally, notice that $\partial q/\partial \beta$ always takes the opposite sign of $\partial q/\partial r$, because all that matters in the model is the ratio r/β . Q.E.D.

CLAIM 2. With exogenous breakdowns, $\partial q/\partial M \leq 0$; with no exogenous breakdowns, there exists $\mu = \mu(r)$, with $\mu > 0$ for small r , such that $\partial q/\partial M \leq 0$ if and only if $M \geq \mu$.

Proof. With exogenous breakdowns, the result follows from differentiating the equilibrium condition $S(q) = 0$, where $S(q)$ is defined in the proof of proposition 2. With no exogenous breakdowns, differentiation of $T(q) = 0$ implies that $\partial q/\partial M$ has the same sign as

$$\gamma = \gamma(M, r) \equiv (1 - 2M)(u - c)(u' - c') - r(uu' - cc').$$

There exists a unique $\mu = \mu(r)$ such that $\gamma = 0$ if $M = \mu$, because γ is decreasing in M whenever $\gamma = 0$. The critical μ satisfies

$$\mu = \frac{1}{2} - \frac{r(uu' - cc')}{2(u' - c')(u - c)}.$$

By construction, $\partial q/\partial M \leq 0$ if and only if $M \geq \mu$. To show $\mu > 0$ for small r , note that $\mu \rightarrow \mu_0$ as $r \rightarrow 0$, where

$$\mu_0 = \frac{1}{2} - \frac{M(1 - M)[u(q^*) - c(q^*)]}{2[Mu(q^*) + (1 - M)c(q^*)]} > 0.$$

Q.E.D.

Proof of Proposition 2

Consider unconstrained monetary equilibria. The analogous condition to (10) is $S(q) = 0$, where

$$S(q) \equiv \varphi(q)u'(q) - \psi(q)c'(q),$$

and $\varphi(q)$ and $\psi(q)$ are defined in (8) and (9). For $q > 0$, $S(q) = 0$ can be rewritten $c'(q)/u'(q) = \varphi(q)/\psi(q)$. The left-hand side is zero at $q = 0$ and strictly increasing. The right-hand side is positive at $q = 0$, zero at $q = \bar{q}$, and strictly decreasing for all q in $(0, \bar{q})$. By continuity, there exists a unique solution to $S(q) = 0$ in $(0, \bar{q})$. This establishes the existence of a unique unconstrained equilibrium. Since $S(\bar{q}) < 0$, there are no constrained equilibria. Finally, to show that equilibrium does not necessarily satisfy $u'(q) > c'(q)$, consider the limiting case of $r \rightarrow 0$. Then $S(q) = 0$ implies $(1 - M)u'(q) =$

$M'c'(q)$, so q can be greater or less than q^* , depending on the value of M . Q.E.D.

Proof of Proposition 3

The generalization of (10) is

$$\begin{aligned} T(q) &= [(r + M)(1 - M)u(q) - \Phi c(q)]u'(q) \\ &\quad - [\Phi u(q) - M(r + 1 - M)c(q)]c'(q) \\ &\quad + [u'(q)(1 - M) - c'(q)(r + 1 - M)]\Omega. \end{aligned}$$

By an argument analogous to that in proposition 1, there is a unique solution to $T(q) = 0$ in $[0, \bar{q}]$. If $r = 0$, then $T(q) = 0$ at $q = q^*$, where $q^* > \underline{q} > 0$. As r increases, the solution q to $T(q) = 0$ decreases monotonically to zero and \underline{q} increases. Hence, there exists a unique $\bar{r} > 0$ (which depends on M and x) such that $T(\underline{q}) = 0$.

If $r > \bar{r}$, then there are no unconstrained equilibria since the solution to $T(q) = 0$ is less than \underline{q} . Also, when $Q = \underline{q}$ is taken as given, the unconstrained Nash solution is less than \underline{q} ; and when $Q = \bar{q}$ is taken as given, the unconstrained Nash solution is less than \bar{q} (since T is negative at \underline{q} and \bar{q}). Hence, neither \underline{q} nor \bar{q} is a constrained equilibrium. We conclude that if $r > \bar{r}$, then there exist neither unconstrained nor constrained monetary equilibria.

Now suppose $r < \bar{r}$. Then the solution in $[0, \bar{q}]$ to $T(q) = 0$ satisfies $\underline{q} < q < \bar{q}$, and so it is an unconstrained equilibrium. Also, when $Q = \underline{q}$ is taken as given, the unconstrained Nash solution is greater than \underline{q} , and when $Q = \bar{q}$ is taken as given, the unconstrained Nash solution is less than \bar{q} , since T is positive at \underline{q} and negative at \bar{q} . Hence, \underline{q} is a constrained equilibrium but \bar{q} is not. We conclude that if $r < \bar{r}$, there exist exactly two monetary equilibria, one unconstrained and one constrained. Q.E.D.

Proof of Proposition 4

The generalization of (11) is

$$S(q) = \varphi(q)u'(q) - \psi(q)c'(q) - (1 - M)\Omega[u'(q) + c'(q)].$$

Let \hat{r} (which depends on M and x) be the smallest value of r such that $S(q)$ is nonnegative for some q in $[\underline{q}, \bar{q}]$. Note that $S(q)$ is monotonically decreasing in r , and $S(q) < 0$ for all q in $[\underline{q}, \bar{q}]$ when r is large. Also, at least when x is not too big, $S(q) > 0$ for some q in $[\underline{q}, \bar{q}]$ at $r = 0$. Hence, when x is not too big, $\hat{r} > 0$.

Now note the following: when $Q = \underline{q}$ is taken as given, the unconstrained Nash solution is less than \underline{q} ; when $Q = \bar{q}$ is taken as given, the unconstrained Nash solution is less than \bar{q} , since S is negative at \underline{q} and \bar{q} . This means that there are no constrained equilibria. The fact that $S(\underline{q}) < 0$ and $S(\bar{q}) < 0$ also implies that if $S(q) = 0$ has a solution in $[\underline{q}, \bar{q}]$, it generically has (an even number of) multiple solutions. This means that there are multiple unconstrained monetary equilibria (generically an even number) when $r < \hat{r}$ and no monetary equilibria when $r > \hat{r}$. Q.E.D.

References

- Aiyagari, S. Rao, and Wallace, Neil. "Existence of Steady States with Positive Consumption in the Kiyotaki-Wright Model." *Rev. Econ. Studies* 58 (October 1991): 901–16.
- . "Fiat Money in the Kiyotaki-Wright Model." *Econ. Theory* 2 (1992): 447–64.
- Azariadis, Costas. *Intertemporal Macroeconomics*. Cambridge: Blackwell, 1993.
- Binmore, Kenneth G.; Rubinstein, Ariel; and Wolinsky, Asher. "The Nash Bargaining Solution in Economic Modelling." *Rand J. Econ.* 17 (Summer 1986): 176–88.
- Burdett, Kenneth; Coles, Melvyn; Kiyotaki, Nobuhiro; and Wright, Randall. "Buyers and Sellers." Working Paper no. 93-03. Philadelphia: Univ. Pennsylvania, Center Appl. Res. Econ. and Soc. Sci., 1993.
- Coles, Melvyn, and Wright, Randall. "Dynamic Bargaining Theory." Staff report. Minneapolis: Fed. Reserve Bank, 1994.
- Cuadras-Morato, Xavier. "Commodity Money in the Presence of Goods of Heterogeneous Quality." *Econ. Theory* 4 (1994): 579–91.
- Diamond, Peter A. "Money in Search Equilibrium." *Econometrica* 52 (January 1984): 1–20.
- Diamond, Peter A., and Yellin, Joel. "Pricing and the Distribution of Money Holdings in a Search Economy, II." In *New Approaches to Monetary Economics*, edited by William A. Barnett and Kenneth J. Singleton. Cambridge: Cambridge Univ. Press, 1987.
- . "Inventories and Money Holdings in a Search Economy." *Econometrica* 58 (July 1990): 929–50.
- Jones, Robert A. "The Origin and Development of Media of Exchange." *J.P.E.* 84, no. 4, pt. 1 (August 1976): 757–75.
- Kiyotaki, Nobuhiro, and Wright, Randall. "On Money as a Medium of Exchange." *J.P.E.* 97 (August 1989): 927–54.
- . "A Contribution to the Pure Theory of Money." *J. Econ. Theory* 53 (April 1991): 215–35.
- . "A Search-Theoretic Approach to Monetary Economics." *A.E.R.* 83 (March 1993): 63–77.
- Li, Victor E. "Inventory Accumulation in a Search-Based Monetary Economy." *J. Monetary Econ.* (in press).
- Marimon, Ramon; McGrattan, Ellen; and Sargent, Thomas J. "Money as a Medium of Exchange in an Economy with Artificially Intelligent Agents." *J. Econ. Dynamics and Control* 14 (May 1990): 329–73.
- Matsuyama, Kiminori; Kiyotaki, Nobuhiro; and Matsui, Akihiko. "Toward a Theory of International Currency." *Rev. Econ. Studies* 60 (April 1993): 283–307.
- Nash, John F. "The Bargaining Problem." *Econometrica* 18 (April 1950): 155–62.
- Oh, Seonghwan. "A Theory of a Generally Acceptable Medium of Exchange and Barter." *J. Monetary Econ.* 23 (January 1989): 101–19.
- Osborne, Martin J., and Rubinstein, Ariel. *Bargaining and Markets*. San Diego: Academic Press, 1990.
- Ostroy, Joseph M., and Starr, Ross M. "The Transaction Role of Money." In *Handbook of Monetary Economics*, vol. 1, edited by Benjamin M. Friedman and Frank H. Hahn. Amsterdam: North-Holland, 1990.
- Rubinstein, Ariel. "Perfect Equilibrium in a Bargaining Model." *Econometrica* 50 (January 1982): 97–109.

- Rubinstein, Ariel, and Wolinsky, Asher. "Equilibrium in a Market with Sequential Bargaining." *Econometrica* 53 (September 1985): 1133–50.
- Shi, Shouyong. "Money and Prices: A Model of Search and Bargaining." *J. Econ. Theory* (in press).
- Townsend, Robert M. "Models of Money with Spatially Separated Agents." In *Models of Monetary Economies*, edited by John H. Kareken and Neil Wallace. Minneapolis: Fed. Reserve Bank, 1980.
- Trejos, Alberto. "Monetary Exchange under Private Information: A Note." Working Paper no. 93-16. Philadelphia: Univ. Pennsylvania, Center Appl. Res. Econ. and Soc. Sci., 1993.
- Trejos, Alberto, and Wright, Randall. "Search, Bargaining, Money and Prices." Working Paper no. 93-13. Philadelphia: Univ. Pennsylvania, Center Appl. Res. Econ. and Soc. Sci., 1993. (a)
- . "Search, Bargaining, Money and Prices: Recent Results and Policy Implications." *J. Money, Credit and Banking* 25, no. 3, pt. 2 (August 1993): 558–76. (b)
- Williamson, Steven, and Wright, Randall. "Barter and Monetary Exchange under Private Information." *A.E.R.* 84 (March 1994): 104–23.
- Wolinsky, Asher. "Matching, Search, and Bargaining." *J. Econ. Theory* 42 (August 1987): 311–33.
- Wright, Randall. "Search, Evolution and Money." *J. Econ. Dynamics and Control* (in press).
- Zhou, Ruilin. "Currency Exchange in a Random Search Model." Manuscript. Philadelphia: Univ. Pennsylvania, 1993.