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DOES AND SHOULD A COMMODITY MEDIUM OF EXCHANGE HAVE RELATIVELY LOW STORAGE COSTS?*

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I provide new existence and welfare results for a version of the Kiyotaki-Wright model. I construct an equilibrium where all agents use mixed strategies. Consequently, an object with a higher storage cost must have a higher acceptability. Therefore, the endogenous transaction pattern corresponds to the observation that money is dominated in rate of return by other assets (e.g., bonds), something that is a central issue in monetary economics. Furthermore, at least in a neighborhood of equal storage costs, the equilibrium that I construct Pareto dominates alternative equilibria in which better objects are widely accepted.

1. INTRODUCTION

This paper provides new existence and welfare results for the original version of the Kiyotaki-Wright model (1989) of commodity money (i.e., the model with three consumption goods having different storage costs and no fiat money). In particular, for open sets of parameter values and initial conditions, I construct an equilibrium where all agents use mixed trading strategies. This means, of course, that all agents are indifferent between accepting and rejecting goods that are not their consumption goods, and an immediate implication of this indifference is that a good with a higher storage cost must have a higher acceptability; that is, "liquidity" has to compensate for bad intrinsic properties bringing about some version of Gresham's

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law (see Aiyagari 1989). The equilibrium thereby generates an endogenous transaction pattern corresponding to the observation that in the real world there are usually lower rates of return on assets with higher acceptability (like money) than on assets with lower acceptability (like bonds), something that has been a central issue in monetary economics at least since Hicks (1935).

Moreover, at least for parameters in the neighborhood of equal storage costs, I establish that the equilibrium I construct Pareto dominates other alternative equilibria in which lower-storage-cost objects have higher acceptability. Intuitively, in an equilibrium where low-storage-cost objects are also more widely accepted, they dominate other objects in terms of rate of return and liquidity, and so agents are very reluctant to trade them away. In the equilibrium where high-storage-cost objects are more widely acceptable, agents are less reluctant to trade away any particular object, the resulting distribution of goods is closer to being symmetric, and therefore, the frequency of trade and consumption is higher. Hence the results challenge the traditional and ostensibly natural view that good intrinsic properties (such as divisibility, portability, durability, and so on) are either necessary or desirable for objects to play the role of money (see, for example, Ostroy-Starr 1989, p. 39).

Previous work on the Kiyotaki-Wright model (1989) has focused mainly on steady states and has established the existence of multiple steady states. However, as is well known, multiple steady states are neither necessary nor sufficient for multiple equilibria. Moreover, welfare comparisons among steady states are of dubious significance because transitions paths are ignored. Even more important, from generic initial conditions it may be impossible to approach a given steady state (see Renero 1998). Hence it is important to note that the results in this paper pertain to equilibria and not just to steady states.

The rest of the paper is organized as follows: In the next section I describe briefly the physical environment and the equilibrium concept. In Section 3 I present the existence and welfare results. In Section 4 I make some concluding remarks. Technical proofs are left to the Appendix.[2]

2. PHYSICAL ENVIRONMENT AND EQUILIBRIUM DEFINITION

I will use the physical environment, notation, and essentially the equilibrium concept of the Aiyagari-Wallace (1992) exposition of the Kiyotaki-Wright (1989)

[2] The version of the model studied here adopts the restriction that agents play a unique strategy for each opportunity set, which is useful because, for example, it can be used to show that the number of steady states is generically finite (see Kehoe, Kiyotaki, and Wright 1993). In a related paper (Renero 1994), in a general version of the model, I relax this restriction and show the following: For an open set of parameters such that storage costs are not too different across goods, there exist equilibria in which the object with the highest storage cost is universally accepted and agents play strategies close to "always trade." Such an equilibrium may dominate equilibria in which objects with lower storage costs are universally accepted, as well as the mixed-strategy equilibria analyzed here. The mixed strategy equilibria analyzed here cannot come close to "always trade" because the restriction that agents play a unique strategy for each opportunity set implies that if an agent trades good i for good j with a high probability, then he or she must trade good j for good i with a low probability.

model. For the reader's convenience, I give next a brief description of these items, assuming that there are no fiat objects.

2.1. *The Physical Environment.* Time is discrete and represented by positive integers. There exist three perfectly durable goods indexed by the set $\{1,2,3\}$. Each good is indivisible. There are three types of infinitely lived agents indexed by the set $\{1,2,3\}$. Type i consumes good i and produces good $i+1$ (modulo 3). There is a $[0,\frac{1}{3}]$ continuum of agents for each type.

Agents maximize expected discounted utility with discount factor $\rho \in (0,1)$. In any period, a type i agent's utility of neither consuming nor storing anything is zero, the utility of consuming one unit of good i without storing anything is $u_i > 0$, and the utility of not consuming and storing one unit of good j from the given period until the next period is $-c_{ij} \leq 0$. After consuming one unit of good i, agent of type i produces one unit of good $i+1$ that appears at the beginning of next period. At the beginning of the initial period $t=1$, each agent is endowed with one unit of a good $j \in \{1,2,3\}$. Finally, each period each agent is paired randomly with one other agent. It is assumed that paired agents know each other's type and current inventory but not trading histories.

2.2. *Definition of Equilibrium.* Next, I define a class of Nash equilibria with rational expectations. In particular, I assume that the strategies are *symmetric* (i.e., all agents of the same type in the same situation use the same strategy) and that trading strategies are *nondiscriminatory* (i.e., willingness to trade does not depend on the type of agents one meets). Moreover, we assume that agents do not dispose of goods and do not postpone consumption. We require that in equilibrium the actions of each agent are individually optimizing given the actions of the other agents and the inventory distribution of stocks. The notation presumes that each agent starts each period with one unit of a good. That is, we assume additionally that agents never give a good away to another agent for nothing.[3]

Therefore, the timing of an agent's activities in period t is that he or she starts with one good, meets another agent, ends up with some good after the meeting, and then stores or consumes and produces to start at period $t+1$ with some good.

I give next notation for the trading strategies of the agents and other items. The probability of choosing to trade good j for good k by those who are type i, hold good j, and meet another agent with good k at period t is denoted by $s_{ij}^{k}(t)$. The vector of these over (j,k) is denoted by $s_i(t)$, and the vector of these over (i,j,k) is denoted by $s(t)$. In this paper I deal with mixed strategies that restrict agents to play the same strategy for each opportunity set, and consequently, the equalities

$$s_{ij}^{k}(t) + s_{ik}^{j}(t) = 1$$

have to be satisfied.[4]

[3] In fact, I can ignore gift giving because, as noticed by Aiyagari and Wallace (1991), it is never an equilibrium strategy given the following small and otherwise innocuous change to the model: Let agents derive some small enough utility from consuming any good.

[4] I owe to Neil Wallace the interpretation of these restrictions.

The symbol $p_{ij}(t)$ denotes the proportion of agents who are type i and hold good j at the start of period t. The symbol $p(t)$ denotes the vector of these elements or the distribution of inventories; that is, $p(t)$ is the state of the system. The law of motion of the sequence $\{p(t)\}$ is given by

$$(2.1)\qquad p_{ij}(t+1) = (1-\delta_{ij})\Big[p_{ij}(t) - p_{ij}(t)\sum_k \sum_{1\neq j} p_{k1}(t) s^1_{ij}(t) s^j_{k1}(t) + \sum_k \sum_{1\neq j} p_{i1}(t) p_{kj}(t) s^j_{i1}(t) s^1_{kj}(t)\Big] + \delta_{i+1,j}\Big[\sum_k \sum_1 p_{i1}(t) p_{kj}(t) s^i_{i1}(t) s^1_{ki}(t)\Big]$$

where $\delta_{kn} = 1$ if $k = n$ and 0 otherwise.

I explain next the first bracket on the RHS of the Eq. (2.1). The first term is the proportion of agents who are type i and hold good j at the start of period t and, consequently, before the random meetings at period t; the second term is the proportion of agents who are type i and held good j before the random meetings and traded for a different good; and the third term is the proportion of agents who are type i and who did not hold good j before the random meetings and who traded for good j at period t. Finally, the term in the second bracket on the RHS of the Eq. (2.1) is the proportion of agents who are type i and who ended up after the random meetings at period t with good i. Notice that this is the fraction of agents who produce good $i+1$ and that if $j = i$, then $p_{ij}(t+1) = 0$ because agents do not postpone consumption.

Let $v_{ij}(t)$ denote the expected discounted utility of agent of type i ending with good j after trade at period t but before consuming, storing, or disposing, and let $v(t) \in \mathbf{R}^{3(3)}$ denote the vector of expected discounted utilities. Given the sequence $\{p(t+1), s(t+1)\}$, there exists a unique bounded sequence of vectors $\{v(t)\}$ that satisfies the dynamic-programming equations

(2.2)

$$v_{ij}(t) = -c_{ij} + \rho \sum_k \sum_1 p_{k1}(t+1)\big\{s^1_{ij}(t+1) s^j_{k1}(t+1) v_{i1}(t+1) + \big[1 - s^1_{ij}(t+1) s^j_{k1}(t+1)\big] v_{ij}(t+1)\big\} \quad \text{if } i \neq j$$

and

(2.3)

$$v_{ii}(t) = u_i + \rho \sum_k \sum_1 p_{k1}(t+1)\big\{s^1_{i,i+1}(t+1) s^{i+1}_{k1}(t+1) v_{i1}(t+1) + \big(1 - s^1_{i,i+1}(t+1) s^{i+1}_{k1}(t+1)\big) v_{i,i+1}(t+1)\big\}$$

The existence and uniqueness of the bounded sequence $\{v(t)]$ follow from the fact that the RHS of Eqs. (2.2) and (2.3) are contractions as functions of the components of the vector $v(t+1)$.

The individual optimizing conditions for trading strategies are

$$v_{ij}(t) \geq v_{ik}(t) \qquad \text{if } s_{ij}^{k}(t) = 0 \tag{2.4}$$

$$v_{ij}(t) \leq v_{ik}(t) \qquad \text{if } s_{ij}^{k}(t) = 1 \tag{2.5}$$

$$v_{ij}(t) = v_{ik}(t) \qquad \text{if } 0 < s_{ij}^{k}(t) < 1 \tag{2.6}$$

The condition about optimality of consumption after acquiring the consumption good is

$$u_i + c_{i,i+1} + v_{i,i+1}(t) \geq -c_{ii} + \rho\left[u_i + c_{i,i+1} + v_{i,i+1}(t+1)\right] \tag{2.7}$$

The condition about optimality of nondisposal is

$$v_{ij}(t) \geq 0 \tag{2.8}$$

Therefore, we have the next definitions.

Definition 1. A *symmetric equilibrium* from $p(1)$, in which agents (1) do not play discriminatory strategies, (2) do not dispose of goods, and (3) do not postpone consumption, is a path $\{p(t+1), s(t)\}$ such that (a) Eq. (2.1) holds, and (b) there exists a bounded sequence $\{v(t)\}$ such that Eqs. (2.2) and (2.3) and (2.4) through (2.8) hold.

Definition 2. A *symmetric steady state* in which agents (1) do not play discriminatory strategies, (2) do not dispose of goods, and (3) do not postpone consumption is a constant (p, s) such that $[p(t+1), s(t)] = (p, s)$ for all $t \geq 1$ satisfies Definition 1 when $p(1) = p$.

Notice that the equilibrium condition (b) requires that the sequence $\{s_i(t)\}$ be the best response of any agent of type i taken as given the path $\{p(t+1), s(t)\}$.

3. Equilibria and Welfare

We can expect some asymmetry of inventory stocks if a good is universally accepted and exchanged only for the holder's consumption good. The inventories of this good should be greater than those of the other goods. Such is the case of the pure-strategy steady states analyzed by Kiyotaki and Wright (1989), in which the least or the second least costly to-store good is universally accepted. (Steady states like these exist for the case of three goods, at least for some parameters). However, the agents producing this universally accepted good do not have a great opportunity of consuming because they face a relatively small stock of their consumption good. Because the fraction of agents who consume for each type is equal in a steady state, one may conjecture that an "asymmetric" steady state implies a smaller fraction of agents who consume than a "near symmetric" steady state.

For the case of three goods and no fiat objects, I will next prove the existence of a "near symmetric" steady state for trading strategies that satisfy $s_{ij}^k + s_{ik}^j = 1$ (and consequently are bounded away from always trade). In this steady state, the acceptance rate of goods in trade for the respective consumption good of each type of agent varies directly with their storage cost because all agents are randomizing trading strategies. In fact, to say that there exists a steady state in which agents are playing mixed strategies is to say that there exists a steady state in which the acceptance rate of goods in trade for the respective consumption good of each type of agent varies directly with the storage cost. Moreover, the fraction of agents who consume is higher in that mixed-strategy steady state than in the steady states in which the least or the second least is universally accepted. One can further conjecture the existence of multiple equilibria and the desirability of "near symmetric" equilibrium paths at least if all goods are nearly equally costly to store for each type of agent. I also will prove this conjecture.

3.1. *Existence of a Mixed-Strategy Equilibrium.* Always-trade strategies s^{AT} imply the "symmetric" inventory distribution p^c given by

$$p^c \equiv \begin{bmatrix} p_{11}^c & p_{12}^c & p_{13}^c \\ p_{21}^c & p_{22}^c & p_{23}^c \\ p_{31}^c & p_{32}^c & p_{33}^c \end{bmatrix} \equiv \begin{bmatrix} 0 & \frac{2}{9} & \frac{1}{9} \\ \frac{1}{9} & 0 & \frac{2}{9} \\ \frac{2}{9} & \frac{1}{9} & 0 \end{bmatrix}$$

Indeed, s^{AT}, which is a vector whose components are equal to 1, does not satisfy the equality $s_{ij}^k + s_{ik}^j = 1$. But the vector s^c of trading strategies given by

$$s^c \equiv \left(s_{12}^3, s_{13}^2, s_{23}^1, s_{21}^3, s_{31}^2, s_{32}^1\right) \equiv \left(\tfrac{2}{3}, \tfrac{1}{3}, \tfrac{2}{3}, \tfrac{1}{3}, \tfrac{2}{3}, \tfrac{1}{3}\right)$$

does satisfy this restriction and implies the stationary inventory distribution p^c. Further, the fraction of agents who consume is $7/27$, which is higher than in the pure-strategy steady states in which the least or the second least costly to-store good is universally accepted.

The trading strategies s^c are *not* individually optimal in an open set of parameters. In particular, the components of s^c are best responses of agents given (p^c, s^c) if and only if for every type of agent all goods, with exception of their respective consumption good, are equally costly to store (see Eqs. 3.3 and 3.4 in Kehoe, Kiyotaki, and Wright 1993). I will prove next that if storage costs are near to equality, there exists a convergent equilibrium path with strategies close to (p^c, s^c).

The proof proceeds in two steps, as in Renero (1998), where storage costs bounded away from equality are dealt with. First, given the existence of a steady state for some parameters, I find an open set of parameters containing those parameters for which a steady state exists. I accomplish this by using the implicit function theorem[5] and using as my reference the steady state (p^c, s^c). Second, I find an open set of parameters and initial conditions for which an equilibrium path converges.

[5] Kehoe et al. (1993, pp. 310–311) employ this technique using a numerical approximation to an "asymmetric" steady state.

The next proposition, whose proof is left to the Appendix, implies that for an open set of parameters and initial conditions there exists a mixed-strategy equilibrium. By construction, this equilibrium converges in two periods to the steady state.

PROPOSITION 3.1. *Assume three goods and* $c_{ij} < \frac{7}{27}\rho(u_i + c_{i,i+1})$. *There exists an open set* $\mathscr{V}$ *of parameters* (ρ, u_i, c_{ij}) *(a set that includes equal storage costs) and a neighborhood* $\mathscr{O}$ *of* p^c *such that for any parameters* (ρ, u_i, c_{ij}) *in* $\mathscr{V}$ *and any* $p(1) \in \mathscr{O}$, *(i) there exists a Definition* 2 *steady state* (p, s) *in which*

$$s_{i,i+1}^{i+2} \in (0,1) \qquad \text{and} \qquad s_{ij}^{k} + s_{ik}^{j} = 1$$

and (ii) there exists a unique strategy vector $s(1)$ *satisfying the Eq.* (2.1) *for* $p(t+1) = p$, $p(t) = p(1)$, $S(t) = S(1)$, *and satisfying*

$$s_{ij}^{k}(1) + s_{ik}^{j}(1) = 1$$

Notice that if agents are playing mixed strategies, the vector of trading strategies $s(1)$ is not determined, and I take advantage of this indeterminacy to select an $s(1)$ such that the path converges in two periods. Hence, for an open set of parameters and initial conditions, there exists a locally unique steady state that is approachable by a two-period-convergent equilibrium path from any of these initial conditions and in which agents are playing mixed strategies.

3.2. *Existence of Pure-Strategy Equilibria.* For an open set of parameters and initial conditions, I will next prove (1) that there exists one and only one equilibrium in which the least costly to-store good is universally accepted and (2) that there exists possibly another equilibrium and only one in which the second least costly to-store good is universally accepted. Moreover, these equilibria converge and are characterized by the fact that agents play pure trading strategies. I will review next the associated steady states.

Kiyotaki and Wright (1989) show that there exists a so-called fundamental steady state (p^f, s^f) in which the *least* costly to-store object is universally accepted if either $c_{i1} < c_{i2} < c_{i3}$ or $c_{i1} < c_{i3} < c_{i2}$.[6] The vector s^f is given by

$$s^f = \left(s_{12}^3, s_{13}^2, s_{23}^1, s_{21}^3, s_{31}^2, s_{32}^1\right) = (1,0,1,0,0,1)$$

and p^f is given by

$$p^f = \begin{bmatrix} p_{11}^f & p_{12}^f & p_{13}^f \\ p_{21}^f & p_{22}^f & p_{23}^f \\ p_{31}^f & p_{32}^f & p_{33}^f \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 0 & 2^{-1/2} & 1 - 2^{-1/2} \\ 2 - 2^{1/2} & 0 & 2^{1/2} - 1 \\ 1 & 0 & 0 \end{bmatrix}$$

[6] If all agents rank storage costs c_{ij} the same way, then without loss of generality, one can assume that good 1 has the lowest storage cost, and then these are the only two interesting versions of the model (everything else is a relabeling).

The pair (p^f, s^f) is a steady state for parameters in an open set. In particular, the strategies of agents of type 1 are optimal if

$$(c_{13}-c_{12})\rho^{-1}(u_1+c_{12})^{-1} < \left(p_{31}^f - p_{21}^f\right) = \tfrac{1}{3}(2^{1/2}-1)$$

For an open set of parameters, Kiyotaki and Wright (1989) also prove that there exists a so-called speculative steady state (p^s, s^s) in which the *second least* costly to-store good is universally accepted if $c_{i1} < c_{i3} < c_{i2}$. The trading-strategy vector s^s is given by

$$s^s = \left(s_{12}^3, s_{13}^2, s_{23}^1, s_{21}^3, s_{31}^2, s_{32}^1\right) = (1,0,0,1,1,0)$$

and the inventory distribution p^s is given by

$$p^s = \begin{bmatrix} p_{11}^s & p_{12}^s & p_{13}^s \\ p_{21}^s & p_{22}^s & p_{23}^s \\ p_{31}^s & p_{32}^s & p_{33}^s \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 0 & 2^{1/2}-1 & 2-2^{1/2} \\ 0 & 0 & 1 \\ 2^{-1/2} & 1-2^{-1/2} & 0 \end{bmatrix}$$

The next proposition, whose proof is left to the Appendix, implies that for an open set of parameters and initial conditions $p(1)$, there exists an equilibrium path $\{p(t+1), s^f\}$, that is, an equilibrium in which the least costly to-store good is universally accepted and only pure strategies are played. Moreover, this equilibrium converges and is the only one in which the least costly is universally accepted.

PROPOSITION 3.2. *Assume three goods and either $c_{i1} < c_{i2} < c_{i3}$ or $c_{i1} < c_{i3} < c_{i2}$. If the initial condition $p(1)$ satisfies $p_{31}(1) > p_{21}(1)$, there exists an open set $\mathscr{W}$ of parameters (ρ, u_i, c_{ij}) (a set that includes equal storage costs) such that for any parameters (ρ, u_i, c_{ij}) in $\mathscr{W}$, $\{p(t+1), s^f\}$ from $p(1)$ is a Definition 1 equilibrium and the only equilibrium in which $s_{ij}^1(t) = 1$, $j \neq i$. Moreover, there exists a neighborhood $\mathscr{O}$ of p^c such that if $p(1) \in \mathscr{O}$, $\{p(t+1), s^f\}$ converges to (p^f, s^f).*

The next proposition, whose proof is left to the Appendix, implies that for an open set of parameters and initial conditions $p(1)$, there exists an equilibrium path $\{p(t+1), s^s\}$, that is, an equilibrium in which the second least costly to-store good is universally accepted and only pure strategies are played. Moreover, this equilibrium converges and is the only one in which the second least costly to-store good is universally accepted.

PROPOSITION 3.3. *Assume three goods and $c_{i1} < c_{i3} < c_{i2}$. If the initial condition $p(1)$ satisfies $p_{23}(1) > p_{13}(1)$, and $p_{32}(2) = p_{12}(1)p_{31}(1) + [1-p_{23}(1)]p_{32}(1) > 0$, there exists an open set $\mathscr{X}$ of parameters (ρ, u_i, c_{ij}) (a set that includes equal storage costs) such that for any parameters (ρ, u_i, c_{ij}) in $\mathscr{X}$, $\{p(t+1), s^s\}$ from $p(1)$ is a Definition 1 equilibrium and the only equilibrium in which $s_{ij}^3(t) = 1$, $j \neq i$, if $s_{ij}^k(t) + s_{ik}^j(t) = 1$. Moreover, there exists a neighborhood $\mathscr{O}$ of p^c such that if $p(1) \in \mathscr{O}$, $\{p(t+1), s^s\}$ converges to (p^s, s^s).*

3.3. *Some Welfare Comparisons of Multiple Equilibria.* The next proposition establishes that for an open set of parameters and initial conditions, there exist multiple equilibria. In particular, there exists an equilibrium in which the most costly to-store good has the highest acceptance rate and which coexists with and Pareto dominates to the other equilibria, discussed earlier, in which *less* costly to-store goods are universally accepted. This is the main result of the paper.

PROPOSITION 3.4. *Assume three goods and either* $c_{i1} < c_{i2} < c_{i3}$ *or* $c_{i1} < c_{i3} < c_{i2}$. *There exists a neighborhood* $\mathscr{P}$ *of* p^c *and an open set* $\mathscr{Y}$ *of parameters* (ρ, u_i, c_{ij}) *(a set that includes equal storage costs) such that for any* $p(1) \in \mathscr{P}$ *and any parameters* (ρ, u_i, c_{ij}) *in* $\mathscr{Y}$, (1) *the path* $\{p(t+1), s^f\}$ *from* $p(1)$ *converges and is a Definition* 1′ *equilibrium and the only equilibrium in which* $s_{ij}^1(t) = 1$, $j \neq i$; (2) *for* $c_{i1} < c_{i3} < c_{i2}$, *the path* $\{p(t+1), s^s\}$ *from* $p(1)$ *converges and is a Definition* 1′ *equilibrium and the only equilibrium in which* $s_{ij}^3(t) = 1$, $j \neq i$, *if* $s_{ij}^k(t) + s_{ik}^j(t) = 1$; *and* (3) *there exists a two-period convergent equilibrium in which agents play mixed strategies satisfying*

$$s_{ij}^k(t) + s_{ik}^j(t) = 1$$

and which is Pareto superior to the equilibria in (1) *and* (2).

PROOF. The existence follows using Propositions 3.1, 3.2, and 3.3. The claim of the Pareto ranking follows by continuity for ρ close enough to 1, since (1) all equilibria converge for $p(1)$ in a neighborhood of p^c and (2) the components of the expected-utility vectors v^f and v^s associated with the steady state (p^f, s^f) and (p^s, s^s), respectively, are smaller than those of the expected-utility vector v^c associated to (p^c, s^c) if for every type of agent all goods are equally to store.

Notice that Proposition 3.4 implies that the existence parameter regions for the three equilibria overlap, but it does say that they are identical. Moreover, there may be parameters where all three equilibria coexist and one of the pure-strategy equilibria dominates.

4. CONCLUSION

We have seen examples of multiple equilibria (from a given initial condition) that correspond to different objects being used as generalized media of exchange. Moreover, we have seen an example of an equilibrium in which objects with poor storage properties are widely accepted. Furthermore, this equilibrium has good welfare properties relative to other alternative equilibria in which better objects are widely accepted.

The technical intuition for the welfare result is that mixed strategies mean more frequent trade for two reasons. First, mixed strategies are closer to always trade than pure strategies. Second, the distribution of goods across agents is a lot more symmetric than, say, in the fundamental equilibrium, where the best good is always accepted and the other goods are accepted a lot less.

As regards further analytical work, a generalization of the results of this paper may be attempted. It seems at least feasible to construct a mixed-strategy steady state for the case of any number of goods, although at the expense of a great deal of nasty algebra.

APPENDIX

The proofs use the following notation: For a given $i \in \{1,2,3\}$, $e[i] \in \operatorname{argmax}(c_{ij}: j \neq i)$; that is, $e[i]$ is one of the most costly to-store goods, with the exception of the respective consumption good for agents of type i:

$$\lambda_i(j) \equiv (c_{i,e[i]} - c_{ij})\rho^{-1}(u_i + c_{i,i+1})^{-1}$$

$$\lambda \equiv [\lambda_i(j)]$$

$$\Lambda \equiv \{\lambda_i(j) \in \mathbf{R}^{(3)(3)}: \lambda_i(j) \geq 0,\ i \neq j; \text{ and for each } i \text{ there exists } e[i] \neq i \text{ such that } \lambda_i(e[i]) = 0\}$$

PROOF OF PROPOSITION 3.1. We first prove the existence of a stationary state with individually optimal mixed trading strategies for a neighborhood of $\lambda = 0$. Let f: $\Pi_{k=1}^3[0,1] \times \Pi_{k=1}^3[0,\frac{1}{3}] \times \Lambda \to \mathbf{R}^3$ be defined by

$$f_1(s,p;\lambda) = p_{31}s_{31}^2 - \lambda_1(3) - p_{31} + (\tfrac{1}{3} - p_{23})s_{23}^1 + \lambda_1(2)$$

$$f_2(s,p;\lambda) = p_{12}s_{12}^3 - \lambda_2(1) - p_{12} + (\tfrac{1}{3} - p_{31})s_{31}^2 + \lambda_2(3)$$

$$f_3(s,p;\lambda) = p_{23}s_{23}^1 - \lambda_3(2) + (\tfrac{1}{3} - p_{12})s_{12}^3 - p_{23} + \lambda_3(1)$$

Let

$$B(p) \equiv \begin{bmatrix} p_{12}p_{23} + p_{13}p_{32} & p_{13}p_{21} & 0 \\ 0 & p_{23}p_{31} + p_{13}p_{21} & p_{21}p_{32} \\ p_{13}p_{32} & 0 & p_{12}p_{31} + p_{21}p_{32} \end{bmatrix}$$

$$b(p) \equiv \begin{bmatrix} p_{13}p_{21} + \frac{1}{3}p_{13} \\ p_{21}p_{32} + \frac{1}{3}p_{21} \\ p_{32}p_{13} + \frac{1}{3}p_{32} \end{bmatrix}$$

Using the identities $p_{i,i+2} = \frac{1}{3} - p_{i,i+1}$, let g: $\Pi_{k=1}^3[0,1] \times \Pi_{k=1}^3[0,\frac{1}{3}] \to \mathbb{R}^3$ be defined by $g(s,p) = B(p)s - b(p)$. Notice that after eliminating variables, the law of motion of inventory distributions becomes

$$H[p,s] \equiv p - g(s,p) \equiv p - B(p)s + b(p)$$

Notice that $[f(\cdot,\cdot;\cdot), g(\cdot,\cdot)]$ is continuously differentiable in a neighborhood of $(s^c, p^c; 0)$, $[f(s^c,p^c;0), g(s^c,p^c)] = 0$, and the determinant of the 6×6 matrix

$D_{s,p}[f(s^c, p^c; 0), g(s^c, p^c)]$ is $49/19683 \neq 0$. Hence, by the implicit function theorem, there exist a neighborhood $\mathscr{N}(\lambda = 0)$ of $\lambda = 0$, a neighborhood $\mathscr{N}(s^c, p^c)$ of (s^c, p^c), and a differentiable function $\gamma: \mathscr{N}(\lambda = 0) \subseteq \Lambda \to \mathscr{N}(s^c, p^c) \subseteq \Pi_{k=1}^3[0,1] \times \Pi_{k=1}^3[0, \frac{1}{3}]$ such that $\{f[\gamma(\lambda); \lambda], g[\gamma(\lambda)]\} = 0$.

Hence, using Eqs. (3.3) and (3.4) in Kehoe, Kiyotaki, and Wright (1993), $\gamma(\lambda) \equiv [s(\lambda), p(\lambda)]$ for $\lambda \in \mathscr{N}(\lambda = 0)$ is a stationary state with individually optimal trading strategies. For convenience, I argue lastly about the optimality of nondisposal to have a steady state.

Now I prove the existence of a neighborhood of p^c such that for any pair (p, p') of inventory distributions in this neighborhood, there exists a trading-strategy vector $\mathscr{S}(p, p')$ satisfying the equation $p' = H[p, \mathscr{S}(p, p')]$. Notice that $B(p)$ is nonsingular if $p_{i,i+1} > 0$. Using the identities $p_{i,i+2} = \frac{1}{3} - p_{i,i+1}$, define $\mathscr{S}$: $\Pi_{k=1}^3(0, \frac{1}{3}] \times \Pi_{k=1}^3(0, \frac{1}{3}] \to \mathbf{R}_+^3$ by

$$\mathscr{S}(p, p') = B^{-1}(p)[b(p) + (p - p')]$$

Notice that $p' = H[p, \mathscr{S}(p, p')]$ by construction and $\mathscr{S}$ is continuous at (p^c, p^c). Then there exists a neighborhood $\mathscr{O}(p^c)$ of p^c such that

$$\mathscr{S}(p, p') \in \Pi_{k=1}^3(0,1)$$

if $p \in \mathscr{O}(p^c)$ and $p' \in \mathscr{O}(p^c)$.

I verify next the existence of a neighborhood of $\lambda = 0$ and a neighborhood of p^c for which there exists a stationary state $\gamma(\lambda) \equiv [s(\lambda), p(\lambda)]$ approachable by a two-period convergent path. Notice that for any $p(1) \in \mathscr{O}(p^c)$ and any $\lambda \in \gamma^{-1}(\Pi_{k=1}^3(0,1), \mathscr{O}(p^c))$, $p(\lambda) = H\{p(1), \mathscr{S}[p(1), p(\lambda)]\}$.

Hence I have proved, by continuity of λ in the parameters (ρ, u_i, c_{ij}), that there exist an open set of parameters and a neighborhood $\mathscr{O}$ of p^c for which there exists a stationary state approachable by a two-period convergent path. Moreover, this stationary state satisfies conditions (2.4) through (2.7). Finally, I argue about the optimality of nondisposal, condition (2.8), to have a steady state.

Notice that the expected-utility vector v^c associated with (p^c, s^c) for $\lambda = 0$ satisfies the inequalities

$$(1 - \rho) v_{ij}^c = -c_{ij} + (7/27)\rho(u_i + c_{i,i+1}) > 0$$

by assumption. That is, condition (2.8) is satisfied by (p^c, s^c) for $\lambda = 0$. Hence the claim follows by continuity of γ at $\lambda = 0$ and continuity of λ in the parameters (ρ, u_i, c_{ij}).

PROOF OF PROPOSITION 3.2. I prove first that $s(t) = s^f$ are the unique best responses under the assumptions made. Note that $s_{ij}^1(t) = 1$, $j \neq i$, implies $v_{i1}(t) > v_{ij}(t)$ and consequently $s_{i1}^j(t) = 0$, $j \neq i$, in any equilibrium. That is, $s_{-1}(t) = s_{-1}^f$ are the unique best responses if $s_{ij}^1(t) = 1$, $j \neq i$. Using the identities $p_{i,i+1} + p_{i,i+2} = \frac{1}{3}$, notice that for $s_{-1}(t) = s^f_{-1}$,

$$\begin{aligned} v_{13}(t) - v_{12}(t) &= c_{12} - c_{13} + \rho(u_1 + c_{12})[p_{31}(t+1) - p_{21}(t+1)] \\ &\quad + \rho[v_{13}(t+1) - v_{12}(t+1)] \\ &\quad \times [1 - p_{31}(t+1) - p_{23}(t+1)s^3_{12}(t+1) \\ &\quad - p_{32}(t+1)s^2_{13}(t+1)] \end{aligned}$$

$$p_{31}(t+1) - p_{21}(t+1) = \tfrac{1}{3}[p_{31}(t) - p_{21}(t)] - p_{21}(t)p_{13}(t) + \tfrac{1}{9} + p_{13}(t)p_{32}(t)s^2_{13}(t)$$

$$p_{12}(t+1) = p_{12}(t)[1 - p_{23}(t)s^3_{12}(t) - p_{31}(t)] + \tfrac{1}{3}p_{31}(t) + p_{13}(t)p_{32}(t)s^2_{13}(t)$$

and the $p_{31}(t)$ converges monotonically to $\frac{1}{3}$. Hence, if $\lambda_1(2)$ is small enough relative to $p_{31}(t+1) - p_{21}(t+1)$, which is bounded away from zero for $p_{31}(1) - p_{21}(1) > 0$, $v_{13}(t) > v_{12}(t)$, and consequently, $s^3_{12}(t) = 1$ and $s^2_{13}(t) = 0$ are the unique best responses. Therefore, $s(t) = s^f$ are the unique best responses under the assumptions made.

Condition (2.7) is also satisfied with strict inequality for parameters that satisfy $u_i + c_{ii} > \rho(u_i + c_{i,i+1})$. Notice that if $p_{31}(1) > 0$, all $p_{ij}(t+4)$, $j \neq i$, but $p_{32}(t)$ is bounded away from zero. This implies enough direct and indirect trade for condition (2.8) to be satisfied if storage costs are small enough. Hence the existence claim follows.

I deal next with the convergence claim. I prove next that the law of motion of the inventory distributions (Eq. 2.1) is a contraction on the inventory distributions for a compact set containing the steady-state inventory distribution p^f. The path $\{p(t+1), s^f\}$ from p^c actually reaches this compact set because at least $p(13)$, which can be computed directly, does. Therefore, since Eq. (2.1) is continuous on $p(t)$, the equilibrium path $\{p(t+1), s^f\}$ converges to (p^f, s^f) for initial conditions in a neighborhood of p^c.

Let the vector s of trading strategies be given by

$$s = (s^3_{12}, s^2_{13}, s^1_{23}, s^3_{21}, s^2_{31}, s^1_{32}) = (1, 0, 1, 0, 0, 1)$$

Plugging these strategies into Eq. (2.1) and using the identities $p_{i,i+1} = \frac{1}{3} - p_{i,i+1}$, notice that

$$p_{13}(t+1) = [\tfrac{1}{3} + p_{21}(t) + p_{32}(t)]p_{13}(t) - \tfrac{1}{3}p_{21}(t) + \tfrac{1}{9}$$

$$p_{21}(t+1) = [\tfrac{1}{3} + p_{13}(t)]p_{21}(t) - \tfrac{1}{3}p_{32}(t) + \tfrac{1}{9}$$

$$p_{32}(t+1) = \tfrac{2}{3}p_{32}(t)$$

Let $S = [0.09, 0.11] \times [0.18, 0.2] \times [0, 0.001] \subset \mathbb{R}^3$ and define $T\colon S \to S$ by

$$T_1(x,y,z) = \left(\tfrac{1}{3} + y + z\right)x - \tfrac{1}{3}y + \tfrac{1}{9}$$

$$T_2(x,y,z) = \left(\tfrac{1}{3} + x\right)y - \tfrac{1}{3}z + \tfrac{1}{9}$$

$$T_3(x,y,z) = \tfrac{2}{3}z$$

Now I am going to show that T is a contraction on S with the sup metric d. Let (x,y,z) and (x',y',z') be in S. Assume that

$$d[(x,y,z),(x',y',z')] = \sup\{|x-x'|, |y-y'|, |z-z'|\} = \delta$$

Since $xy - x'y' = x(y-y') + y'(x-x')$ and similarly for $xz - x'z'$,

$$\begin{aligned}
|T_1(x,y,z) - T_1(x',y',z')| &= \left|\tfrac{1}{3}(x-x') - \tfrac{1}{3}(y-y') + xy - x'y' + xz - x'z'\right| \\
&= \left|\left(\tfrac{1}{3} + y' + z'\right)(x-x') - \left(\tfrac{1}{3} - x\right)(y-y') + x(z-z')\right| \\
&\leq \left(\tfrac{1}{3} + y' + z'\right)\delta + \left(\tfrac{1}{3} - x\right)\delta + x\delta \\
&\leq (0.67 + 0.2 + 0.001)\,\delta = 0.871\,\delta \\
|T_2(x,y,z) - T_2(x',y',z')| &= \left|\tfrac{1}{3}(y-y') - \tfrac{1}{3}(z-z') + xy - x'y'\right| \\
&= \left|\left(\tfrac{1}{3} + x\right)(y-y') + y'(x-x') - \tfrac{1}{3}(z-z')\right| \\
&\leq \left(\tfrac{2}{3} + x + y'\right)\delta \\
&\leq (0.667 + 0.11 + 0.2)\,\delta = 0.977\delta \\
|T_3(x,y,z) - T_3(x',y',z')| &\leq \tfrac{2}{3}\delta
\end{aligned}$$

Hence, for all (x,y,z) and (x',y',z') in S,

$$d[T(x,y,z), T(x',y',z')] \leq 0.98 d[(x,y,z),(x',y',z')]$$

that is, T is a contraction of modulus 0.98.

PROOF OF PROPOSITION 3.3. We prove first that $s(t) = s^s$ are the unique best responses under the assumptions made. Note that $s_{ij}^3(t) = 1$, $j \neq i$, implies that $v_{13}(t) > v_{12}(t)$, and consequently, $s_{12}^3(t) = 1$ and $s_{13}^1(t) = 0$ are the unique best responses for agents of type 1 if $s_{ij}^3(t) = 1$, $j \neq i$. Notice that $s_{21}^3(t) = 1$ implies $s_{23}^1(t) = 0$ because of the strategies assumed. Using the identities $p_{i,i+1} + p_{i,i+2} = \frac{1}{3}$, notice that for $s_{-3}^s(t) = s_{-3}^s$,

$$\begin{aligned}
v_{32}(t) - v_{31}(t) = c_{31} - c_{32} &+ \rho(u_3 + c_{31})[p_{23}(t+1) - p_{13}(t+1)] \\
&+ \rho[v_{32}(t+1) - v_{31}(t+1)] \\
&\quad \times [1 - p_{23}(t+1) - p_{12}(t+1)s_{31}^2(t+1) \\
&\qquad - p_{21}(t+1)s_{32}^1(t+1)]
\end{aligned}$$

$$p_{23}(t+1) - p_{13}(t+1) = \tfrac{1}{3}[p_{23}(t) - p_{13}(t)] - p_{13}(t)p_{32}(t) + \tfrac{1}{9} + p_{32}(t)p_{21}(t)s_{32}^{1}(t)$$

$$p_{31}(t+1) = p_{31}(t)[1 - p_{12}(t)s_{31}^{2}(t) - p_{23}(t)] + \tfrac{1}{3}p_{23}(t) + p_{32}(t)p_{21}(t)s_{32}^{1}(t)$$

and that $p_{23}(t)$ converges monotonically to $\frac{1}{3}$. Hence, if $\lambda_3(1)$ is small enough relative to $p_{23}(t+1) - p_{13}(t+1)$, which is bounded away from zero for $p_{23}(1) - p_{13}(1) > 0$, $v_{32}(t) > v_{31}(t)$, and consequently, $s_{31}^{2}(t) = 1$ and $s_{32}^{1}(t) = 0$ are the unique best responses for agents of type 3. Finally, we have to verify that the trading strategies of agents of type 2, $s_{21}^{3}(t) = 1$ and $s_{23}^{1}(t) = 0$, are best responses. Since for $s(t) = s^s$,

$$v_{23}(t) - v_{21}(t) = c_{21} - c_{23} + \rho(u_2 + c_{23})p_{32}(t+1) + \rho[v_{23}(t+1) - v_{21}(t+1)][1 - p_{12}(t+1) - p_{13}(t+1)]$$

$$p_{32}(t+1) = p_{12}(t)p_{31}(t) + [1 - p_{23}(t+1)]p_{32}(t)$$

$$p_{12}(t+1) = p_{12}(t)[\tfrac{2}{3} - p_{31}(t)] + \tfrac{1}{3}[p_{21}(t) + p_{31}(t)]$$

$s_{21}^{3}(t) = 1$ and $s_{23}^{1}(t) = 0$ are best responses if $\lambda_2(1)$ is less than $p_{32}(t+1)$, which is bounded away from zero given the assumptions. Therefore, $s(t) = s^s$ are the unique best responses under the assumptions made.

Condition (2.7) is also satisfied with strict inequality for parameters that satisfy $u_i + c_{ii} > \rho(u_i + c_{i,i+1})$. Notice that if $p_{31}(1) > 0$, all $p_{ij}(t+4)$, $j \neq i$, but $p_{21}(t)$ are bounded away from zero. This implies enough direct and indirect trade for condition (2.8) to be satisfied if storage costs are small enough. Hence the existence claim follows.

The proof of the convergence claim is similar to the proof of Proposition 3.2.

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