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Existence of Steady States with Positive Consumption in the Kiyotaki-Wright Model

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We prove the general existence of steady states with positive consumption in an N goods and fiat money version of the Kiyotaki-Wright model by admitting mixed strategies. We also show that there always exists a steady state in which everyone accepts a least costly-to-store object. In particular, if fiat money is one such object, then there always exists a monetary steady state. We also establish some other properties of steady states and comment on the relationship between steady states and (incentive) feasible allocations.

I. INTRODUCTION

In Kiyotaki-Wright (1989), the authors describe steady-state patterns of exchange among indivisible and durable, but costly-to-store objects in a model in which people meet pairwise and at random. They study versions with three goods and with and without a fourth object, fiat money, and consider only pure trading strategies. In the version without fiat money they find, among other things, that there is a region of the parameter space for which there is no pure strategy steady state. Here we study an N good plus fiat money version of their model and allow mixed trading strategies. The purpose is to show the general existence of non-monetary (with fiat money not used) and monetary (with fiat money used) steady states with positive consumption and with agents neither freely disposing of goods nor giving gifts.[1] We also establish some trading-pattern properties of these steady states, and explore the relationship between steady states analysed here and some notions of incentive feasible allocations.

The outline of this paper is as follows. In Section II, we describe the physical environment of the model. In Section III, we set up the notation, describe the individual stationary decision problems, and define monetary and non-monetary steady states. Our existence results are established in Sections IV-VII. First, in Section IV, we use a fixed point argument to show that restricted steady states (in dominant strategies) exist when free disposal is disallowed and agents are prohibited from accepting fiat money. Then in Section V, we give sufficient conditions (in terms of the parameters of the model) for these restrictions to be non-binding and to guarantee positive consumption, thus implying

1. Kehoe, Kiyotaki and Wright (1989) have examples of steady states with positive consumption in mixed strategies for parameters for which there are no such steady states in pure strategies.

that any Section IV fixed point is a positive consumption (non-monetary) steady state. Roughly speaking, the sufficient conditions require that the utility costs of storing be relatively low compared to the utility of consuming. In Section VI, we establish some trading-pattern properties of non-monetary steady states. One of these is that if there is a least costly-to-store good and the sufficient conditions of Section V hold, then there always exists a non-monetary steady state in which everyone accepts the least costly-to-store good. The logic of this result is used in Section VII to establish the existence of a monetary steady state whenever fiat money is a least costly-to-store object and a condition analogous to that of Section V holds.

Lastly, in Section VIII, we discuss the relationship between steady-state allocations and a notion of stationary incentive feasible allocations. The discussion is motivated by Kiyotaki-Wright's observation that in some examples, always-trade strategies lead to allocations that Pareto dominate some steady-state allocations, even though these (always-trade) strategies are not individually optimal. We show that there is a concept of feasibility, which includes incentive compatibility and individual rationality restrictions, that implies that the set of steady-state allocations and a comparable set of stationary feasible allocations coincide. According to this concept of feasibility, the allocation resulting from always-trade strategies is not feasible and necessarily, at least one steady state is optimal.

II. THE MODEL

Time is discrete. There are $N+1$ indivisible and storable objects—N (consumption) goods and fiat money. There is a $[0, 1/N]$ continuum of each of N types of infinitely-lived agents. (We will generally, but not always, index objects by Latin letters—$i, j, \ldots$, taking integer values $0, 1, 2, \ldots, N$ with object 0 being fiat money, and agent types by Greek letters—$\alpha, \beta, \ldots$, taking integer values $1, 2, \ldots, N$.)

An agent of type α maximizes the expected value of discounted utility, with discount factor $\rho \in (0, 1)$ not dependent on α.[2] Date t utility of type α is an increasing, time-independent function of α's date t consumption of good $i = \alpha$ and a decreasing function of the amounts of objects α stores from t to $t+1$.

It turns out that we need to evaluate date t utility at only the following $N+3$ points. We let u_α be the date t utility for type α of consuming one unit of good $i = \alpha$ at t and not storing anything from t to $t+1$, we let $-c_{\alpha i}$ be the date t utility for type α of not consuming at t and of storing one unit of object i from t to $t+1$, and we let 0 be the date t utility of neither consuming nor storing.

For each unit of good $i = \alpha$ consumed by a type α agent at date t, the agent produces one unit of good $j = \alpha + 1$ (modulo N) which appears at date $t+1$. That is, there is a linear technology of the form: each unit of consumption of good i at t gives rise to one unit of good $i+1$ at $t+1$.

At each date, each agent is paired at random with one other agent. Moreover, it is assumed that paired agents do not know each other's trading histories. This is plausible because with a continuum of agents the probability of these agents having met before or of either having met others who had met others . . . who had met the current trading partner, is zero. Trading partners are assumed to know each other's type and current inventory. Finally, the initial condition, although strictly speaking not used, should be taken to be that each agent begins with exactly one unit of some object.

The pairwise meetings and the specialized preference and technology assumptions are intended to imply the need for indirect trades—trades in which agents acquire

2. Nothing important hinges on the discount factors being identical across agent types.

something other than their consumption good. They do if $N>2$, because then a pairwise meeting between two agents who have their produced goods does not give rise to a double coincidence in consumed goods.

This model is the Kiyotaki-Wright model except in one respect. They assume that agents can store at most one unit and can at any time choose between storing the object they have and disposing of it and producing their production good. We assume that consumption at t is the sole input into production at $t+1$ and that there is no storage capacity. The difference does not matter for the existence question. Their non-monetary steady states correspond to ours without any moneyholders; the monetary steady states are the same. The difference matters for comparing welfare between monetary and non-monetary steady states, something we do not undertake in this paper.

III. DEFINITION OF A STEADY STATE

We use a notation that presumes that each person enters each period with one unit of some object. Thus, the notation presumes that agents never dispose of any object nor do they give it away to another agent for nothing. Our notation also presumes that trading strategies are non-discriminatory (willingness to trade does not depend on the type of agent one meets) and symmetric (all agents of a given type in the same trading situation use the same strategy). We show that equilibria of this kind exist and that these remain equilibria even if agents are permitted to freely dispose and strategies are permitted to be discriminatory.

We let $s^j_{\alpha i}$ be the strategy of agent α holding object i who meets with an opportunity to trade i for j. We interpret it as the probability that α is willing to trade. Accordingly, we assume $s^j_{\alpha i}\in I\equiv[0,1]$. We let $s\in I^{(N+1)(N+1)N}$ denote the vector of these strategies. We assume that the consumption storage strategies are also symmetric and let $t_\alpha\in I$ be the probability that a type α agent will consume good α given that he ends up with good α after trading. Then, $(1-t_\alpha)$ is the probability that he will store good α. We let $t\in I^N$ denote the vector of these strategies.

The timing of various activities within a period is shown in Figure 1 and described below. Period t extends from date t to date $t+1$. At date t, each agent begins with one unit of some object. At time A within period t, each agent meets another agent to trade. At time B, each agent ends up with some object after the trading round. At time C, each agent decides whether to store the object he has, or to consume it (if it is his consumption good) and produce his production good. Thus, he starts period $t+1$ with one unit of some object. We let $p_{\alpha i}$ be the proportion who are type α agents and who hold object i at the start of any period. These proportions must satisfy the following

$$\sum_i p_{\alpha i}=1/N. \tag{1}$$

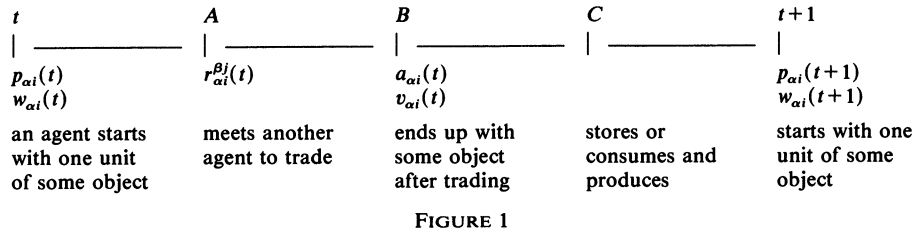

FIGURE 1

We let m denote the fraction of agents who hold money so that $\sum_\alpha p_{\alpha 0} = m$. We let $p \in I^{N(N+1)}$ denote the vector of $p_{\alpha i}$'s.

For any agent, the probability of meeting a type α agent holding object i is $p_{\alpha i}$. We let $a_{\alpha i}$ denote the fraction of agents who are type α and end up with object i after trading, i.e. at time B in Figure 1. In a steady state, the $p_{\alpha i}$, $a_{\alpha i}$, and $s^j_{\alpha i}$ satisfy the following:

$$a_{\alpha i}(s, p) = p_{\alpha i} - p_{\alpha i} \sum_\beta \sum_{j \neq i} p_{\beta j} s^j_{\alpha i} s^i_{\beta j} + \sum_\beta \sum_{j \neq i} p_{\alpha j} p_{\beta i} s^i_{\alpha j} s^j_{\beta i}, \tag{2a}$$

$$p_{\alpha i} = (1 - t_\alpha \delta_{\alpha i}) a_{\alpha i}(s, p) + t_\alpha \delta_{i,\alpha+1} a_{\alpha\alpha}(s, p), \tag{2b}$$

where $\delta_{ij} = 1$ if $i = j$ and 0 otherwise.

If (p, s, t) on the right-hand sides of (2a) and (2b) pertain to one date, then the right-hand side function in (2b) gives $p_{\alpha i}$ at the next date. On the right-hand side of (2a), the first term is the proportion who are type α and hold i prior to trade; the second term consists of the fraction of those who trade for a different object; and the third term is the proportion who are type α and trade some other object for object i. The explanation for (2b) is the following. If i is neither α nor $\alpha+1$ (which is α's produced good), then necessarily $p_{\alpha i}$ equals $a_{\alpha i}$. If i is α, then $p_{\alpha i}$ is the stored fraction $(1 - t_\alpha)$ of $a_{\alpha i}$. Lastly, if i is $\alpha+1$ then $p_{\alpha i}$ is $a_{\alpha i}$ plus the consumed fraction t_α of $a_{\alpha\alpha}$.

We now define the individual stationary decision problems in terms of optimal values. Let $r^{\beta j}_{\alpha i} \in R$ be the optimal value (expected discounted utility) of type α with object i who meets a type β with object j, i.e. at time A in Figure 1. Let $v_{\alpha i} \in R$ be the optimal value for type α ending with object i after trading but before consuming or storing, i.e. at time B in Figure 1. Lastly, let $w_{\alpha i} \in R$ denote the optimal value for type α beginning a period with object i. We let $v \in R^{N(N+1)}$ be the vector of $v_{\alpha i}$'s.

The values satisfy the following version of Bellman's equation:

$$r^{\beta j}_{\alpha i} = \max_{x \in I} [x s^i_{\beta j} v_{\alpha j} + (1 - x s^i_{\beta j}) v_{\alpha i}] = v_{\alpha i} + s^i_{\beta j} \max_{x \in I} [x(v_{\alpha j} - v_{\alpha i})], \tag{3a}$$

$$w_{\alpha i} = \sum_\beta \sum_j p_{\beta j} r^{\beta j}_{\alpha i}, \tag{3b}$$

$$v_{\alpha i} = \begin{cases} \rho w_{\alpha i} - c_{\alpha i} & \text{if } i \neq \alpha, 0, \\ \max [0, \rho w_{\alpha i} - c_{\alpha i}] & \text{if } i = 0, \\ \max_{z \in I} [z(u_\alpha + \rho w_{\alpha, i+1}) + (1 - z)(\rho w_{\alpha i} - c_{\alpha i})] & \text{if } i = \alpha. \end{cases} \tag{3c}$$

Definition. A *steady state with positive consumption* consists of (s, p, t, v) such that

$$(s, p, t) \text{ satisfies equation (2)}, \tag{4a}$$

$$(s, p, v) \text{ satisfies equation (3)}, \tag{4b}$$

$$z = t_\alpha \text{ attains } v_{\alpha\alpha}, \tag{4c}$$

$$x = s^j_{\alpha i} \text{ attains } r^{\beta j}_{\alpha i} \text{ for all } \beta, \tag{4d}$$

$$\text{(positive consumption) } t_i a_{ii} > 0, \tag{4e}$$

$$\text{(free disposal) } v_{\alpha i} \geqq 0. \tag{4f}$$

and either (for a monetary steady state)

$$\sum_\alpha p_{\alpha 0} = m, \tag{4g}$$

$$p_{\alpha 0} s^j_{\alpha 0} s^0_{\beta j} p_{\beta j} > 0 \quad \text{for some } (\alpha, \beta, j) \text{ with } j \neq 0, \tag{4h}$$

or (for a non-monetary steady state)

$$p_{\alpha 0} = m/N, \tag{4g'}$$

$$p_{\alpha 0} s^j_{\alpha 0} s^0_{\beta j} p_{\beta j} = 0 \quad \text{for all } (\alpha, \beta, j) \text{ with } j \neq 0. \tag{4h'}$$

Remark. Given that no one else gives objects away, an agent's expected discounted utility from disposing of any object is 0. Therefore, condition (4f) implies that there is no gain from disposing of any object. Note that condition (4f) is equivalent to $r_{\alpha i}^{\beta j} \geqq 0$. This follows from (3a) because, $r_{\alpha i}^{\beta j} \geqq v_{\alpha i} = r_{\alpha i}^{\alpha i}$. In (4e), $t_i a_{ii}$ represents consumption of good i. In a steady state this must equal production of good i which in turn must equal consumption of good $(i-1)$, and so on. This may be verified by summing (2b) over α and using (2a) to note that $\sum_\alpha p_{\alpha i} = \sum_\alpha a_{\alpha i}$. Condition (4h) defines monetary trade; it says that some trade of money for goods occurs. For a steady state to be non-monetary, condition (4h′) requires that money never trade for goods. It follows that any initial distribution of money holdings will be maintained. We assume that in a non-monetary steady state money is equally distributed over types; this is condition (4g′). From equations (3), it is easy to see that in a non-monetary steady state $v_{\alpha 0} = \max[0, -c_{\alpha 0}/(1-\rho)]$. If $c_{\alpha 0} > 0$, then this is to be interpreted to mean that money holders have disposed of money and attained zero expected discounted utility. Accordingly, in a non-monetary steady state we interpret $p_{\alpha 0}$ as the proportion who hold at most money.[3]

IV. EXISTENCE OF RESTRICTED NON-MONETARY STEADY STATES

We pursue the strategy outlined in the introduction for our existence result. We prohibit goods holders from trading for money and we disallow disposal of goods. A fixed point argument is used to establish the existence of such a steady state (in dominant strategies) which satisfies conditions (4a)-(4c), (4d) except possibly for $j=0$, and (4g′). We then give a sufficient condition involving utilities for any such fixed point to also satisfy the remaining part of (4d), (4e), and (4f). Absent some such conditions, the fixed point could have agents unwilling to trade, even for their consumption good.

Let,

$$b_\alpha = \min[0, -c_{\alpha 0}, -c_{\alpha 1}, \ldots, -c_{\alpha N}, u_\alpha]/(1-\rho), \tag{5a}$$

$$B_\alpha = \max[0, -c_{\alpha 0}, -c_{\alpha 1}, \ldots, -c_{\alpha N}, u_\alpha]/(1-\rho), \tag{5b}$$

$$V_\alpha = [b_\alpha, B_\alpha], \tag{5c}$$

$$V = \bigtimes_\alpha \bigtimes_i V_\alpha, \tag{5d}$$

$$\bar{S} = \{s \in I^{(N+1)(N+1)N} \mid s_{\alpha i}^0 = 0\}, \tag{5e}$$

$$\bar{P} = \{p \in I^{N(N+1)} \mid p_{\alpha i} \text{ satisfy (1) and } p_{\alpha 0} = m/N\}. \tag{5f}$$

We now define four mappings.[4] The first, $\mu: \bar{S} \times \bar{P} \times V \to V$, is defined by way of (3a)-(3c) as follows:

$$r_{\alpha i}^{\beta j} = \begin{cases} \text{right-hand side of (3a)}, & j \neq 0, \\ v_{\alpha i}, & j = 0, \end{cases} \tag{6a}$$

3. In non-monetary steady states, those endowed with money are unable to trade for goods. Nevertheless, meetings occur pairwise at random among *all* agents. This is consistent with the assumed exogeneity of the meeting process in this model (people do not choose any aspect of who to meet and when to meet). Although money holders may attain zero discounted utility, that need not be interpreted as zero consumption. All significant aspects of the model are unaffected by assuming, for example, that every person is endowed each period with some non-durable amount of consumption which does not serve as an input and which, by itself, implies zero utility.

4. For the present analysis of non-monetary steady states it is possible to omit 0 and $c_{\alpha 0}$ on the right sides of (5a) and (5b). However, including these is convenient since then the V_α's and V do not have to be redefined for the analysis of monetary steady states.

$$w_{\alpha i} = \sum_\beta \sum_j p_{\beta j} r_{\alpha i}^{\beta j}, \tag{6b}$$

$$(\mu(s, p, v))_{\alpha i} = \begin{cases} \text{right-hand side of (3c)}, & i \neq 0, \\ \max [0, -c_{\alpha 0}/(1-\rho)], & i = 0. \end{cases} \tag{6c}$$

It follows from the Berge maximum theorem that (6a) defines a continuous function $r_{\alpha i}^{\beta j}: \bar{S} \times V \to V_\alpha$ and hence (6b) defines a continuous function $w_{\alpha i}: \bar{S} \times \bar{P} \times V \to V_\alpha$. Applying the theorem again to (6c) shows that μ is a continuous function.

Next we are interested in the maximizers in (3a) for $j \neq 0$ and in (3c) for $i = \alpha$. The first equality in (3a) suggests that, in general, the maximizing trading strategy correspondence for (α, i) facing (β, j) will depend on $s_{\beta j}^i$, $v_{\alpha i}$, and $v_{\alpha j}$. However, the second equality in (3a) shows that there exists a maximizing trading strategy correspondence for (α, i) facing j that is independent of the strategy chosen by (β, j) and independent of type β. For this reason, we call this a dominant trading strategy correspondence and denote it by $\sigma_{\alpha i}^j: V \to I$. Inspection of the second equality in (3a) shows that

$$\sigma_{\alpha i}^j = \begin{cases} \{1\} & \text{if } v_{\alpha j} - v_{\alpha i} > 0, \ j \neq 0, \\ I & \text{if } v_{\alpha j} - v_{\alpha i} = 0, \ j \neq 0, \\ \{0\} & \text{if } v_{\alpha j} - v_{\alpha i} < 0, \ j \neq 0. \end{cases} \tag{7a}$$

For $j = 0$ we have the restriction

$$\sigma_{\alpha i}^0 = \{0\}. \tag{7b}$$

We also define the following consumption/storage strategy correspondences derived from maximization in (3c) for $i = \alpha$. This is denoted $\tau_\alpha: \bar{S} \times \bar{P} \times V \to I$. Inspection of (3c) for $i = \alpha$ reveals that

$$\tau_\alpha = \begin{cases} \{1\} & \text{if } u_\alpha + \rho w_{\alpha,\alpha+1} > \rho w_{\alpha\alpha} - c_{\alpha\alpha}, \\ I & \text{if } u_\alpha + \rho w_{\alpha,\alpha+1} = \rho w_{\alpha\alpha} - c_{\alpha\alpha}, \\ \{0\} & \text{if } u_\alpha + \rho w_{\alpha,\alpha+1} < \rho w_{\alpha\alpha} - c_{\alpha\alpha}. \end{cases} \tag{8}$$

Inspection of (7) and (8) and application of the Berge maximum theorem shows that the correspondences $\sigma_{\alpha i}^j$ and τ_α are each non-empty, convex-valued and upper hemicontinuous.

Finally, let $\pi_{\alpha i}: \bar{S} \times \bar{P} \times I^N \to R$ be given by: $\pi_{\alpha i}(s, p, t) =$ right-hand side of (2b). It follows that $\pi_{\alpha i}(\cdot)$ is a continuous function.

We can now show the existence of restricted non-monetary steady states. Let

$$\sigma = \bigtimes_\alpha \bigtimes_i \bigtimes_j \sigma_{\alpha i}^j: V \to \bar{S}, \tag{9a}$$

$$\pi = \bigtimes_\alpha \bigtimes_i \pi_{\alpha i}: \bar{S} \times \bar{P} \times I^N \to \bar{P}, \tag{9b}$$

$$\tau = \bigtimes_\alpha \tau_\alpha: \bar{S} \times \bar{P} \times V \to I^N, \tag{9c}$$

$$\psi = (\sigma, \pi, \tau, \mu): \bar{S} \times \bar{P} \times I^N \times V \to \bar{S} \times \bar{P} \times I^N \times V. \tag{9d}$$

Proposition 1. *There exists a fixed point for the correspondence ψ; i.e. there exists $(s, p, t, v) \in \bar{S} \times \bar{P} \times I^N \times V$ such that $(s, p, t, v) \in \psi(s, p, t, v)$.*

Proof. From definitions (9) and earlier remarks, we have that ψ is non-empty, convex-valued, and upper hemicontinuous. Since $\bar{S} \times \bar{P} \times I^N \times V$ is non-empty, compact and convex, Kakutani's fixed point theorem applies. ‖

V. EXISTENCE OF NON-MONETARY STEADY STATES WITH POSITIVE CONSUMPTION

Note that a restricted non-monetary steady state (a Proposition 1 fixed point) satisfies conditions (4a)-(4c), and (4d) for $j \neq 0$ in the definition of a steady state. We now give a sufficient condition for the fixed point to satisfy the remaining conditions for a non-monetary steady state. The argument, which is rather long, proceeds as follows. We first show, using part of the sufficient condition, that the fixed point is such that people want to trade for and consume their consumption good (Proposition 2). That implies that the fixed point is such that no one holds their own consumption good; $p_{ii} = 0$ for all i. A consequence is that the proportion of each type holding some other good is bounded below. This, in turn, implies that there is a trading route using the lower bound proportions and involving no more than N trades that produces a positive lower bound on the rate of consumption for some type (Proposition 3). This immediately implies the same bound for all types and the same bound for the proportion of each type holding their production good, a lower bound for $p_{i,i+1}$ for all i. Finally, the uniform bound on $p_{i,i+1}$ implies a trading route for anyone starting with any good to their consumption good in no more than N trades; namely, trade the good held to the person who consumes that good and holds their production good and then repeat such trades, trading good i for good $i+1$, until the consumption good is obtained. This and the entire sufficient condition imply a lower bound on expected utility for any type holding any good, a bound which assures that agents do not dispose of their goods (Proposition 4) and do not want to trade for fiat money (Proposition 5).

First we introduce some notation. Let

$$\bar{p} = (1-m)/N(N-1), \tag{10}$$

$$A_\alpha(\theta, n) = \theta^n[u_\alpha - \rho c_{\alpha,\alpha+1}/(1-\rho)] + (1-\theta^n)\min(0, -C_\alpha)/(1-\rho), \tag{11}$$

where $C_\alpha = \max_{i \neq 0} c_{\alpha i}$. It will turn out that $p_{\alpha i} \geqq \bar{p}$ for some $i \neq 0, \alpha$. $A_\alpha(\theta, n)$ is a weighted average of two terms. The first is expected utility from consuming and producing and storing the produced good forever. It is a lower bound on $v_{\alpha\alpha}$. The second term is the minimum of zero and the expected utility of storing the most costly-to-store good forever. One part of the sufficient condition we use is

(C1) $$A_\alpha(\bar{\theta}, N) > \max(0, -c_\alpha)/(1-\rho),$$

where $c_\alpha = \min_{i \neq 0} c_{\alpha i}$ and

$$\bar{\theta} = \rho\bar{p}/[1-\rho(1-\bar{p})]. \tag{12}$$

Note that since $0 < \bar{\theta} < 1$, (11) and (C1) imply

$$u_\alpha - \rho c_{\alpha,\alpha+1}/(1-\rho) = A_\alpha(\bar{\theta}, 0) > A_\alpha(\bar{\theta}, 1) > \cdots > A_\alpha(\bar{\theta}, N), \tag{13}$$

$$u_\alpha + c_{\alpha,\alpha+1} = A_\alpha(\bar{\theta}, 0) + c_{\alpha,\alpha+1}/(1-\rho) \geqq A_\alpha(\bar{\theta}, 0) + c_\alpha/(1-\rho) > 0. \tag{14}$$

We now show that if (C1) holds, then in any restricted non-monetary steady state agents with goods always want to trade for and consume their consumption good.

Proposition 2. *Let* (s, p, t, v) *be a restricted non-monetary steady state and suppose that condition* (C1) *holds. Then* (a) $s^{\alpha}_{\alpha i}=1$ *for* $i\neq 0$ *and* (b) $t_{\alpha}=1$.

Proof. (a) Suppose not. Then by (3a) for some (α, i) with $i\neq\alpha, 0$; $v_{\alpha i}\geqq v_{\alpha\alpha}$. Without loss of generality, let this i be such that $v_{\alpha i}\geqq v_{\alpha j}$ for all $j\neq 0$. Then by (3a), $r^{\beta j}_{\alpha i}=v_{\alpha i}$ for all (β, j). Substituting these values into (3b) we get $w_{\alpha i}=v_{\alpha i}=\rho w_{\alpha i}-c_{\alpha i}$ from (3c). Therefore, $v_{\alpha i}=-c_{\alpha i}/(1-\rho)\leqq -c_{\alpha}/(1-\rho)$. But $v_{\alpha\alpha}\geqq[u_{\alpha}-\rho c_{\alpha,\alpha+1}/(1-\rho)]> -c_{\alpha}/(1-\rho)$ from (13) and (C1). That is, $v_{\alpha\alpha}>v_{\alpha i}$, which contradicts $v_{\alpha i}\geqq v_{\alpha\alpha}$.

(b) Suppose not. Then by (3c) for some α, $v_{\alpha\alpha}=\rho w_{\alpha\alpha}-c_{\alpha\alpha}$. By part (a), $v_{\alpha\alpha}>v_{\alpha i}$ for all $i\neq 0, \alpha$. Therefore, by (3b) and (3a), $w_{\alpha\alpha}=v_{\alpha\alpha}=\rho w_{\alpha\alpha}-c_{\alpha\alpha}$. It follows that $v_{\alpha\alpha}=-c_{\alpha\alpha}/(1-\rho)\leqq -c_{\alpha}/(1-\rho)$, which contradicts the lower bound on $v_{\alpha\alpha}$ used in part (a). ‖

We can now show that (C1) implies that a restricted non-monetary steady state satisfies the positive consumption condition (4e). We begin by setting out an expression for $a_{\alpha\alpha}$ that will be used to establish a positive lower bound for some $a_{\alpha\alpha}$ and hence, in view of the remark following the definition of a steady state, for all $a_{\alpha\alpha}$.

In the appendix, we derive the following equation:

$$\textstyle\sum_{i\neq 0} p_{\alpha i}v_{\alpha i}=[a_{\alpha\alpha}\rho(u_{\alpha}+c_{\alpha,\alpha+1})-\sum_{i\neq 0} p_{\alpha i}c_{\alpha i}]/(1-\rho). \tag{15}$$

This says that the expectation of $v_{\alpha i}$ is a sum of two discounted values: the average probability of consuming, $a_{\alpha\alpha}$, multiplied by the utility implied by consuming, and average storage costs. Since, by (14), $u_{\alpha}+c_{\alpha,\alpha+1}>0$, we can solve (15) for $a_{\alpha\alpha}$ obtaining

$$a_{\alpha\alpha}=\textstyle\sum_{i\neq 0} p_{\alpha i}[(1-\rho)v_{\alpha i}+c_{\alpha i}]/\rho(u_{\alpha}+c_{\alpha,\alpha+1}). \tag{16}$$

Since $v_{\alpha i}\geqq -c_{\alpha i}/(1-\rho)$, (storing good i forever is an option), $a_{\alpha\alpha}$ is a sum of non-negative terms, a fact used below.

Proposition 3. *Let* (s, p, t, v) *be a restricted non-monetary steady state. If condition* (C1) *holds, then* $t_i a_{ii}>0$ *for all* i.

Proof. We do this proof in steps.

Step 1. For some (k, l) there is a loop of the form $(p_{kl}, p_{lm}, \ldots, p_{*\cdot}, p_{\cdot k})$ where each $p_{\cdot\cdot}\geqq\bar{p}$ (see 10) and the length of the loop (denoted K) is at most N. To see this start with any i and let $j\neq 0$ be such that $p_{ij}=\max_{j'\neq 0} p_{ij'}$. Therefore, $p_{ij}\geqq\bar{p}$. By part (b) of Proposition 2, $t_i=1$ and hence (2b) implies that $p_{ii}=0$. Therefore, $j\neq i$. Now choose $k\neq 0$ such that $p_{jk}=\max_{k'\neq 0} p_{jk'}$ and continue the process in the obvious fashion. In at most N steps we obtain a sequence of p's containing a loop of the required form. (Note that the k which begins the loop is not necessarily the same as the arbitrary i that begins the sequence.)

Step 2. $a_{kk}\geqq\bar{p}[(1-\rho)v_{kl}+c_{kl}]/\rho(u_k+c_{k,k+1})]$.

Here we have set $\alpha=k$ in (16) and used as the lower bound for a_{kk} the $i=l$ term in the summation with p_{kl} replaced by $\bar{p}\leqq p_{kl}$.

Step 3. $a_{kk}>0$.

This will be established by deriving a lower bound for v_{kl}. Suppose (i, i') are two of the goods in the Step 1 loop (the second subscripts) with i' directly following i. Then

$$v_{ki}\geqq\rho[p_{ii'}v_{ki'}+(1-p_{ii'})v_{ki}]-c_{ki},$$

where the inequality follows from $r_{ki}^{\beta j} \geqq v_{ki}$ (not trading is always an option) and $r_{ki}^{ii'} \geqq v_{ki'}$ (trading for i' is an option by the definition of the loop since $s_{ii'}^{i} = 1$). This implies

$$v_{ki} \geqq \theta_{ii'} v_{ki'} - (1 - \theta_{ii'}) c_{ki}/(1-\rho), \tag{17}$$

where $\theta_{ii'} = \rho p_{ii'}/[1-\rho(1-p_{ii'})]$. Since $p_{ii'} \geqq \bar{p}$, $\theta_{ii'} \geqq \bar{\theta}$. Therefore, it follows from (17) that

$$v_{ki} \geqq \theta_{ii'} v_{ki'} + (1-\bar{\theta}) \min(0, -C_k)/(1-\rho). \tag{18}$$

Now we use (18) to work backward from v_{kk} to v_{kl}. We know that $v_{kk} \geqq A_k(\bar{\theta}, 0) > 0$. Therefore, using $\theta_{ii'} \geqq \bar{\theta}$, $v_{k\cdot} \geqq \bar{\theta} A_k(\bar{\theta}, 0) + (1-\bar{\theta}) \min(0, -C_k)/(1-\rho) = A_k(\bar{\theta}, 1) > 0$. Proceeding this way and using the recursion

$$A_\alpha(\theta, n) = \theta A_\alpha(\theta, n-1) + (1-\theta) \min(0, -C_\alpha)/(1-\rho),$$

we obtain

$$v_{kl} \geqq A_k(\bar{\theta}, K) \geqq A_k(\bar{\theta}, N), \tag{19}$$

since $K \leqq N$.

Therefore

$$(1-\rho) v_{kl} + c_{kl} \geqq (1-\rho) A_k(\bar{\theta}, N) + c_{kl} \geqq (1-\rho) A_k(\bar{\theta}, N) + c_k.$$

Using this in the Step 2 inequality for a_{kk}, we have

$$a_{kk} \geqq \bar{p}[(1-\rho) A_k(\bar{\theta}, N) + c_k]/\rho(u_k + c_{k,k+1}) \equiv a_k.$$

By (C1) (and (14)), $a_k > 0$ for all k, which completes Step 3.

To complete the proof of Proposition 3, we recall that in a steady state, we must have $a_{ii} = a_{jj}$ for all $i, j \neq 0$. ‖

Now let $\hat{p} = \min_k a_k$. By (C1) $\hat{p} > 0$. We use $\hat{p}$ to state the additional part of the sufficient condition. Let

$$\hat{\theta} = \rho\hat{p}/[1-\rho(1-\hat{p})] \tag{20}$$

and note that, $0 < \hat{\theta} < 1$.

The additional part of the sufficient condition is

(C2) $$A_\alpha(\hat{\theta}, N) > \max[0, -c_{\alpha 0}/(1-\rho)].$$

Proposition 4. *Let (s, p, t, v) be a restricted non-monetary steady state. If conditions* (C1) *and* (C2) *hold, then $v_{\alpha i} > 0$ for all (α, i) with $i \neq 0$ (implying that no one wishes to dispose of a good).*

Proof. We do this proof also in steps.

Step 1. $p_{i,i+1} \geqq \hat{p}$.

This follows from (2b), part (b) of Proposition 2 and Step 3 of Proposition 3, since $p_{i,i+1} = a_{i,i+1} + a_{ii} \geqq 0 + \hat{p}$.

Step 2. For any (i, j) with $i, j \neq 0$, and $i \neq j$ we can construct a loop of the form $p_{ij}, p_{j,j+1}, p_{j+1,j+2}, \ldots, p_{i-1,i}$ where each p (except possibly p_{ij}) is at least as large as $\hat{p}$. Therefore, by repeating the argument of Step 3 in Proposition 3 with $\hat{\theta}$ in place of $\bar{\theta}$ we obtain

$$v_{ij} \geqq A_i(\hat{\theta}, N). \tag{21}$$

We also have that (21) holds for $j = i$. Therefore (C2) implies the result. ‖

Proposition 5. *Let (s, p, t, v) be a restricted non-monetary steady state. If conditions* (C1) *and* (C2) *hold then $s_{\alpha i}^0 = 0$ uniquely attains the maximum in* (3a) *for $j = 0$.*

Proof. It is sufficient to show that $v_{\alpha i} > v_{\alpha 0}$ for all (α, i) with $i \neq 0$. Since the right-hand side of (C2) is $v_{\alpha 0}$ (see (6c)), (21) gives the result. ‖

Theorem 1. *There exists a non-empty open set of parameter values for which there exist non-monetary steady states with positive consumption.*

Proof. By Proposition (2), (3), (4), and (5), any restricted non-monetary steady state for parameters satisfying conditions (C1) and (C2) is a non-monetary steady state with positive consumption. Note that $u_\alpha > 0$ for all α and $c_{\alpha i} = 0$ for all (α, i) satisfy conditions (C1) and (C2). By continuity, there is an ε-neighbourhood of parameter values around the above values which also satisfy (C1) and (C2).[5] ‖

VI. SOME PROPERTIES OF NON-MONETARY STEADY STATES WITH POSITIVE CONSUMPTION

Here we show that Theorem 1 steady states generally display non-trivial trading patterns.

Proposition 6. *If $N > 2$, then in a Theorem* 1 *steady state some types trade for other than their consumption goods.*

Proof. From (16) and Proposition 3, we have that for each i and some $j \neq 0$, $p_{ij} > 0$ and $v_{ij} > -c_{ij}/(1-\rho)$. That is, type i is trading from position j with positive probability and hence is also trading into that position with positive probability. If $j \neq i+1$ then type i is trading for other than his consumption good. So, suppose that $j = i+1$. If i trades for good i, then $p_{\beta i} > 0$ for some type $\beta \neq i$. Now, either $\beta \neq i+1$ or $\beta = i+1$. In the former case β trades for other than β's consumption good. In the latter case β was holding other than β's production good, and so had to trade for it. ‖

Next we show that if there is a good that is least costly to store for all agents, then there is a steady state in which everyone accepts it.

Proposition 7. *If there exists a good, without loss of generality call it good* 1, *that is a least costly-to-store good for all agents and if conditions* (C1) *and* (C2) *hold, then there exists a non-monetary steady state with $s^1_{\alpha i} = 1$, $i \neq \alpha$.*

Proof. The idea of the proof is to restrict agents' strategies so that they always trade for good 1 and then show that this is not binding. It is straightforward to modify (6a) and (7a) accordingly, to find a fixed point as in Proposition 1, and to show that Proposition 2 continues to hold. For this more restricted fixed point, we let $q_{\alpha i}$ be the probability of consuming for type α holding good i. Obviously, $q_{\alpha i} \leqq q_{\alpha 1}$. Since type 1 always accepts good 1, we let $\alpha \neq 1$. Let $i^* \neq 0$, α be such that $v_{\alpha i^*} \geqq v_{\alpha i}$ for all $i \neq 0, \alpha$. It follows that

$$v_{\alpha i^*} = \rho[q_{\alpha i^*} v_{\alpha\alpha} + (1 - q_{\alpha i^*}) v_{\alpha i^*}] - c_{\alpha i^*}$$

$$v_{\alpha 1} \geqq \rho[q_{\alpha 1} v_{\alpha\alpha} + (1 - q_{\alpha 1}) v_{\alpha 1}] - c_{\alpha 1}.$$

5. In an earlier version of this paper, this result was established indirectly by using the upper hemi-continuity of the fixed point (see Proposition 1) in the parameters. The present approach involves much explicit algebra but seems preferable since it is constructive and leads to explicit sufficient conditions in terms of the parameters. These can be used to construct valid examples without relying on hit or miss calculations.

That is,

$$(1-\rho)v_{\alpha i^*} = \rho q_{\alpha i^*}(v_{\alpha\alpha} - v_{\alpha i^*}) - c_{\alpha i^*}$$

$$(1-\rho)v_{\alpha 1} \geqq \rho q_{\alpha 1}(v_{\alpha\alpha} - v_{\alpha 1}) - c_{\alpha 1} \geqq \rho q_{\alpha 1}(v_{\alpha\alpha} - v_{\alpha i^*}) - c_{\alpha 1}.$$

Then, since $v_{\alpha\alpha} - v_{\alpha i^*} > 0$, $-c_{\alpha 1} \geqq -c_{\alpha i^*}$, and $q_{\alpha 1} \geqq q_{\alpha i^*}$, it follows that $v_{\alpha 1} \geqq v_{\alpha i^*}$. This implies $v_{\alpha 1} = v_{\alpha i^*}$, which completes the proof. ‖

Note that the conclusion of Proposition 7 cannot be strengthened to say that any non-monetary steady state is such that everyone accepts the least costly good; Kiyotaki-Wright have a counter-example (see their Figure 6 and Theorem 2).

We now extend the above proposition to monetary steady states. Specifically, we wish to show that if money is a least costly-to-store object for all agents and if slightly modified versions of (C1) and (C2) hold, then there always exists a monetary steady state with positive consumption in which everyone accepts money.

VII. EXISTENCE OF MONETARY STEADY STATES WITH POSITIVE CONSUMPTION

Since the method here is very similar to that for non-monetary steady states, we only outline the proof. We restrict agents to always accept money and then give sufficient conditions for this not to be binding. The spaces $\bar{S}$ and $\bar{P}$ are replaced by the following:

$$\hat{S} = \{s \in I^{(N+1)(N+1)N} \mid s^0_{\alpha i} = 1,\ i \neq \alpha\},$$

$$\hat{P} = \{p \in I^{N(N+1)} \mid p_{\alpha i} \text{ satisfy (1) and } \textstyle\sum_\alpha p_{\alpha 0} = m\}.$$

The mapping $\mu : \hat{S} \times \hat{P} \times V \to V$ is defined by:

$$r^{\beta j}_{\alpha i} = \begin{cases} \text{right-hand side of (3a)}, & j \neq 0, \text{ or } (i,j) = (\alpha, 0), \\ s^i_{\beta j} v_{\alpha j} + (1 - s^i_{\beta j}) v_{\alpha i} & j = 0,\ i \neq \alpha, \end{cases} \tag{22a}$$

$$w_{\alpha i} = \textstyle\sum_\beta \sum_j p_{\beta j} r^{\beta j}_{\alpha i}, \tag{22b}$$

$$(\mu(s,p,v))_{\alpha i} = \begin{cases} \text{right-hand side of (3c)}, & i \neq 0, \\ \rho w_{\alpha i} - c_{\alpha i}, & i = 0. \end{cases} \tag{22c}$$

The dominant trading strategy correspondences $\sigma^j_{\alpha i}$ are defined as in (7a) for $j \neq 0$, and (7b) is modified to the following:

$$\sigma^0_{\alpha i} = \{1\},\ i \neq \alpha. \tag{23}$$

The consumption/storage strategy correspondences τ_α are defined exactly as before in (8). Finally, $\pi_{\alpha i}$ is taken to be the right-hand side of (2b). The argument of Proposition 1 can be applied to the modified mapping to produce a fixed point.

The modified version of (C1), labeled (C1)′, is obtained from (C1) by replacing C_α with $\hat{C}_\alpha$ the max taken over all objects, by replacing c_α with $\hat{c}_\alpha$, the min taken over all objects, and by replacing $\bar{p}$ in (12) with $\bar{p}' = \min(\bar{p}, 1/N^2)$. The (C1)′ condition is written

(C1)′ $$A'_\alpha(\bar{\theta}, N) > \max(0, -\hat{c}_\alpha)/(1-\rho)$$

where $A'_\alpha(\bar{\theta}, N)$ is $A_\alpha(\bar{\theta}, N)$ with $\hat{C}_\alpha$ in place of C_α.

We now add the assumption that money is a least costly-to-store object for all agents. That is,

(C3) $$\hat{c}_\alpha = c_{\alpha 0}.$$

With this assumption, $v_{\alpha i} \geqq -c_{\alpha i}/(1-\rho)$ continues to hold even though agents with goods are being forced to accept money, because taking and holding money forever only lowers storage costs. Proposition 2 can now be easily extended to show that agents holding any object will always want to trade for and consume their consumption good. Then we can use (C3) and modify Proposition 7 to show that the restriction that agents always take money is not binding. It only remains to show that in this steady state there is positive consumption and that the free disposal condition (4f) is satisfied.

It is easy to modify (16) to obtain

$$a_{\alpha\alpha} = \textstyle\sum_i p_{\alpha i}[(1-\rho)v_{\alpha i} + c_{\alpha i}]/\rho(u_\alpha + c_{\alpha,\alpha+1}).$$

We now indicate the steps involved in extending Proposition 3 (under conditions (C1)′ and (C3)) to the present case.

Let i be such that $p_{i0} \leqq p_{\alpha 0}$ for all α. Then $p_{i0} \leqq m/N$. This implies that there exists $j \neq 0, i$ such that $p_{ij} \geqq \bar{p}'$ (see (10)). Let $k \neq j$ be such that $p_{jk} \geqq \bar{p}'$. There are two possibilities for k, $k=0$, $k \neq 0$.

$k=0$: Here type j gets to consume with probability no less than p_{ij} because type i accepts money. This implies $v_{jk} \geqq \rho[p_{ij}v_{jj} + (1-p_{ij})v_{jk}] - c_{jk}$ or $v_{jk} \geqq \theta_{ij}v_{jj} + (1-\theta_{ij})c_{jk}/(1-\rho)$. Using $\theta_{ij} \geqq \bar{\theta}$ and $v_{jj} \geqq A_j'(\bar{\theta}, 0))$, this implies as in the derivation of (18), that

$$v_{jk} \geqq A_j'(\bar{\theta}, 1) \quad \text{for some } (j, k) \text{ for which } p_{jk} \geqq \bar{p}'. \tag{24}$$

This permits the derivation of a positive lower bound for a_{jj} and hence, for $a_{\alpha\alpha}$ for all α, as in Proposition 3.

$k \neq 0$: Here choose l such that $p_{kl} \geqq \bar{p}'$. If $l=0$, then the previous argument is used to obtain (24) for v_{kl}. If $l \neq 0$, then continue the chain of p's in the same manner.

Eventually, either we find a loop of the sort in Step 1 of Proposition 3 or we conclude that (24) holds. In the former case, the argument in Steps 2 and 3 of Proposition 3 can be used to reach the desired conclusion.

We can now easily extend Proposition 4 to show that the free disposal condition is satisfied. For this, condition (C2) is modified to the following

(C2)′ $$A_\alpha'(\hat{\theta}, N) > 0.$$

We can now show that under conditions (C1)′, (C3), and (C2)′, $v_{\alpha i} \geqq 0$. Clearly, the argument of Steps 1 and 2 of Proposition 4 implies that $v_{\alpha i} \geqq 0$, $i \neq 0$. Since the constraint that agents always accept money is not binding, $v_{\alpha 0} \geqq v_{\alpha i} \geqq 0$, $i \neq \alpha$. Thus the following result has been established.

Theorem 2. *If conditions* (C1)′, (C3), *and* (C2)′ *hold, then there always exists a monetary steady state with positive consumption in which everyone accepts money.*

VIII. (INCENTIVE) FEASIBLE, OPTIMAL AND EQUILIBRIUM ALLOCATIONS

It is natural to ask if all or any of the steady states we have shown to exist are in some sense optimal. Kiyotaki-Wright demonstrate that there are parameters for which any pure-strategy steady state is Pareto-dominated (in a sense we need not go into) by a

simple scheme called always-trade. In this scheme any two agents who meet exchange whatever they have. Moreover, as Kiyotaki-Wright suggest, always-trade is itself almost never a steady state. (In the Appendix we show that always-trade is a steady state if and only if all goods are equally costly.) As they also note (p. 948), "Unfortunately, these (always-trade) strategies are not implementable; in a given match, trade may not be in an individual's self-interest, and he has incentive to reject offers of high-storage-cost goods . . . " In what follows we elaborate on these issues.

We will show that there is a reasonable concept of feasibility, one which includes incentive compatibility and individual rationality restrictions, that implies that the set of steady-state allocations and a comparable set of stationary (in the sense of constant-over-time p and v) feasible allocations coincide. According to this concept of feasibility, always-trade is not feasible and, necessarily, at least one steady state is optimal.[6]

As a way of introducing the feasibility concept, consider a planning or mechanism design problem at date 0. Whatever the objective, the following three constraints define our feasibility concept: (i) the obvious physical resource constraint in each pairwise meeting; (ii) each agent's trading history is private information; (iii) at each date, each agent is free to keep his endowment or to dispose of it (sequential individual rationality). This concept of feasibility has as an obvious consequence the following proposition.

Proposition 8. *The set of positive consumption stationary allocations that satisfy* (i)-(iii) *and are consistent with* (a) *no discrimination, and* (b) *no gift giving is the same as the set of steady-state allocations.*[7]

That the set of steady-state allocations is a subset of such stationary allocations is clear. That any such stationary allocation is a steady-state allocation follows from the fact that individual rationality in the context of pairwise meetings with each agent holding only one indivisible object is equivalent to optimization as defined in (3a).

We now show that always-trade is not feasible (does not satisfy (i)-(iii)) unless all objects are equally costly to store. Suppose to the contrary that there is a mechanism satisfying (i)-(iii) that gives rise to always-trade and that person α with object i meets someone with object j, which is not α's consumption good and which is more costly to store for α than is object i. By (ii), future allocations are independent of current and past trades. Therefore, under always-trade the agent's future trading opportunities and, hence, future expected discounted utility are the same no matter which object he carries over. Since object i is less costly to store than object j, agent α would be better off not trading. Therefore, always-trade violates condition (iii).

Note that if the planner could keep track of individual histories, then always-trade could be feasible. In effect, agents could be "punished" in the future for sending "wrong" messages. For example, the planner could set up a game with the following rules: (1) In the first round of pairwise meetings, people trade if everyone sends a "yes" message;

6. A steady-state allocation is a (p, v) for which there exists an (s, t) such that (s, p, t, v) is a steady state. Comparable stationary allocations are defined below.

7. A stationary allocation satisfying (b) is a (p, v) for which there exist probabilities of trade when (α, i) meets (β, j), denoted $g_{\alpha i}^{\beta j}$, and t (probabilities of consuming) such that (p, t, g) satisfies (2) when $s_{\alpha i}^{j} s_{\beta j}^{i}$ is replaced by $g_{\alpha i}^{\beta j}$ and (p, v, t, g) satisfies (3) when the maximizations are dropped and $xs_{\beta j}^{i}$ in (3a) and z in (3c) are replaced by $g_{\alpha i}^{\beta j}$ and t_α, respectively. Such a (p, v) also satisfies (a) if there exists a vector y in $I^{(N+1)(N+1)N}$ such that $g_{\alpha i}^{\beta j} = y_{\alpha i}^{j} y_{\beta j}^{i}$ for all (α, i) and (β, j).

if someone sends a "no" message, then he and his partner do not trade. (2) In subsequent rounds, there is no trade in a pairwise meeting if one of the agents had sent a "no" message at any time previously. Otherwise, rule (1) is used. These rules do not violate (iii).

According to our notion of feasibility, there always exists a steady state which is optimal.[8] We also know that not any steady state is optimal. Aiyagari-Wallace (1989) display examples in which monetary equilibria are Pareto Superior to non-monetary equilibria, in much the same way that monetary equilibria can be Pareto Superior to non-monetary equilibria in overlapping-generations models.

We also know that the optimal allocation satisfying the Proposition 8 conditions can be one that displays the somewhat paradoxical feature that trade does not occur in some pairwise meetings in which one agent wants to trade and the other is indifferent. Our equilibrium concept permits this to happen and all the steady states for the class of examples referred to in Footnote 1 have this feature. Therefore, Proposition 8 implies that one of these is optimal. This class of examples also implies that condition (iii) cannot be strengthened to require group rationality—the requirement that no coalition of agents (in this model there are only coalitions of two agents) be able to improve by trading among themselves—without making empty the set of feasible allocations satisfying the conditions in Proposition 8. Further, (iii) cannot be strengthened even if strategies are permitted to be discriminatory. This follows from the fact that when $N=3$ (and there is no fiat money) there is no loss of generality in not allowing such strategies. In that case, each type α can meet only two other types. If they have the same good, which is the only situation that gives scope to discriminatory policies, then that good must be good $i=\alpha$ since it cannot be the consumption good of either of the other types. Hence, type α wants to trade for it.

The equivalence between feasible and equilibrium allocations is more general than Proposition 8 may suggest. We have stated the equivalence between feasible and equilibrium allocations only for the very restrictive kind of equilibrium (steady-state) we have defined. If the definition of equilibrium is broadened, then the class of feasible allocations can be broadened comparably. Among the ways to broaden the equilibrium concept are the following: define equilibrium paths from arbitrary initial conditions (see Aiyagari-Wallace (1989)); allow for discriminatory strategies; allow for the occurrence of free disposal and zero consumption. For any of the equilibrium concepts, there will be equivalence between equilibrium and comparably chosen feasible allocations that satisfy (i)-(iii) and no gift-giving.

The no gift-giving requirement cannot be dropped without changing the model. There can, in fact, be steady states with gift giving and gift giving can be feasible (satisfy (i)-(iii)). An example of this, from Aiyagari-Wallace (1989), is an $N=2$ mixed-strategy steady state in which agents with goods are indifferent between accepting and rejecting fiat money which is costless to store. Agents holding fiat money can be interpreted as holding nothing and an agent giving up another object to get fiat money can be interpreted as giving a gift. However, gift-giving is not robust to the following small and otherwise innocuous change in the model: allow agents to derive at least some utility from consuming any good. For such a model, the equivalence between equilibrium and feasible allocations holds without any proviso about gift-giving because gift-giving is neither feasible nor an equilibrium strategy in such a model.

8. The set of steady states is closed since if (s_n, p_n, t_n, v_n) is a steady state for each n and $(s_n, p_n, t_n, v_n) \to (s, p, t, v)$, then (s, p, t, v) is a steady state. Since the set of admissible (s, p, t, v) is compact, it follows that the set of steady states is compact. Therefore, any continuous welfare criterion will imply that an optimal steady state exists.

APPENDIX

1.

Here we fill in the details involved in deriving equation (15).

Note that, by part (b) of Proposition 2, we can set $t_\alpha = 1$ in (2b) which implies that $p_{\alpha\alpha} = 0$ and that $p_{\alpha,\alpha+1} = a_{\alpha\alpha} + a_{\alpha,\alpha+1}$. Then we can use equations (3c) and (3b) to obtain

$$\sum_{i\neq 0} p_{\alpha i} v_{\alpha i} = -\sum_{i\neq 0} p_{\alpha i} c_{\alpha i} + \rho \sum_{i\neq 0} \sum_\beta \sum_j p_{\alpha i} p_{\beta j} r_{\alpha i}^{\beta j}. \tag{A1}$$

Now,

$$\begin{aligned}
\sum_i \sum_\beta \sum_j p_{\alpha i} p_{\beta j} r_{\alpha i}^{\beta j} &= \sum_i \sum_\beta p_{\alpha i} p_{\beta i} r_{\alpha i}^{\beta i} + \sum_i \sum_\beta \sum_{j\neq i} p_{\alpha i} p_{\beta j} r_{\alpha i}^{\beta j} \\
&= \sum_i \sum_\beta p_{\alpha i} p_{\beta i} v_{\alpha i} + \sum_i \sum_\beta \sum_{j\neq i} p_{\alpha i} p_{\beta j} [s_{\alpha i}^j s_{\beta j}^i v_{\alpha j} + (1 - s_{\alpha i}^j s_{\beta j}^i) v_{\alpha i}] \\
&= \sum_i v_{\alpha i} [\sum_\beta p_{\alpha i} p_{\beta i} + \sum_\beta \sum_{j\neq i} p_{\alpha i} p_{\beta j} (1 - s_{\alpha i}^j s_{\beta j}^i)] + \sum_i \sum_\beta \sum_{j\neq i} p_{\alpha i} p_{\beta j} s_{\alpha i}^j s_{\beta j}^i v_{\alpha j} \\
&= \sum_i v_{\alpha i} [a_{\alpha i} - \sum_\beta \sum_{j\neq i} p_{\alpha j} p_{\beta i} s_{\alpha j}^i s_{\beta i}^j] + \sum_i \sum_\beta \sum_{j\neq i} p_{\alpha i} p_{\beta j} s_{\alpha i}^j s_{\beta j}^i v_{\alpha j} \quad \text{(using (2a))} \\
&= \sum_i a_{\alpha i} v_{\alpha i} - \sum_i \sum_\beta \sum_j p_{\alpha j} p_{\beta i} s_{\alpha j}^i s_{\beta i}^j v_{\alpha i} + \sum_i \sum_\beta p_{\alpha i} p_{\beta i} s_{\alpha i}^i s_{\beta i}^i v_{\alpha i} \\
&\quad + \sum_i \sum_\beta \sum_j p_{\alpha i} p_{\beta j} s_{\alpha i}^j s_{\beta j}^i v_{\alpha j} - \sum_i \sum_\beta p_{\alpha i} p_{\beta i} s_{\alpha i}^i s_{\beta i}^i v_{\alpha i} \\
&= \sum_i a_{\alpha i} v_{\alpha i} \\
&= \sum_{i\neq\alpha, i\neq\alpha+1, i\neq 0} a_{\alpha i} v_{\alpha i} + a_{\alpha\alpha} v_{\alpha\alpha} + a_{\alpha,\alpha+1} v_{\alpha,\alpha+1} + a_{\alpha 0} v_{\alpha 0} \\
&= \sum_{i\neq\alpha, i\neq\alpha+1, i\neq 0} a_{\alpha i} v_{\alpha i} + a_{\alpha\alpha}(u_\alpha + c_{\alpha,\alpha+1} + v_{\alpha,\alpha+1}) + a_{\alpha,\alpha+1} v_{\alpha,\alpha+1} + a_{\alpha 0} v_{\alpha 0} \\
&= \sum_{i\neq 0} p_{\alpha i} v_{\alpha i} + a_{\alpha\alpha}(u_\alpha + c_{\alpha,\alpha+1}) + p_{\alpha 0} v_{\alpha 0} \quad \text{(using (2))}.
\end{aligned}$$

Therefore,

$$\sum_{i\neq 0} \sum_\beta \sum_j p_{\alpha i} p_{\beta j} r_{\alpha i}^{\beta j} = \sum_{i\neq 0} p_{\alpha i} v_{\alpha i} + a_{\alpha\alpha}(u_\alpha + c_{\alpha,\alpha+1}).$$

Substituting the right-hand side into (A1) and re-arranging yields (15).

2.

Proposition 9. (a) *There is a Theorem* 1 *steady state with* $s_{\alpha i}^j = 1$ *for all* (α, i, j) *with* $\alpha \neq i$; $j \neq 0$, *if and only if* $c_{\alpha i}$ *(for* $i \neq \alpha, 0$*) is independent of* i. (b) *There is a Theorem* 2 *steady state with* $s_{\alpha i}^j = 1$ *for all* (α, i, j) *with* $\alpha \neq i$, *if and only if* $c_{\alpha i}$ *(for* $\alpha \neq i$*) is independent of* i.

Proof. We only give the proof for Theorem 1 steady states. The proof for Theorem 2 steady states is similar.

(i) *Sufficiency*: Let $c_\alpha^* = c_{\alpha i}$ for $i \neq \alpha, 0$. It can be verified that the following is a steady state: $s_{\alpha i}^j = 1$ for all (α, i, j), $\alpha \neq i, j \neq 0$; $t_\alpha = 1$ for all α;

$$p_{\alpha i} = \begin{cases} 2(1-m)/N^2 & \text{if } i = \alpha + 1, \\ (1-m)/N^2 & \text{if } i \neq \alpha + 1;\ i \neq 0, \end{cases}$$

$$r_{\alpha i}^{\beta j} = \begin{cases} \rho w_{\alpha j} - c_\alpha^* & \text{if } j \neq \alpha, j \neq 0, \\ u_\alpha + \rho w_{\alpha,\alpha+1} & \text{if } j = \alpha; \end{cases}$$

and

$$w_{\alpha i} = (1-m)[u_\alpha - c_\alpha^*(N-1)]/(1-\rho)N, \qquad \alpha \neq i,\ i \neq 0.$$

(ii) *Necessity.* For $i, j \neq \alpha, 0$, $s_{\alpha i}^j = s_{\alpha j}^i = 1$ and (3c) imply

$$\rho w_{\alpha j} - c_{\alpha j} = \rho w_{\alpha i} - c_{\alpha i} \equiv \lambda_\alpha. \tag{A2}$$

Then (3b) implies that for $k \neq 0$,

$$\begin{aligned}
w_{\alpha k} &= \sum_\beta [p_{\beta\alpha}(u_\alpha + \rho w_{\alpha,\alpha+1}) + \sum_{l\neq\alpha} p_{\beta l} \lambda_\alpha] \\
&= \sum_\beta [p_{\beta\alpha}(u_\alpha + c_{\alpha,\alpha+1} + \lambda_\alpha) + \lambda_\alpha (1/N - p_{\beta\alpha})] \\
&= [(u_\alpha + c_{\alpha,\alpha+1}) \sum_\beta p_{\beta\alpha} + \lambda_\alpha].
\end{aligned}$$

Hence, $w_{\alpha k}$ does not depend on k for $k \neq 0$. Then, by (A2), $c_{\alpha j}$ does not depend on j for $j \neq \alpha, 0$. ‖

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