ON THE NUMERICAL CONSTRUCTION OF HYPERBOLIC STRUCTURES FOR COMPLEX DYNAMICAL SYSTEMS

Jennifer Suzanne Lynch Hruska, Ph.D.

Cornell University 2002

Our main interest is using a computer to rigorously study ϵ-pseudo orbits for polynomial diffeomorphisms of $\mathbb{C}^{\mathbf{2}}$. Periodic ϵ-pseudo orbits form the *ϵ-chain recurrent set*, $\mathcal{R}_\epsilon$. The intersection $\cap_{\epsilon>0}\mathcal{R}_\epsilon$ is the chain recurrent set, $\mathcal{R}$. This set is of fundamental importance in dynamical systems.

Due to the theoretical and practical difficulties involved in the study of $\mathbb{C}^{\mathbf{2}}$, computers will presumably play a role in such efforts. Our aim is to use computers not only for inspiration, but to perform rigorous mathematical proofs.

In this dissertation, we develop a computer program, called *Hypatia*, which locates $\mathcal{R}_\epsilon$, sorts points into components according to their ϵ-dynamics, and investigates the property of *hyperbolicity* on $\mathcal{R}_\epsilon$. The output is either "yes", in which case the computation *proves* hyperbolicity, or "not for this ϵ", in which case information is provided on numerical or dynamical obstructions.

A diffeomorphism f is *hyperbolic on a set* X if for each x there is a splitting of the tangent bundle of x into an *unstable* and a *stable* direction, with the unstable (stable) direction expanded by f (f^{-1}). A diffeomorphism is *hyperbolic* if it is hyperbolic on its chain recurrent set.

Hyperbolicity is an interesting property for several reasons. Hyperbolic diffeomorphisms exhibit *shadowing* on $\mathcal{R}$, *i.e.*, ϵ-pseudo orbits are δ-close to true orbits.

Thus they can be understood using combinatorial models. Shadowing also implies *structural stablity*, *i.e.*, in a neighborhood in parameter space the behavior is constant. These properties make hyperbolic diffeomorphisms amenable to computer investigation via ϵ-pseudo orbits.

We first discuss *Hypatia* for polynomial maps of $\mathbb{C}$. We then extend to polynomial diffeomorphisms of $\mathbb{C}^{\mathbf{2}}$. In particular, we examine the class of Hénon diffeomorphisms, given by

$$H_{a,c}\colon (x,y) \to (x^2 + c - ay, x).$$

This is a large class of diffeomorphisms which provide a good starting point for understanding polynomial diffeomorphisms of $\mathbb{C}^{\mathbf{2}}$. However, basic questions about the complex Hénon family remain unanswered.

In this work, we describe some Hénon diffeomorphisms for which *Hypatia* verifies hyperbolicity, and the obstructions found in testing hyperbolicity of other examples.