Baby Julia sets and combinatorial models

(some work in progress)

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Thank you Han and Erlend for the invitation!

Goal/Overview

Explore and make combinatorial models for the dynamics on the Julia set of different types of hyperbolic (or Axiom A) rational and/or polynomial maps of $\mathbb{C}, \hat{\mathbb{C}}, \mathbb{C}^2, \mathbb{CP}^n$, etc.

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Ultimate goal to show how a complicated map ‘contains’ dynamics (J) of simpler maps by comparing IMG models.

Expanding Multivalued Dynamical Systems (MDS)

Let $\mathcal{X} = (\iota, f)\colon X^1 \to X^0$ be an expanding multi-valued dynamical system, such that X^0 consists of finitely many arcwise connected components (Ishii-Smillie). E.g., f is a rational map on $\hat{\mathbb{C}}$ with finite postcritical set, P, X^0 is $\mathbb{C}$ minus a neighborhood of P, $X^1 = f^{-1}(X^0)$ and ι is inclusion.

Let $(\iota, f)\colon X^{n+1} \to X^n$ be the successive pullbacks, so points in X^n can be identified with length n orbits in X^0. Let $X^{+\infty}$ be the limiting space, points with infinite orbits in X^0, and $\hat{f}\colon X^{+\infty} \to X^{+\infty}$ the shift map, induced by f. (So $X^{+\infty}$ is J).

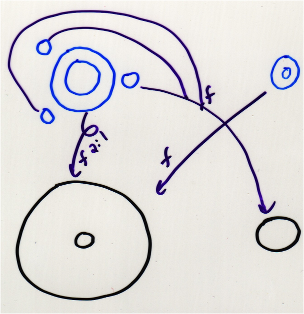
Theorem [Ishii-Smillie]: The topological conjugacy class of $\hat{f}$ is completely determined by homotopy information.

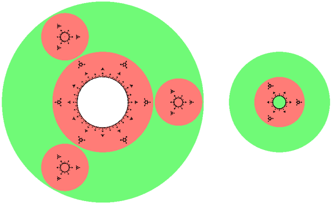
Wire Models

More precisely, If $(\iota_f, f)\colon X^1 \to X^0$ and $(\iota_g, g)\colon Y^1 \to Y^0$ are homotopy equivalent expanding systems (i.e., there are semi-conjugacies $h_k : X^k \to Y^k$ for $k = 0, 1$ (i.e., $\iota_g h_1 = h_0 \iota_f$ and $g h_1 = h_0 f$), and vice-versa), then the limiting systems $\hat{f}\colon X^\infty \to X^\infty$ and $\hat{g}\colon Y^\infty \to Y^\infty$ are topologically conjugate.

Thus one can describe $\hat{f}\colon X^\infty \to X^\infty$ (i.e., f on the Julia set) via a "wire model" $(\iota, f)\colon Y^1 \to Y^0$ which is homotopy equivalent to $(i, f)\colon X^1 \to X^0$.
One approach is to use an IMG type model....
(Nekrashevych, connected case; Ishii/Smillie, disconnected)

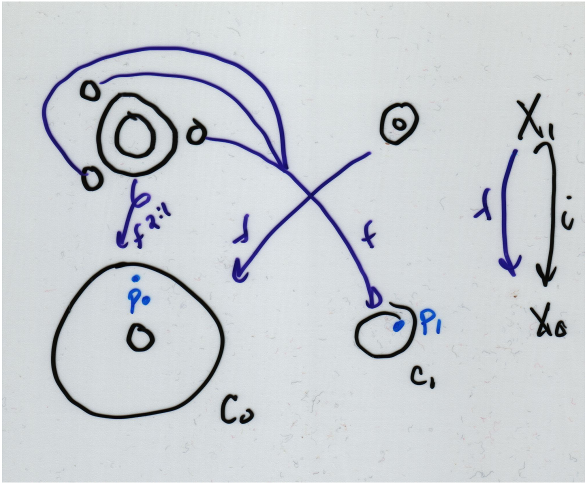
For example, based on a “fake” cubic polynomial with one critical point escaping and one fixed (left), we will derive the expanding system below:



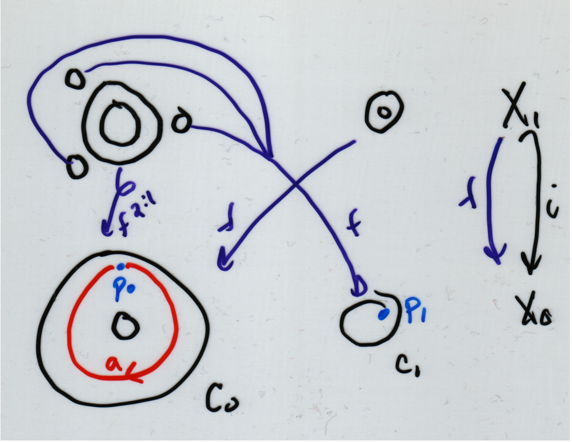


Construction

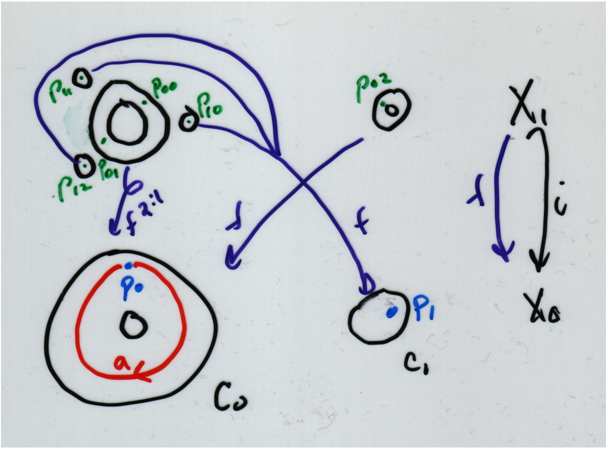
1. In each connected component C_k of X^0, choose a basepoint p_k.



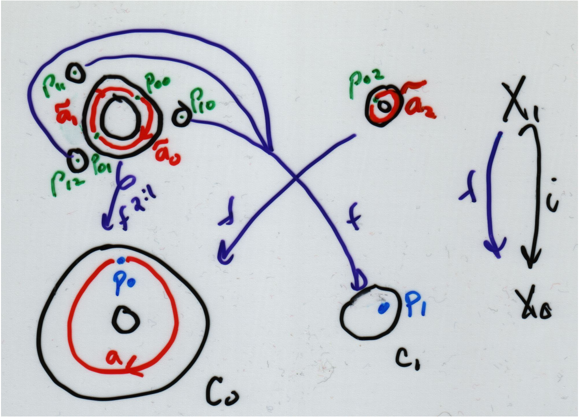
2. Choose and label generators for each $\pi_1(C_k, p_k)$, in X^0.



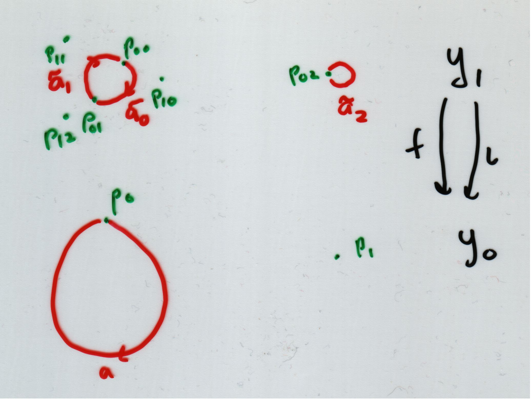
3. For each p_k in X^0, let $\{p_{km}\}$ in X_1 be all the preimages under f of p_k.



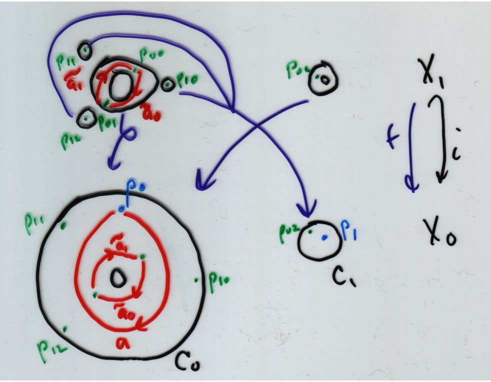
4. For each k, and each $v \in \pi_1(C_k, p_k)$, let $\tilde{v}_m = f^{-1}|_m(v)$ in X^1 be the lift of v based at p_{km}.



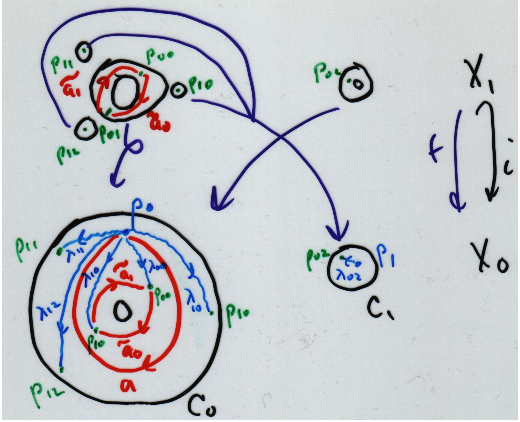
5. Now we begin defining $(\iota, f)\colon Y^1 \to Y^0$. Start with Y^0 as the chosen generators of $\pi_1(C_k, p_k)$, for all k, and Y^1 all the lifts under f of these generators. So elements of Y^n naturally are included in X^n, and the map f sends Y^1 to Y^0.



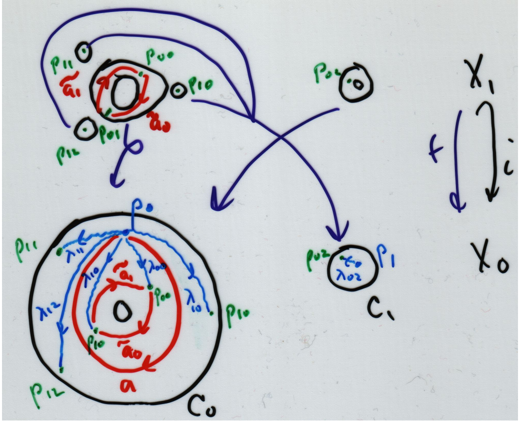
6-a But how do we define $\iota\colon Y^1 \to Y^0$? We want a map which is homotopy equivalent to the inclusion $i\colon X^1 \to X^0$, but it can't just be inclusion, after all, lifts of loops based at p_k are not necessarily loops and are based at the preimages of p_k. *(oops, forgot $\tilde{a}_2$ in pic.)*



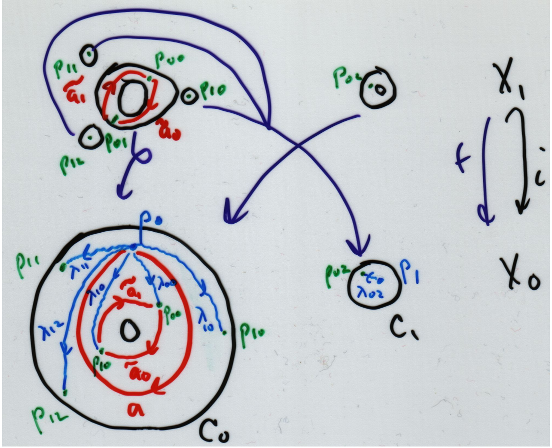
6-b Solution: If p_{km} is in C_j, choose a path λ_{km} in X^0 going from p_j to p_{km}, (so $f(p_{km})$ is not p_j, rather p_j is the basepoint in the component containing p_{km}). Now λ_{km} defines a homotopy from $i(Y_1)$ to Y_0.



7. That is, for each $\tilde{v}_m = f^{-1}|_m(v)$ in Y^1, if $\tilde{v}_m$ is a path from p_{km} to p_{kl}, then $\iota(\tilde{v}_m) := \bar{\lambda}_{kl} * \tilde{v}_m * \lambda_{km}$ is an element of $\pi_1(C_k, p_k)$, (going from p_k to p_{km}, then p_{km} to p_{kl}, then p_{kl} to p_k).



- E.g., $\iota(\tilde{a}_0) = \bar{\lambda}_{01} * \tilde{a}_0 * \lambda_{00} = a$, but $\iota(\tilde{a}_1) = \bar{\lambda}_{00} * \tilde{a}_1 * \lambda_{01} = e_0$. (Also, $\iota(\tilde{a}_2)$ is in C_1 so it's trivial, e_1.)



Iterated Monodromy Group (IMG)

We can store this information via a homomorphism $\phi = (\phi_1, \ldots, \phi_m)$ such that for each conn. comp. C_k of X^0, if U_j denotes a connected component of X^1, we have

$$\phi_k : \; \pi_1(C_k, p_k) \; \to \prod_j S(B_j) \wr \pi_1(U_j) = S(B_j) \ltimes \pi_1(U_j)^{d_k}.$$

where B_j is the set of d_j labels on preimages of baseponts in U_j, and $S(B)$ is the group of permutations.

Each map ϕ_j is a *wreath recursion*, and ϕ is defined on the free product of fundamental groups. The **Iterated Monodromy Group** is defined as the quotient of this free product of fundamental groups by the Kernal of ϕ.

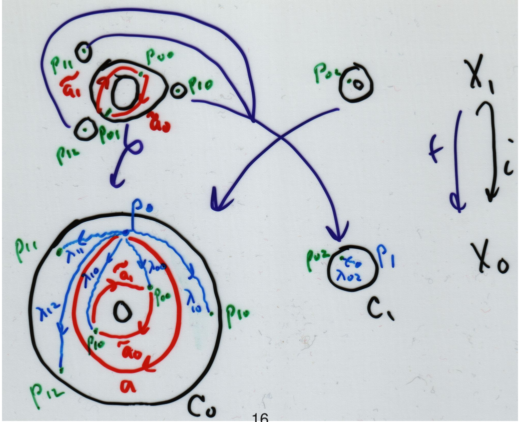
This homomorphism ϕ is determined by its action on generators, so a finite amount of info. encodes the topological conjugacy class: our combinatorial model. (Ishii)

E.g., in the toy cubic above,

$\phi(a) = [(p_{00}p_{01}) \cdot (a, e_0); (id)(e_1); \emptyset; \emptyset; \emptyset]$

$\phi(e_0) = [(id)(e_0, e_0); (id)(e_1); \emptyset; \emptyset; \emptyset]$

$\phi(e_1) = [\emptyset; \emptyset; (id)(e_0); (id)(e_0); (id)(e_0)]$.



Combining IMG's

An IMG for a disconnected system contains an underlying SSFT (subshift of finite type) representing the component graph, as well as IMG's of any "simpler" maps whose dynamics is contained in the whole system.

E.g., the above map contains an annulus mapped into itself as by $z \mapsto z^2$, and the IMG for z^2 is $\phi(a) = \sigma(a, e)$, which is equal to $\phi_1(a)$ above.

One goal of ours is to describe how the IMG's of certain polynomial skew products (and other maps) consist of combinations of SSFT and IMG's of simpler and/or one-dimensional maps...

Polynomial Skew Products of $\mathbb{C}^2$

- A *polynomial skew product of* $\mathbb{C}^2$ is a map of the form

$$f(z,w) = (p(z), q(z,w)),$$

with p, q polynomials same degree $d \geq 2$.

- f preserves vertical lines, or fibers, i.e.,
$f\colon \{z\} \times \mathbb{C} \mapsto \{p(z)\} \times \mathbb{C}$
$\Rightarrow f^n|_{\{z\}\times\mathbb{C}}$ is a composition of polynomial maps of $\mathbb{C}$ (at each iterate, the "fiber map" depends on the base point $p^n(z)$).

- Trivial examples are products $f(z,w) = (p(z), q(w))$, every fiber map same, e.g., $(z,w) \mapsto (z^2, w^2 - 6)$

Dynamics of $f(z, w) = (p(z), q(z, w))$

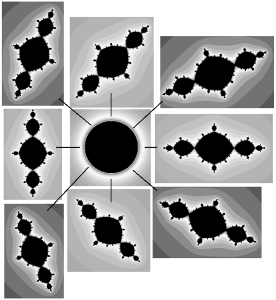
- $K_p, J_p, \ldots$ sets corresponding to "base" polynomial p
- K = points with bounded orbits (under f)
- Vertical Dynamics: set
  - $q_z(w) = q(z, w)$
  - $K_z = K \cap (\{z\} \times \mathbb{C})$
  - $J_z = \partial K_z$ and
  - $J_2 = \bigcup_{z \in J_p} J_z = \overline{\{ \text{ repelling periodic points } \}}$
- K, J_2 are totally invariant
  and $q_z(K_z) = K_{p(z)}$ and $q_z(J_z) = J_{p(z)}$
- For a product, $q_z(w) = q(w)$, so $K_z = K_q$ all z

Axiom A polyn. skew prod.

(1) products of hyperbolic polynomials: for $f(z,w) = (p(z), q(w))$, have $J_2 = J_p \times J_q$;

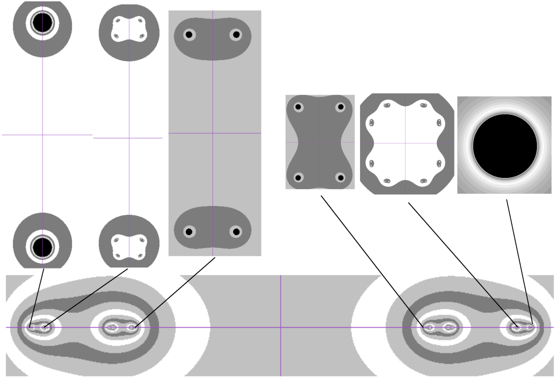
(2) small perturbations: fiber J's vary continuously

(3) hyperbolic polyn. twisted over S^1: $f(z,w) = (z^2, w^2 + az)$, where $w^2 + a$ is hyperbolic (with DeMarco and with Roeder).



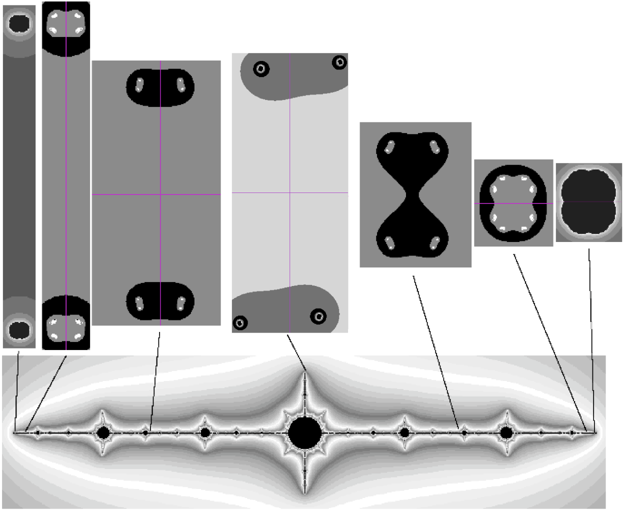
Here $a = -1$, map is semiconjugate to product $(z,w) \mapsto (z^2, w^2 - 1)$.

(4) J_p Cantor set, fibers vary: (gen. with DeMarco)
Let $f(z, w) = (z^2 - 6, w^2 + 3 - z)$ (Jonsson ex.), combines dynamics of w^2+ small with w^2+ big (in general, any two hyperbolic polyn. maps of $\mathbb{C}$):



We show slices of K. The lower figure is the z-plane, and above (from right to left) are the fibers: $z = 3$ (K_z a disk), 2.83 (cantor), 1.73, -1.73, (preimages of K_{-3}, 4 disks), -2.83, (cantor), and -3, (K_z two disks, mapped onto K_3).

(5) J_p aeroplane-like, fibers vary: (with DeMarco)



Shown are slices of K. The lower figure is the z-plane, and above (from right to left) are the fibers:

$z \approx 1.92$, with K_z a quasidisk; $z = 1.8, 1, 0, -1, -1.8$, each with K_z of empty interior; finally

$z \approx -1.92$, with K_z two quasidisks, mapped onto the rightmost fiber.

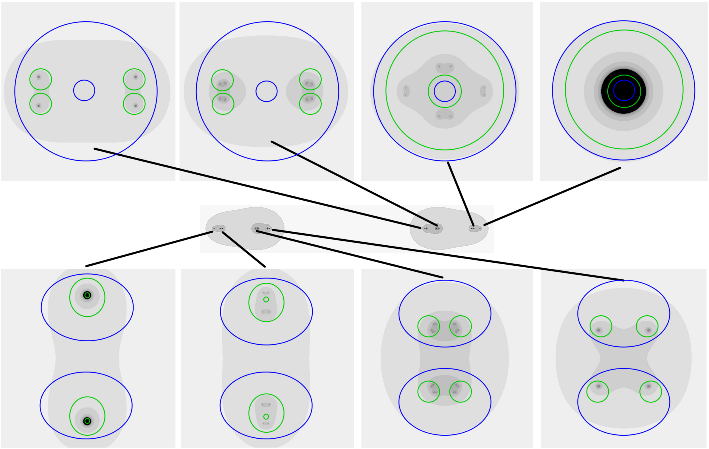
1-dimensional combinatorial description(s)?

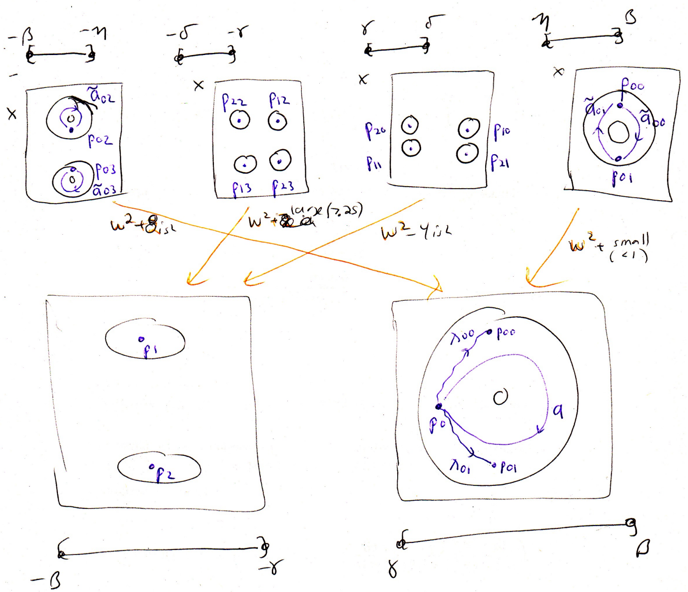
The Axiom A polynomial skew products above may be inherently two-dimensional, in that they are in different hyperbolic components from products, BUT these examples still appear to consist of some combination of hyperbolic polynomial maps of $\mathbb{C}$.

So one would hope to be able to describe the dynamics in part using the dynamics of the one-dimensional pieces....

We do so by comparing IMG wire models for polyn. skew prod. versus some polynomial maps of $\mathbb{C}$.

The IMG construction need not be restricted to $\mathbb{C}$. For example, can be carried out for the polynomial skew product of type (4) above: (X^0 in blue, X^1 in green)




x
$-\beta$
$-\eta$
$-\delta$
$-\gamma$
γ
δ
η
β
$\tilde{a}_{02}$
p_{02}
p_{03}
$\tilde{a}_{03}$
p_{22}
p_{12}
p_{13}
p_{23}
p_{20}
p_{10}
p_{11}
p_{21}
p_{00}
$\tilde{a}_{01}$
$\tilde{a}_{00}$
p_{01}
w^2+ small (<1)
p_1
p_2
λ_{00}
p_0
a
λ_{01}

The IMG is then:

$\phi(a) = [(p_{00}, p01)(e_0, a); (id)(e_1); (id)(e_2); \emptyset; \cdots; \emptyset]$
$\phi(e_0) = [(id)(e_0, e_0); (id)(e_1); (id)(e_2); \emptyset; \cdots; \emptyset]$
$\phi(e_1) = [\emptyset; \emptyset; \emptyset; (id)(e_0); (id)(e_0); \emptyset; \emptyset; (id)(e_1); (id)(e_2); \emptyset; \emptyset;]$
$\phi(e_2) = [\emptyset; \emptyset; \emptyset; \emptyset; \emptyset; (id)(e_0); (id)(e_0); \emptyset; \emptyset; (id)(e_1); (id)(e_2)].$

which note $\phi_1(a)$ contains the IMG for $z \mapsto z^2$,
$\phi(a) = \sigma \cdot (a, e)$,

and is related to the underlying dynamics of $z \mapsto z^2 - R$, which has $\phi(e_0) = [\emptyset; (id)(e_0); (id)(e_1); \emptyset], \phi(e_1) = [(id)(e_0); \emptyset; \emptyset; (id)(e_1)]$. (in a way which needs a bit more machinery to make precise, I'll omit details).

Now, one can re-arrange (i.e., homotope) the components of X^0 and X^1 into a common plane, by first contracting each component of X^0 in the z-plane to a point, ending up with a map similar looking to the toy cubic example presented above. Any choice results in a map conjugate on J.

Theorem

For any Axiom A polynomial skew product $f(z,w) = (p(z), q(z,w))$ such that J_p is a cantor set, there is an expanding multi-valued dynamical system $\mathcal{X} = (\iota, g, X^1, X^0)$ such that $X^1 \subset X^0 \subset \mathbb{C}$ and such that the limiting map $\hat{g} : X^\infty \to X^\infty$ is conjugate to f on its Julia set.

May not be realized by actual rational maps of $\hat{\mathbb{C}}$. One candidate family is Singularly Perturbed Rational Maps...

Singularly Perturbed Rational Maps (SPRM) of $\mathbb{C}$

Consider the family of rational maps

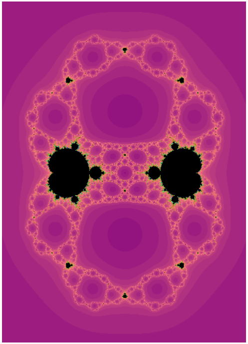
$$R_{n,c,a}(z) = z^n + c + \frac{a}{z^n},$$

where c, a are complex and $a \neq 0$, and $n \geq 2$ is an integer.

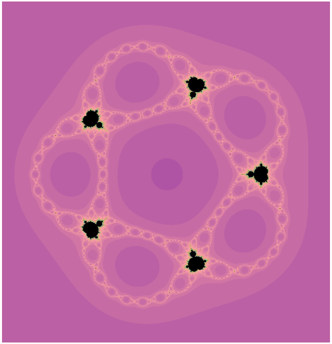
Degree $2n$; crit points $0, \infty, a^{1/2n}$; crit. values $c \pm 2\sqrt{a}$.

- J, the repellor, lives in an annulus
  - Case $c = 0$: one free critical orbit. (Devaney, et. al.)
  - Case $c \neq 0$: two free critical orbits (Devaney and Russell studied $R_{3,i,a}$ for a small).
- We'll fix n and c, then study the family as a varies.
- For a fixed n and c, set $M_{n,c}$ to be the set of a-values such that at least one critical orbit is bounded.

$M_{n,c}$ in $c = 0$ case



$M_{3,0}$



$M_{6,0}$

Outline of SPRM cases

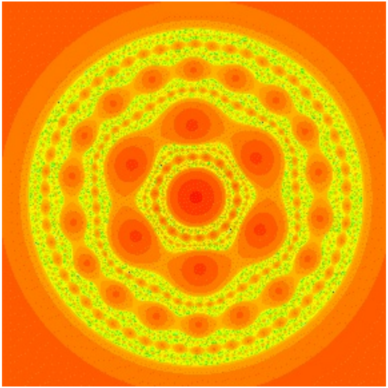
Try for $n = 3$, all cases $n \geq 3$ similar:

1. Case $c = 0$
   1. $a \notin M$, a small: McMullen domain, J disconn. but not cantor

McMullen domain

Proposition

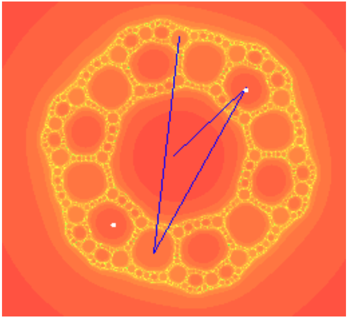
The rational map $R(z) = z^3 + a/z^3$ for small a is topologically conjugate, on its Julia set (which lives in $\mathbb{C}$), to the product $f(z, w) = (z^2 - R, w^3)$ for large R, on its Julia set (which lives in $\mathbb{C}^2$).



Outline of SPRM cases

Try for $n = 3$, all cases $n \geq 3$ similar:

1. Case $c = 0$
   1. $a \notin M$, a small: McMullen domain, J disconn. but not cantor
   2. $a \notin M$, a large: Sierpinski Hole, J conn.

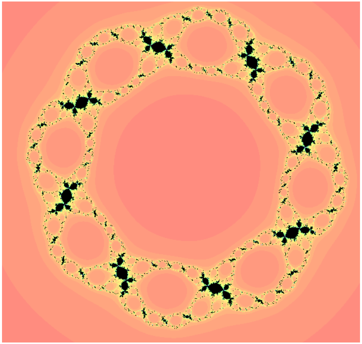


Is there a polyn. skew product (or other map of $\mathbb{C}^2$ conjugate to this (expand circles in normal direction)?

Outline of SPRM cases

Try for $n = 3$, all cases $n \geq 3$ similar:

1. Case $c = 0$
   1. $a \notin M$, a small: McMullen domain, J disconn. but not cantor
   2. $a \notin M$, a large: Sierpinski Hole, J conn
   3. $a \in M$: in a large baby M, J conn



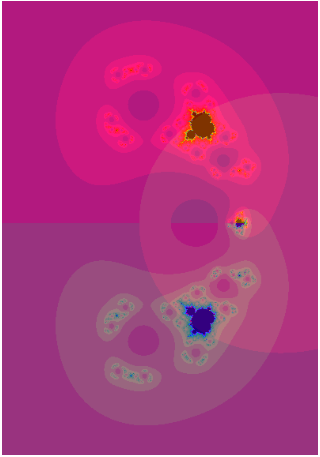
K for $n = 4, c = 0, a = 0. - 4222 + 0.147i$.

When $c = 0$, for a in M the Julia set contains baby Julia sets of one (when $c = 0$) quadratic polynomial. (IMG?)

Outline of SPRM cases

Try for $n = 3$, all cases $n \geq 3$ similar:

1. Case $c = 0$
   1. $a \notin M$, a small: McMullen domain, J disconn. but not cantor
   2. $a \notin M$, a large: Sierpinski Hole, J conn.
   3. $a \in M$: in a large baby M, J conn.
2. Case $c \neq 0$, and 'large', e.g. $c = i$.



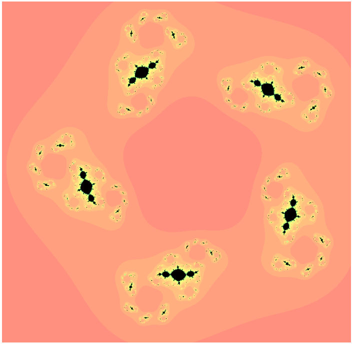
M for $n = 3, c = i$.

Coloration: One crit point gets red if escapes, other gets blue, both get black if bounded, then rgb values averaged.

Outline of SPRM cases

Try for $n = 3$, all cases $n \geq 3$ similar:

1. Case $c = 0$
   1. $a \notin M$, a small: McMullen domain, J disconn. but not cantor
   2. $a \notin M$, a large: Sierpinski Hole, J conn.
   3. $a \in M$: in a large baby M, J conn.
2. Case $c \neq 0$, and 'large', e.g. $c = i$.
   1. a large, one crit orbit bounded, one escapes



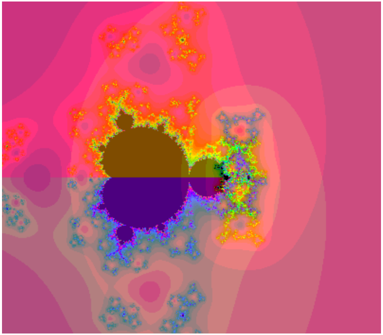
K for $n = 5, c = i, a = 0.0157 + 0.072i$.

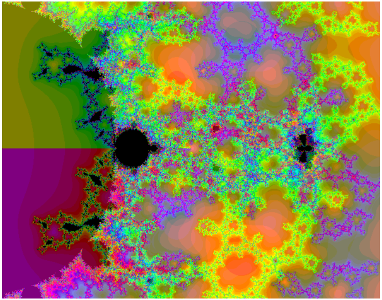
One critical orbit escapes, other is bounded. (IMG?)

Outline of SPRM cases

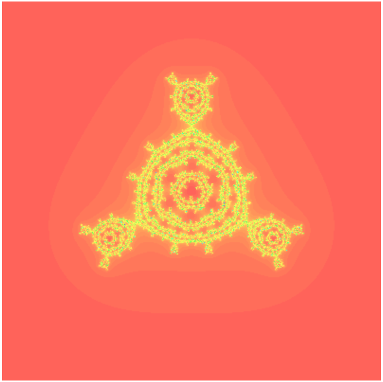
Try for $n = 3$, all cases $n \geq 3$ similar:

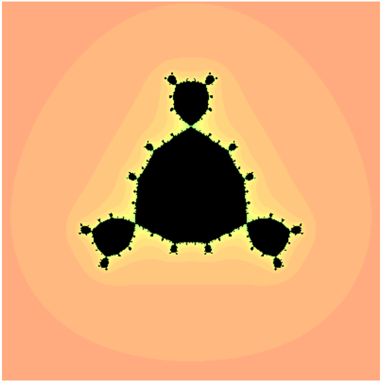
1. Case $c = 0$
   1. $a \notin M$, a small: McMullen domain, J disconn. but not cantor
   2. $a \notin M$, a large: Sierpinski Hole, J conn.
   3. $a \in M$: in a large baby M, J conn.
2. Case $c \neq 0$, and 'large', e.g. $c = i$.
   1. a large, one crit orbit bounded, one escapes
   2. a small, compare to rat $(z \mapsto z^3 + i)$, various cases (mcmullen and mixed J's),





Zooms of $M_{n=3,c=i}$ near $a = 0$.

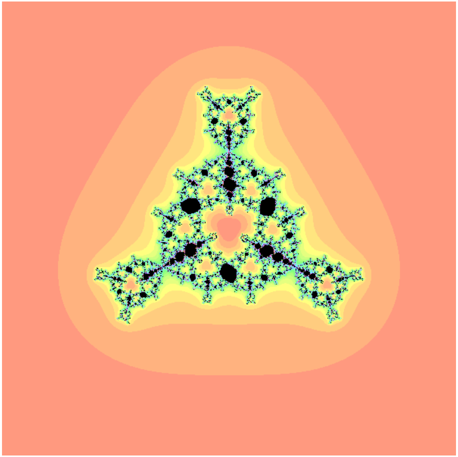




Left: $n = 3$, $c = i$, a tiny (Russell) (IMG?)

Right: J set for $z^3 + i$, the Rat.

Now *a* small but in black! (IMG: compare to basillica, circle, and rat).

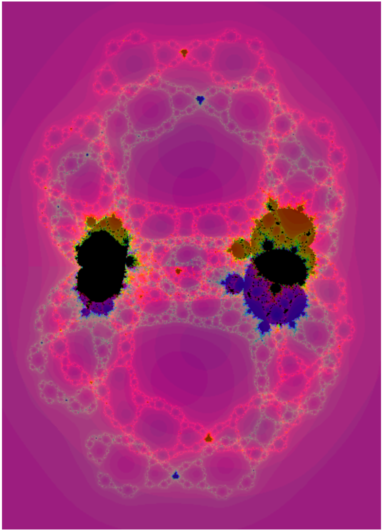


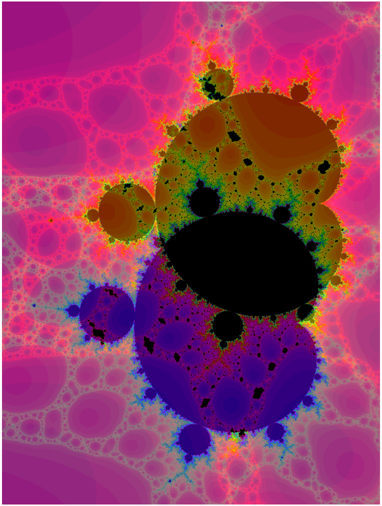
Outline of SPRM cases

Try for $n = 3$, all cases $n \geq 3$ similar:

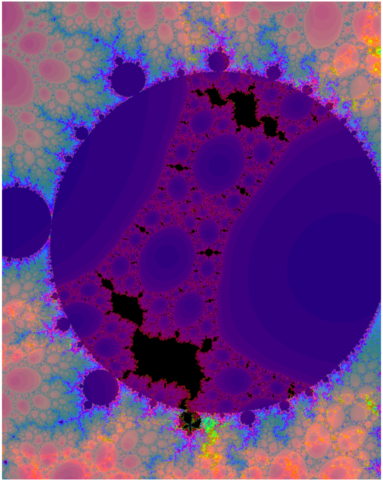
1. Case $c = 0$
   1. $a \notin M$, a small: McMullen domain, J disconn. but not cantor
   2. $a \notin M$, a large: Sierpinski Hole, J conn.
   3. $a \in M$: in a large baby M, J conn.
2. Case $c \neq 0$, and 'large', e.g. $c = i$ or $c = 0.25$.
   1. a large, one crit orbit bounded, one escapes
   2. a small, various cases, compare to rat $(z \mapsto z^3 + i)$
3. Case $c \neq 0$, and small, e.g. $c = 0.01 + 0.05i$.

M for $c \neq 0$ but small, and zoom in:

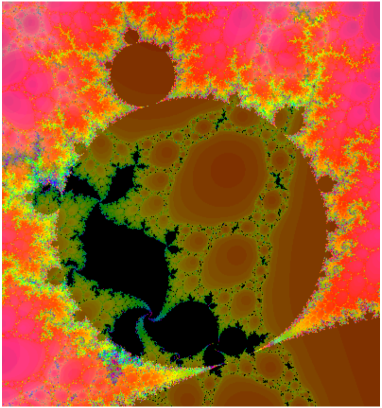




Blue basillica bulb:



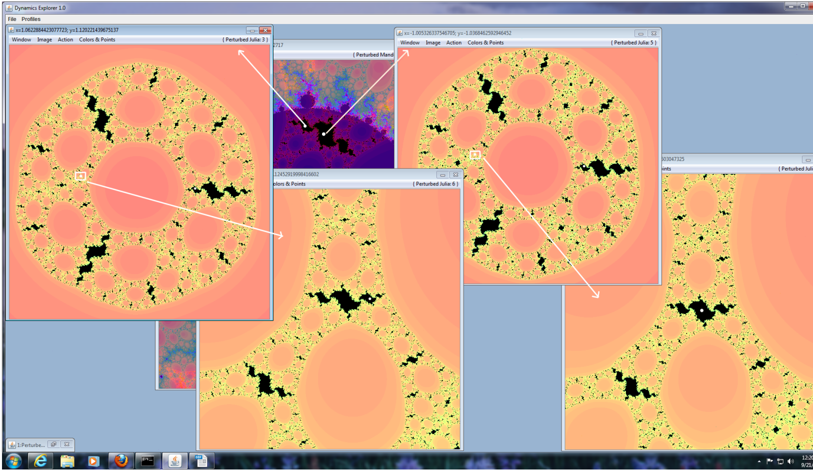
Red rabbit bulb:



Theorem

In the bifurcation locus of $R_{n,c,a}$ for c sufficiently small, there exist homeomorphic copies of Julia sets of quadratic polynomials.

Dynamics causing this:



(IMG's, differ subtly?)

Future

1. Pin down details, generalize.
2. In Hénon family, look for baby Julia sets in parameter bifurcation locus.

Want to help? Let me know, inquiries welcome!

Thanks!!

Thanks for your attention!

Older pics made with “Fractalasm”, newer pics made with “Dynamics Explorer” (sourceforge).

Questions/Comments?

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