"Homotopy Pseudo-Orbits and Iterated Monodromy Groups"

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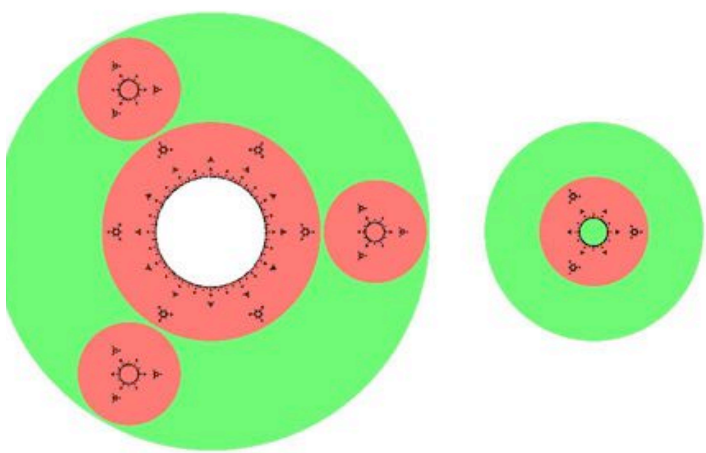
Setting

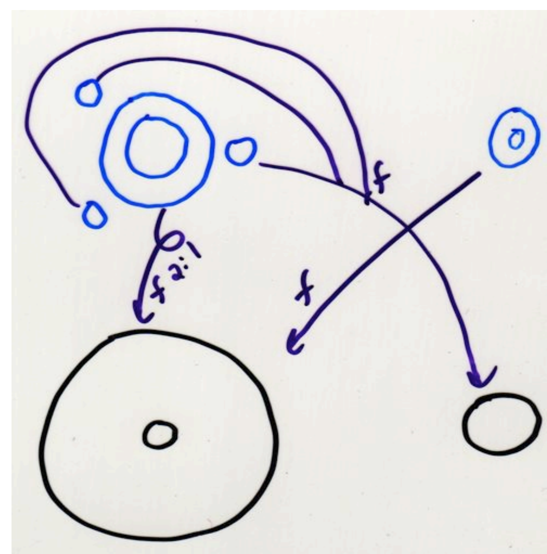
- [Ishii-Smillie] Let X_0, X_1 be "nice" (locally contractible, finitely generated fundamental group...) compact metric spaces, and $\iota, f : X_1 \to X_0$ two maps such that:
  - Given $x', y' \in X_0$ with $d_1(x', y') < \epsilon$, and $x \in f^{-1}(x')$, there is a unique preimage $y = f^{-1}(y')$ such that $d_2(x, y) < \epsilon$ **"Local homeomorphism"**; and
  - There exist $\epsilon > 0$ and $\lambda > 1$ s.t. if $d_2(x, y) \leq \epsilon$, then $d_1(f(x), f(y)) \geq \lambda\, d_1(\iota(x), \iota(y))$ **"Expansion"**.
  - Then call $(\iota, f) : X_0 \to X_1$ an expanding system.
- We'll use the Ishii-Smillie Homotopy Pseudo-Orbit theory and the Bartholdi-Nekrashevych Iterated Monodromy Groups (IMG) theory to build combinatorial models of expanding systems.

Example

- E.g., if $f : \mathbb{C} \to \mathbb{C}$ is a rational map with finite postrcritical set P, let X_0 be $\mathbb{C}$ minus a neighborhood of P, set $X_1 = f^{-1}(X_0)$, and let ι be the inclusion map.

- Based on a "fake" cubic polynomial with one critical point escaping and one fixed (left), we derive the expanding system on the right: $(i, f) : X_1 \to X_0$, where $i = \iota$ is simple inclusion.





Limiting system

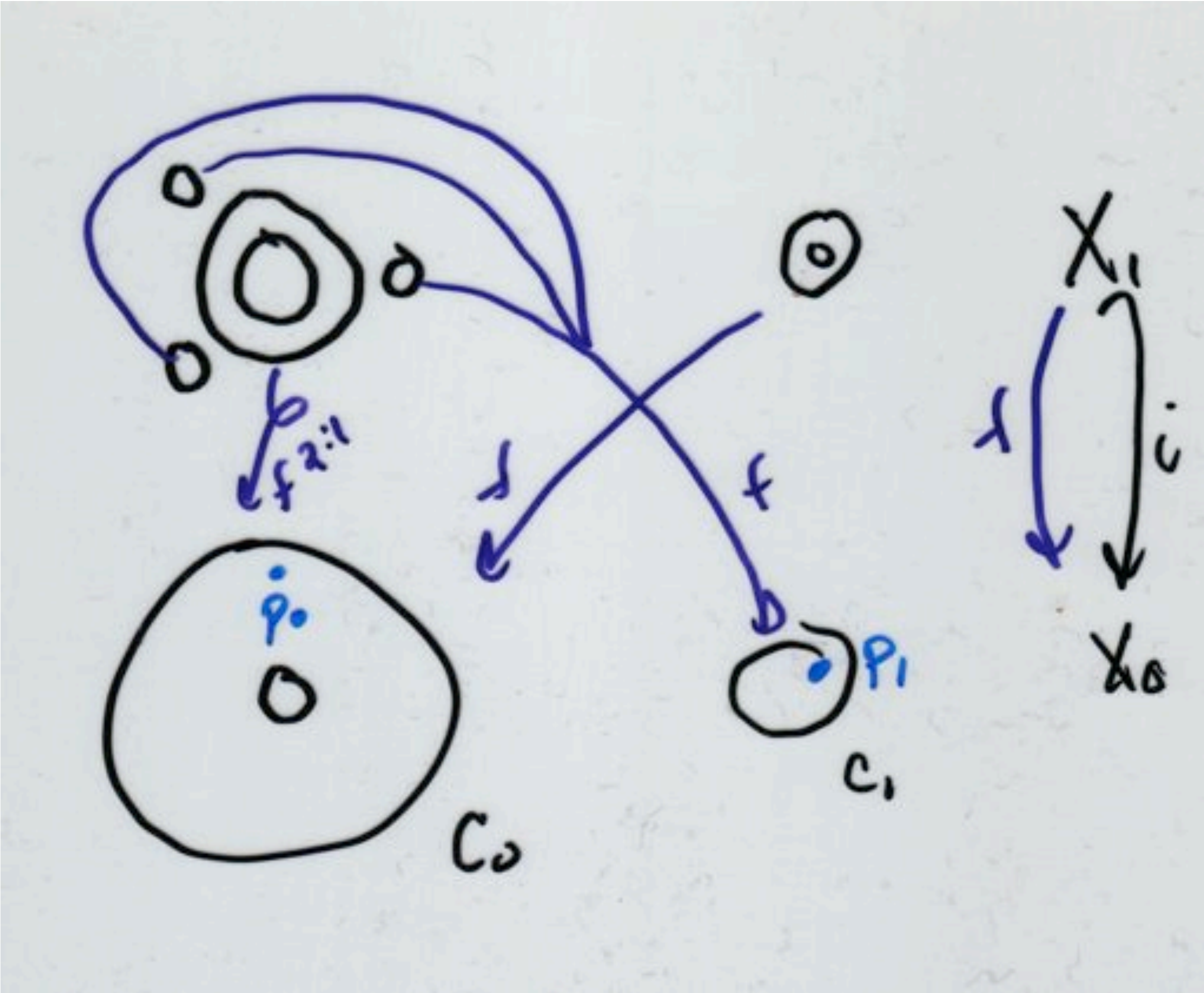
- Let X_n be the set of n-orbits: sequences $a_1, \ldots, a_n \in X_0$ and $b_1, \ldots, b_{n-1} \in X_1$, such that $\iota(b_j) = a_j$ and $f(b_j) = a_{j+1}$. (This is compatible with X_1, X_0.)
- Define $\iota : X_{n+1} \to X_n$ by deleting last terms a_{n+1}, b_{n+1}, and $f : X_{n+1} \to X_n$ by deleting a_1, b_1 and renumbering.
- Let X_∞ be the space of infinite orbits and $f_\infty : X_\infty \to X_\infty$ the shift map. This is an expanding map.
- **HPO Theorem [Ishii-Smillie]:** If $(\iota_f, f)\colon X_1 \to X_0$ and $(\iota_g, g)\colon Y_1 \to Y_0$ are homotopy equivalent expanding systems (i.e., there are semi-conjugacies $h_k : X_k \to Y_k$ for $k = 0, 1$ (i.e., $\iota_g h_1 = h_0 \iota_f$ and $g h_1 = h_0 f$), and vice-versa), then the limiting systems $\hat{f}\colon X_\infty \to X_\infty$ and $\hat{g}\colon Y_\infty \to Y_\infty$ are topologically conjugate.

Goal

- Our goal is to use HPO theory to capture $\hat{f}\colon X_\infty \to X_\infty$ (i.e., f on the Julia set) via a "wire model" $(\iota, f)\colon Y_1 \to Y_0$ which is homotopy equivalent to $(i, f)\colon X_1 \to X_0$.
- Since homotopy equivalence of finite models implies conjugacy of limit systems, we have a lot of flexibility in how we capture the homotopy information about the system $(i, f)\colon X_1 \to X_0$. One approach is to use an IMG type model....

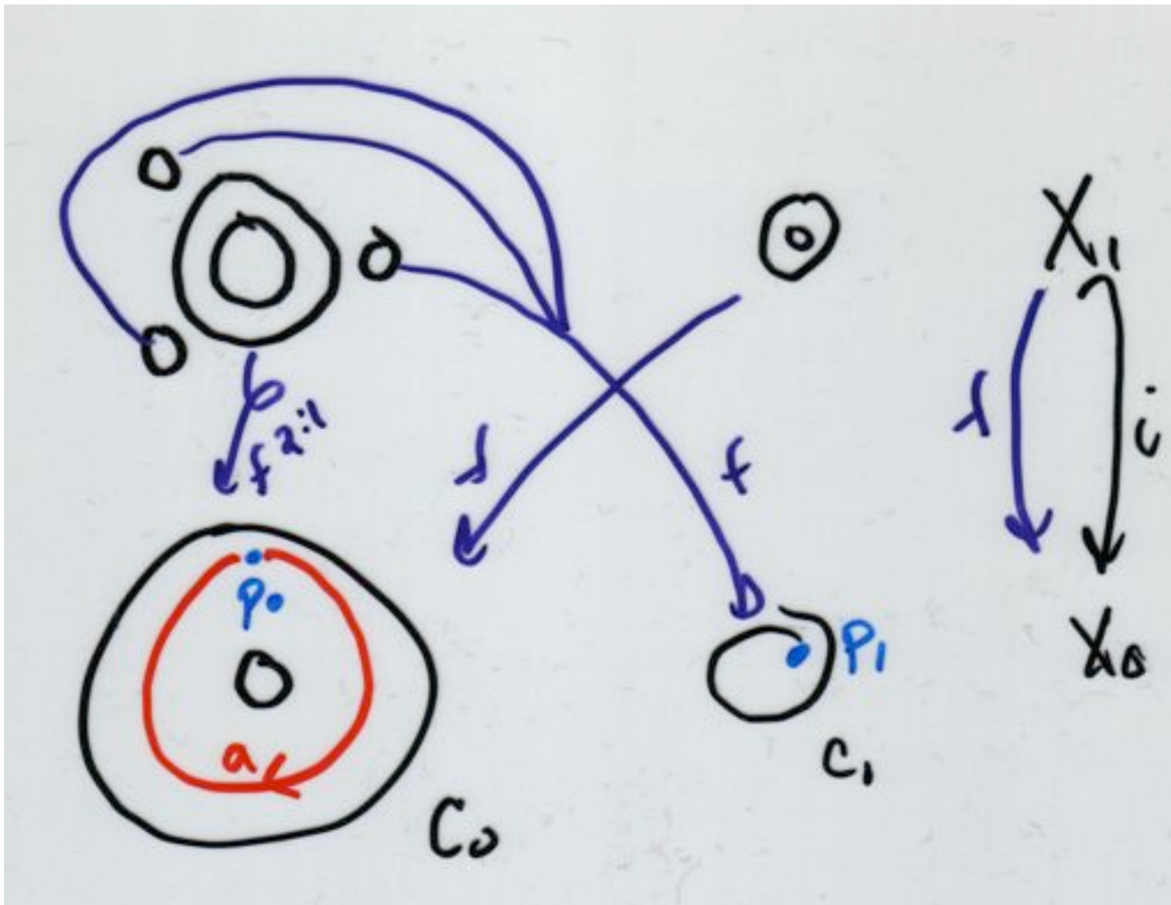
Construction 1

1. In each connected component C_k of X_0, choose a basepoint p_k.



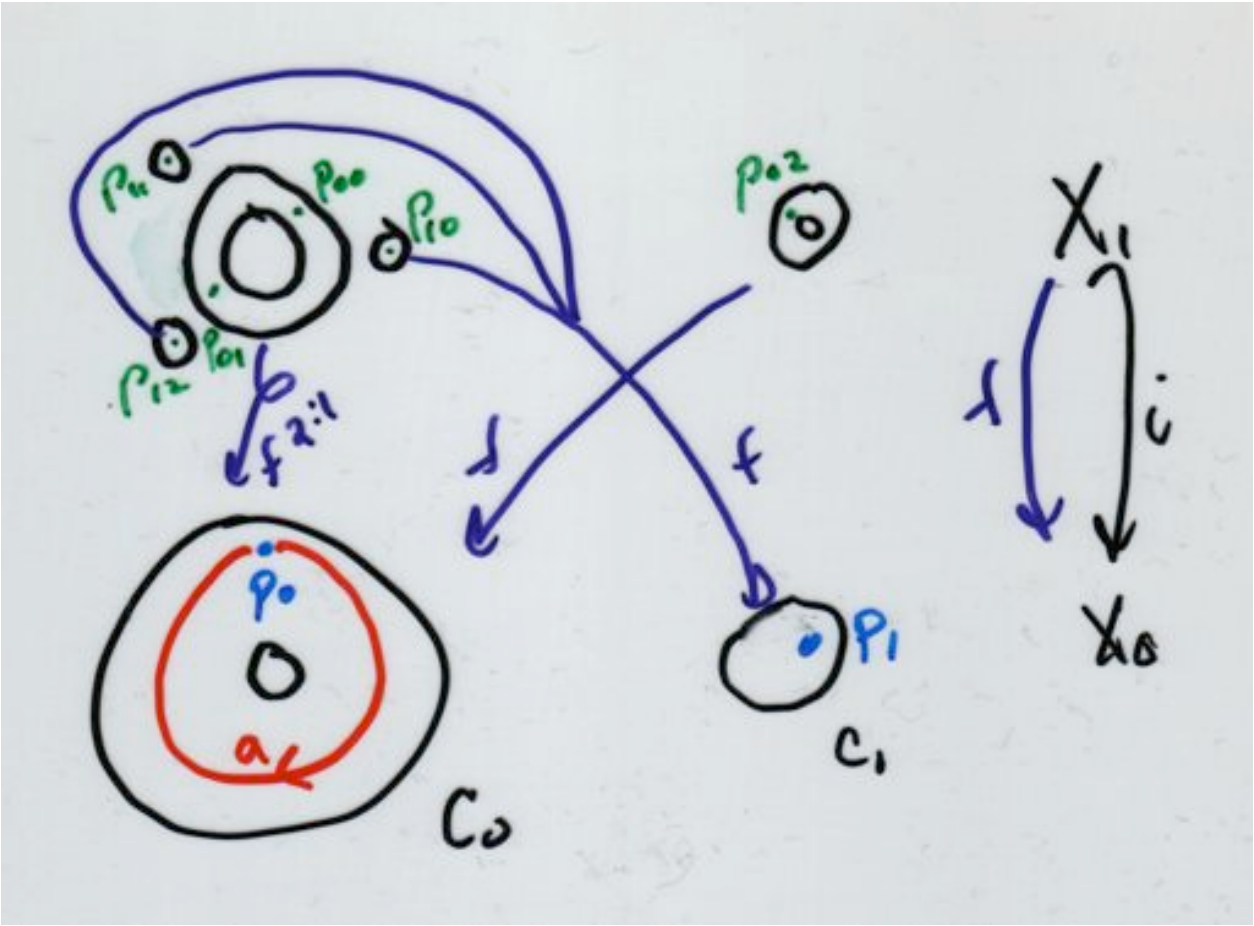
Construction 2

2. Choose and label generators for each $\pi_1(C_k, p_k)$, in X_0.



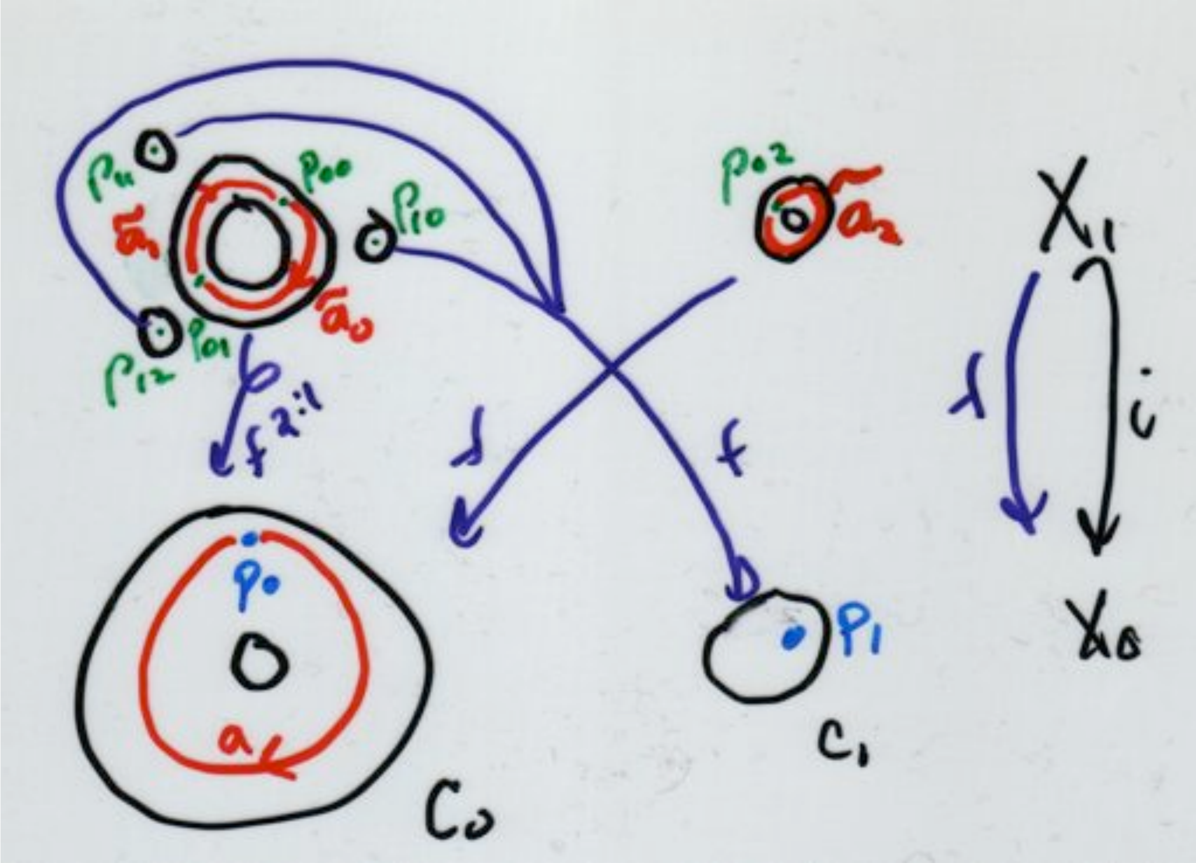
Construction 3

3. For each p_k in X_0, let $\{p_{km}\}$ in X_1 be all the preimages under f of p_k.



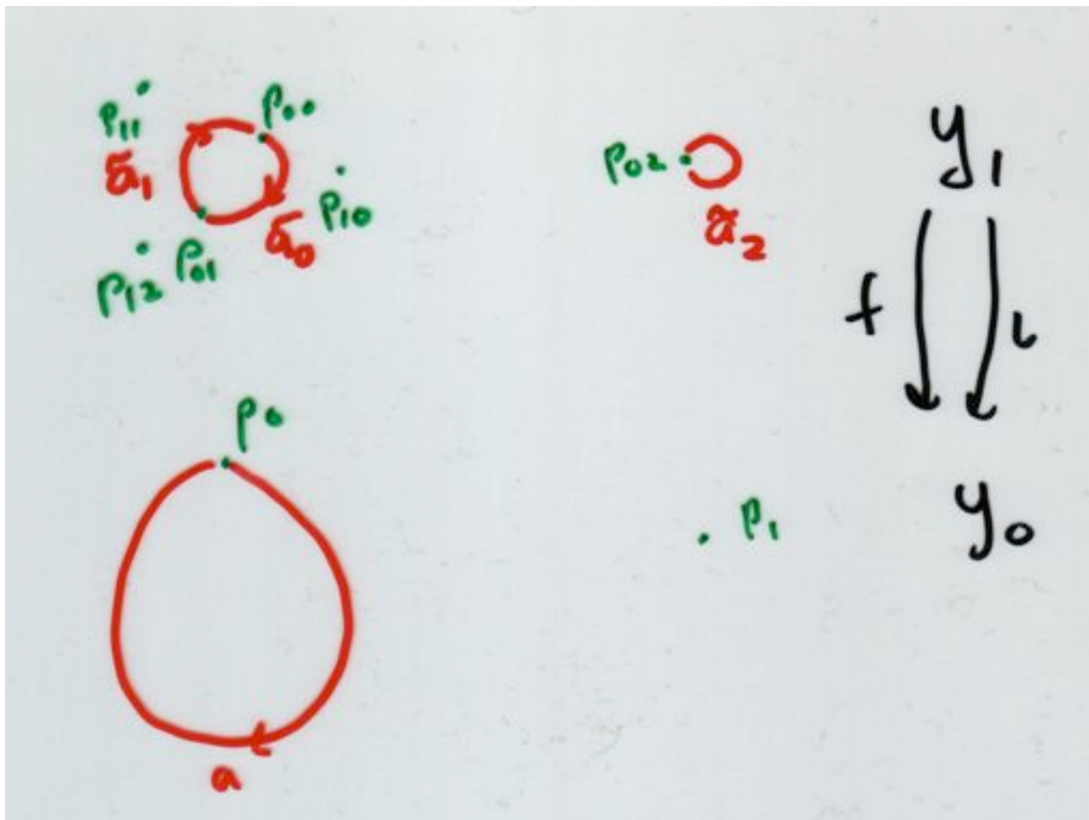
Construction 4

4. For each k, and each $v \in \pi_1(C_k, p_k)$, let $\tilde{v}_m = f^{-1}|_m(v)$ in X_1 be the lift of v based at p_{km}.



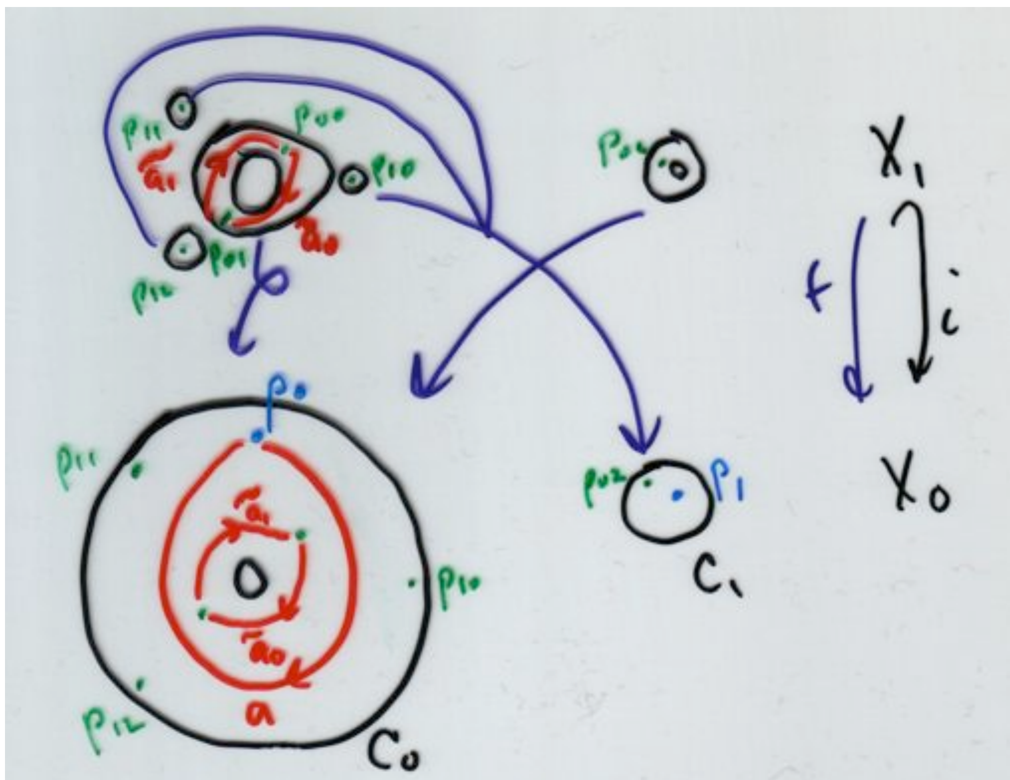
Construction 5

5. Now we begin defining $(\iota, f)\colon Y_1 \to Y_0$. Start with Y_0 as the chosen generators of $\pi_1(C_k, p_k)$, for all k, and Y_1 all the lifts under f of these generators. So elements of Y_n naturally are included in X_n, and the map f sends Y_1 to Y_0.



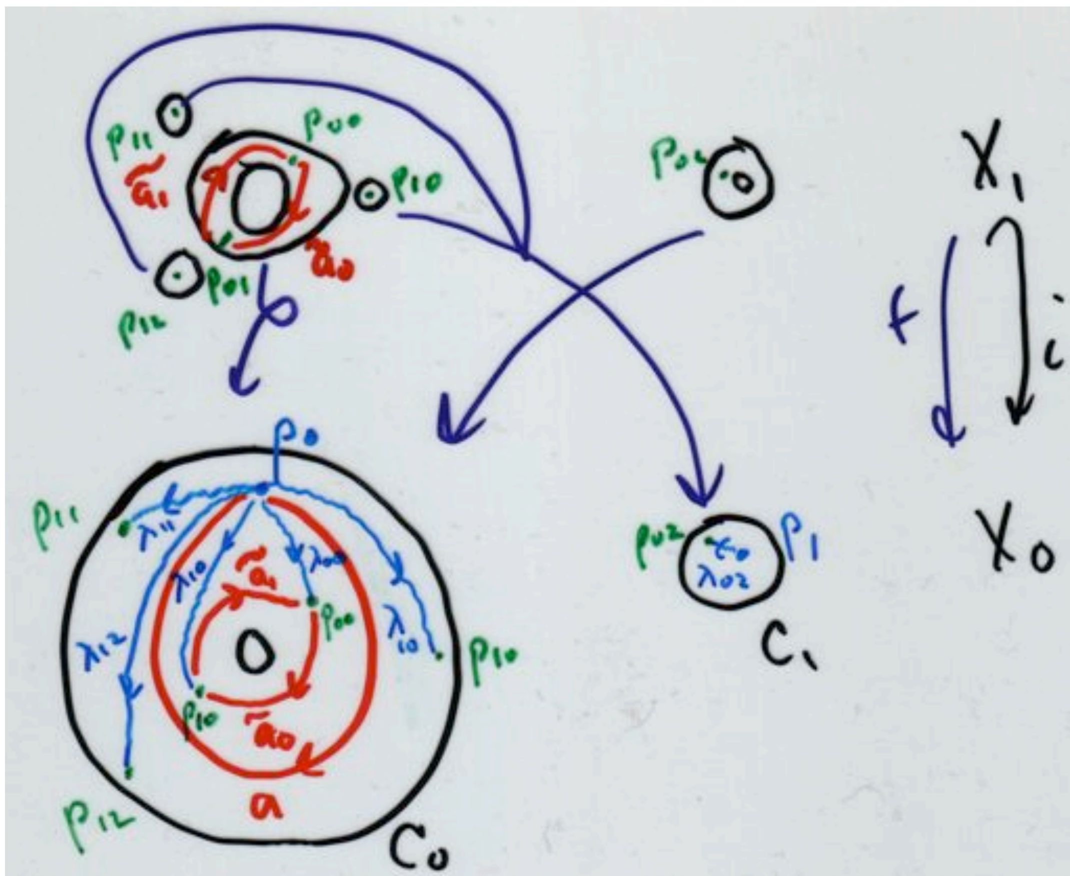
Construction 6

6-a But how do we define $\iota\colon Y_1 \to Y_0$? We want a map which is homotopy equivalent to the inclusion $i\colon X_1 \to X_0$, but it can't just be inclusion, after all, lifts of loops based at p_k are not necessarily loops and are based at the preimages of p_k. *(oops, forgot $\tilde{a}_2$ in pic.)*



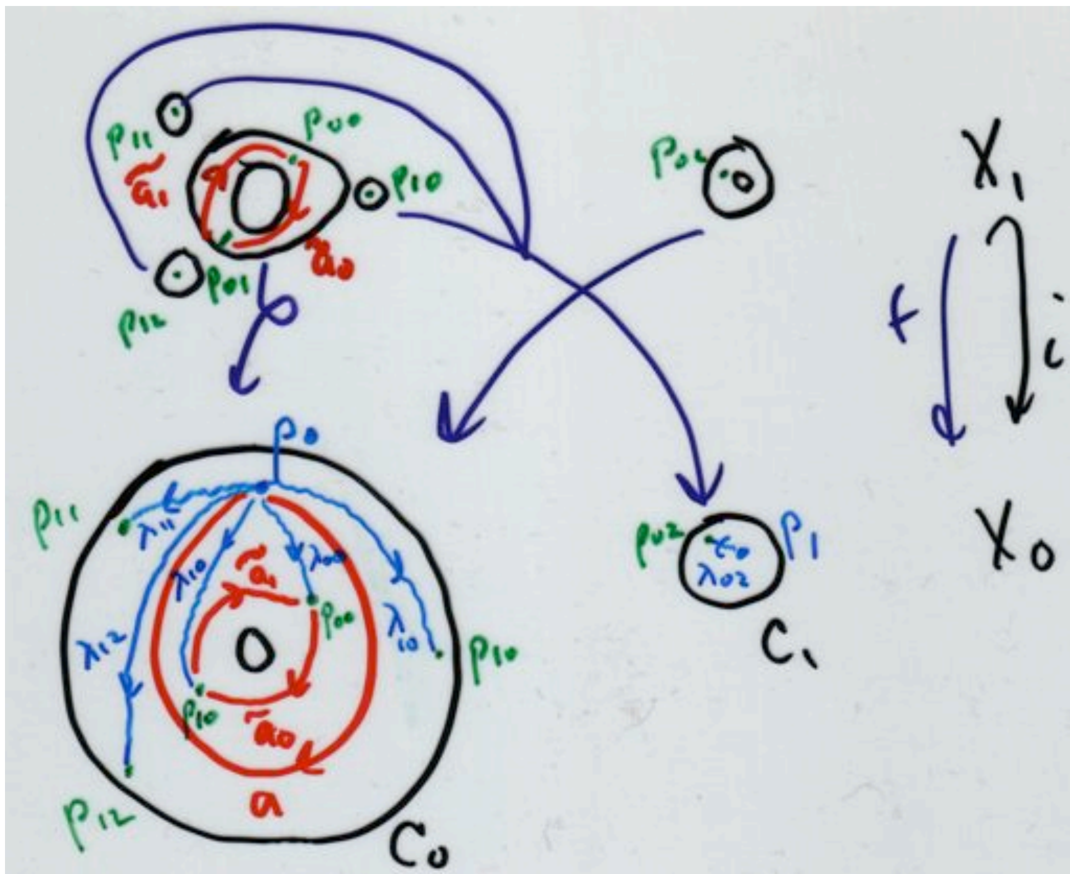
Construction 6 cn'td

6-b Solution: If p_{km} is in C_j, choose a path λ_{km} in X_0 going from p_j to p_{km}, (so $f(p_{km})$ is not p_j, rather p_j is the basepoint in the component containing p_{km}). Now λ_{km} defines a homotopy from $i(Y_1)$ to Y_0.



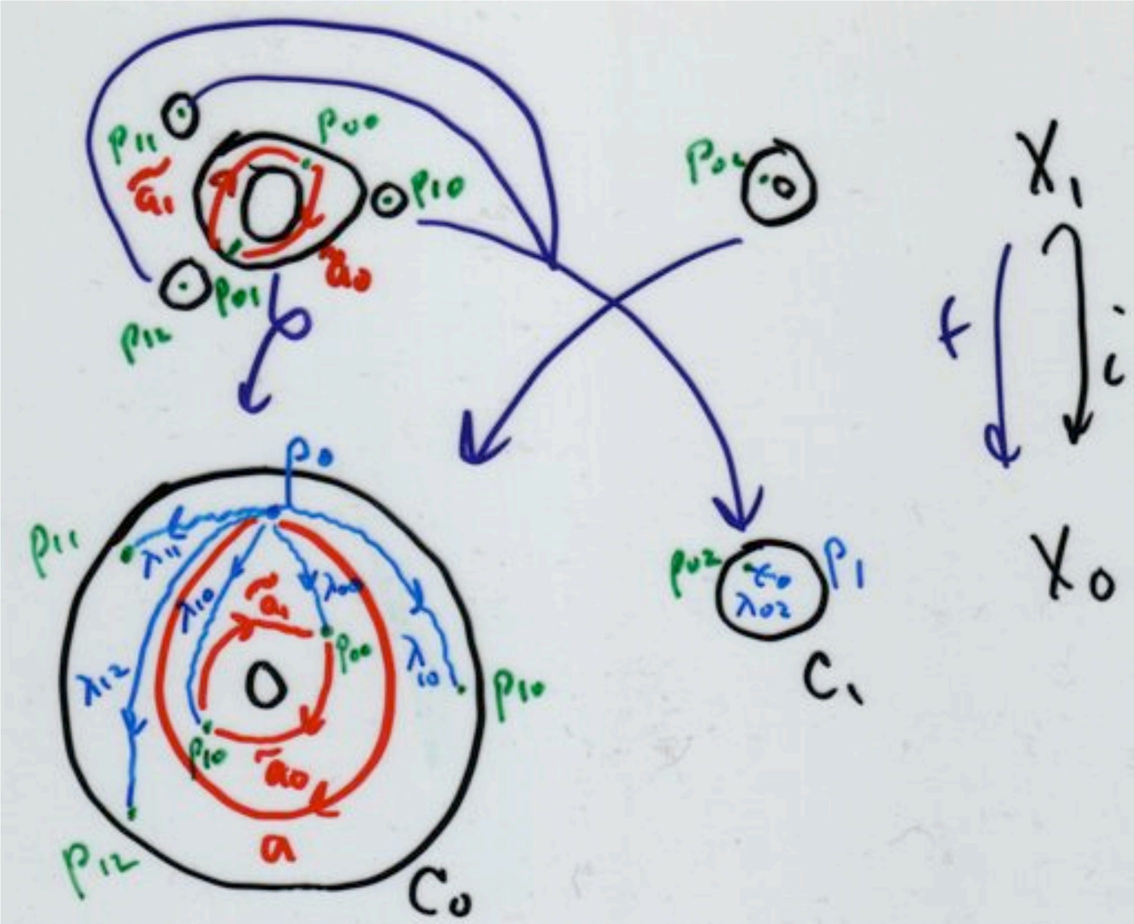
Construction 7

7. That is, for each $\tilde{v}_m = f^{-1}|_m(v)$ in Y_1, if $\tilde{v}_m$ is a path from p_{km} to p_{kl}, then $\iota(\tilde{v}_m) := \bar{\lambda}_{kl} * \tilde{v}_m * \lambda_{km}$ is an element of $\pi_1(C_k, p_k)$, (going from p_k to p_{km}, then p_{km} to p_{kl}, then p_{kl} to p_k).



Construction 7 cn'td

- E.g., $\iota(\tilde{a}_0) = \bar{\lambda}_{01} * \tilde{a}_0 * \lambda_{00} = a$, but $\iota(\tilde{a}_1) = \bar{\lambda}_{00} * \tilde{a}_1 * \lambda_{01} = e_0$. (Also, $\iota(\tilde{a}_2)$ is in C_1 so it's trivial, e_1.)



IMG

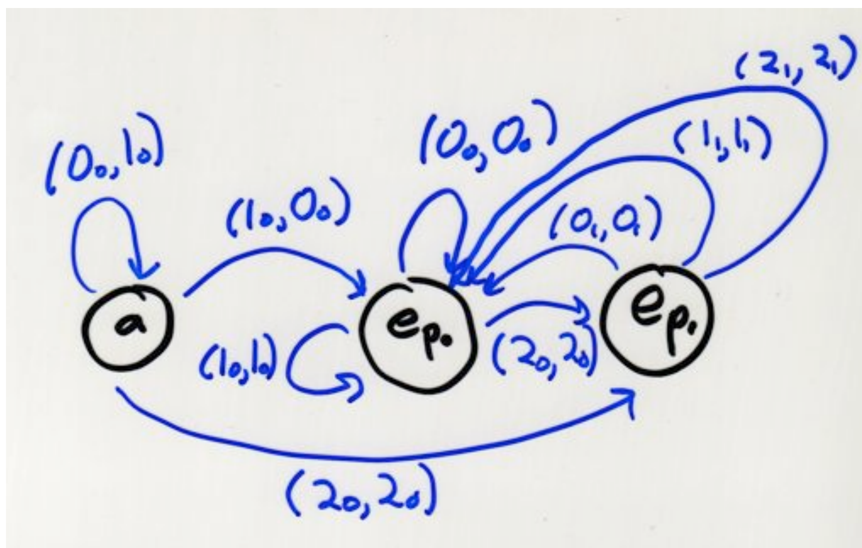
- We can encode the algebraic information of our model $(\iota, f)\colon Y_1 \to Y_0$ using IMG technology: for each v in Y_0, write $v = (\iota(\tilde{v}_0), \ldots, \iota(\tilde{v}_{d-1}))\sigma$, where σ is the permutation on the preimages of the basepoints defined by head to tail for each path $\tilde{v}_m$.
- E.g., $a = (\iota(\tilde{a}_0), \iota(\tilde{a}_1), \iota(\tilde{a}_2)\sigma_a = (a, e_0, e_1)(0_0, 1_0)$,
$e_0 = (e_0, e_0, e_1)()$,
$e_1 = (e_0, e_0, e_0)()$

Generalize Moore Diagram

- The algebraic relations

  E.g., $a = (a, e_0, e_1)(0_0, 1_0), e_0 = (e_0, e_0, e_1)(), e_1 = (e_0, e_0, e_0)()$

  can be encoded in a finite automaton called a (Generalized) Moore Diagram (arrows = ιf^{-1}, labels = σ).



A finite nucleus

- In this simple example, ι mapped each element of Y_1 to an element of the chosen generating set Y_0. But a priori this may not always occur—ι is only guaranteed to map each element of Y_1 into $\pi_1(X_0)$, it may map an element of Y_1 to some combination of elements of Y_0.
- In this case, following IMG theory we add this missing element to Y_0, and re-start. We claim this process terminates, i.e., there is some finite collection of elements of $\pi_1(X_0)$ whose lifts all map by ι back into that same collection. This finite collection is called a *nucleus*.
- There is a very dynamical proof that a finite nucleus exists (basically: f expanding implies lifts of loops eventually shrink), which is very general (for example, it does not require X_n to be connected). Conclusion: a finite expanding system $(\iota, f)\colon Y_1 \to Y_0$ exists, which is homotopy equivalent to $(i, f) : X_1 \to X_0$.

Summary

- To summarize, by HPO theory we can say $\hat{f}\colon X_\infty \to X_\infty$ (i.e. f on J) is conjugate to $\hat{f}\colon Y_\infty \to Y_\infty$, hence the "wire model" $(\iota, f)\colon Y_1 \to Y_0$ (together with its Moore Diagram) provides a combinatorial model for f on J.

- Again, note any other style of "wire" models based on homotopy type would work (for example, instead of loops you could take Y_0 to consist of paths in a 1-skeleton of X_0, like a Hubbard Tree with "feet").