

# Stability of Frictional Sliding With the Coefficient of Friction Depended on the Temperature

Vahid Mortazavi  
Chuanfeng Wang

Michael Nosonovsky  
e-mail: nosonovs@uwm.edu

Department of Mechanical Engineering,  
University of Wisconsin,  
Milwaukee, WI 53201

*Friction-induced instabilities can be caused by different separate mechanisms such as elastodynamic or thermoelastic. This paper suggests another type of instability due to the temperature dependency of the coefficient of friction. The perturbations imposed on the surface temperature field during the frictional sliding can grow or decay. A stability criterion is formulated and a case study of a brake disk is performed with a simple model without including effects of transforming layer and chemical/physical properties change with temperature. The disk is rigid and the coefficient of friction depends on temperature. We show that the mechanism of instability can contribute to poor reproducibility of aircraft disk brake tests reported in the literature. We propose a method to increase the reproducibility by dividing the disk into several sectors with decreased thermal conductivity between the sectors. [DOI: 10.1115/1.4006577]*

## 1 Introduction

It is well known that frictional sliding can be dynamically unstable. Several types of instabilities have been discussed in the literature. If the coefficient of friction decreases with speed, dynamic instabilities (DI) can occur. This is because decreasing frictional resistance to sliding results in the acceleration, which leads to a higher velocity and, in turn, to even lower friction; thus creating a positive feedback loop. Various theories of velocity-dependent friction have been proposed in the literature [1,2]. Adams [3] showed that even for a constant value of the coefficient of friction, the steady sliding of two elastic half-planes can become dynamically unstable, even in the quasi-static limit of very small sliding velocities [3]. The destabilization is in the form of a self-excited elastic wave at the interface between the contacting bodies. The amplitude of these waves grows exponentially with time. As the amplitude of an elastic wave grows, the frictional dissipation increases, leading to the further growth of the wave amplitude and thus creating a positive feedback loop. A similar type of instability may be observed during the sliding of rough surfaces, such as surfaces with periodic wavy profiles [4,5]. Frictional instability can further lead to the formation of self-organized structures and patterns [6–9].

Another type of instability is a result of the interaction between the frictional heating, the thermoelastic distortion, and the contact pressure, and it is referred to as the “thermoelastic instability” (TEI). As the interface temperature grows, the near-surface volumes of the contacting bodies expand, so the contact pressure grows as well. As a result, the friction force increases resulting in excess heat generation and the further growth of the temperature; thus creating another type of a positive feedback. The TEI leads to the formation of “hot spots” or localized high temperature regions at the interface [10]. The TEI occurs for sliding velocities greater than a certain critical value. The coupling between the two types of instabilities constitutes the thermoelastodynamic instability (TEDI) [11].

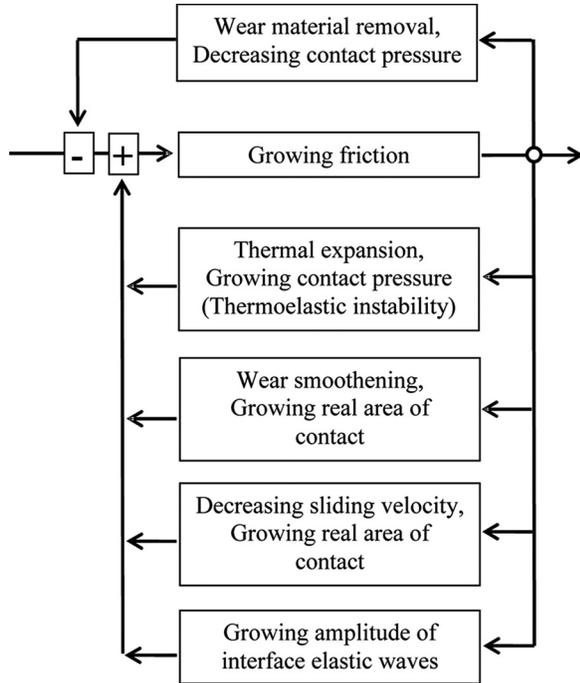
Another mechanism that may provide instability is the coupling between friction and wear. As friction increases, so does also the wear, which may result in an increase of the real area of contact between the bodies and in further increase of friction. The sliding

bodies adjust to each other and the process is known as the self-organization during friction [12]. On the other hand, wear produces smoothening of the surface distorted by the TEI mechanism, and thus the wear and thermal expansion are competing factors, with the wear leading to stabilization of sliding and the thermal expansion leading to destabilization (Fig. 1).

One area where frictional instabilities are particularly important is the design of disk brakes. Frictional instabilities in car disk brakes have been an object of investigation mostly because of the disk brake squeal [13–15]. Aircraft disk brakes often have a similar construction to car disk brakes; however, the instability constitutes even more crucial problem for the aircraft brakes. Aircraft disk brakes are designed to dissipate very large amounts of energy in order to stop the plane within a short time after an aircraft touches the runway. Large amounts of heat are generated in aircraft disk brakes within seconds, resulting in high temperatures as well as very high temperature gradients. The disks should be made of a material which is light, wear-resistant, and able to absorb huge amounts of heat without melting or breaking. In brief, the brake materials should have good heat sinking ability in order to minimize the interface temperature at the braking surface arising out of frictional heat [16]. The materials that provide a compromise between weight, strength, and heat transfer are the carbon-carbon (CC) composites [17,18]. A number of studies have been conducted in the past decade to investigate the tribological performance of the CC composite disk brake material at various sliding velocities, temperatures, and levels of humidity [19–22]. Venkataraman and Sundararajan [16] showed that CC composites exhibit a transition from a low coefficient of friction during the “normal” wear regime, to a high coefficient of friction during the “dusting” wear regime, when the normal pressure times the sliding velocity exceeds a critical value. The transition is associated with the attainment of a critical temperature at the interface between the two CC composite bodies sliding against each other. Yen and Ishihara [23] showed that two types of surface morphology can be distinguished on the sample surface and argued that the TEI is responsible for this effect.

Most theoretical studies of the instabilities have concentrated on investigating the onset of the instability and stability criteria. However, the quantitative study of the unstable motion is also of great practical importance. Due to various safety requirements, the aircraft brake disks should demonstrate highly reproducible performance. However, the instability of frictional sliding between a disk brake and a pad may result in the high sensitivity

Contributed by the Tribology Division of ASME for publication in the JOURNAL OF TRIBOLOGY. Manuscript received August 23, 2011; final manuscript received March 21, 2012; published online August 21, 2012. Editor: Michael Khonsari.



**Fig. 1** Various mechanisms can create positive or negative feedbacks that lead to instabilities during friction

of the tribological system to initial random perturbations. As a result, the time and distance required to stop a plane may vary significantly even under the same conditions (such as the mass and initial velocity of the plane).

In this paper, first, we show that the temperature-dependent coefficient of friction can lead to instability, which is similar to the TEI. We suggest a simple one-dimensional (1D) model to demonstrate that this dependency cannot be neglected. Note that our model ignores the effects of the transforming layer, “third body,” and the change of properties with temperature. Although these effects may have a significant influence on the tribological properties [24], they can be ignored for a simplified model. Then, a stability model with the local coefficient of friction coupled with temperature is developed, and a statistical analysis of the average friction between the brake disk and a pad is performed. The factors that affect the reproducibility of brake tests are discussed. Finally, some measures that can improve the reproducibility are suggested.

## 2 Instability Due to the Temperature Dependency of the Coefficient of Friction

Most of current models of friction-induced instabilities ignore the temperature dependency of the coefficient of friction. These models usually assume that either elastodynamic or thermoelastic effects can give rise to friction-induced instabilities and vibrations. The time scales of these effects differ considerably, so it is usual to neglect the coupling between them, i.e., to neglect the thermal effects in elastodynamic analyses and to use the quasi-static approximation in thermoelastic analyses. In addition, these models assume that the coefficient of friction is constant, i.e., not varying with temperature.

On the other hand, there is experimental evidence that CC composites undergo a transition from a low to a high value of the coefficient of friction depending on the temperature change [16,25,26]. Here we investigate the possibility that the temperature dependency of the coefficient of friction leads to instability.

First we consider the 1D heat conduction equation in rectangular coordinates with a heat generation source resulting from

friction in a slab of length  $L$ . We assume that the 1D region represents a 2D slab with a small thickness  $w$ , so that the heat propagates instantly throughout the thickness and a 1D approximation is valid:

$$\frac{\partial^2 T(x,t)}{\partial x^2} + \frac{1}{wk}g(x,t) = \frac{1}{\alpha} \frac{\partial T(x,t)}{\partial t} \quad \text{in } 0 < x < L, \quad t > 0 \quad (1)$$

$$-k \frac{\partial T}{\partial x} + h(T - T_0) = 0 \quad \text{at } x = 0, \quad t > 0$$

$$k \frac{\partial T}{\partial x} + h(T - T_0) = 0 \quad \text{at } x = L, \quad t > 0$$

$$T(x,0) = F(x) \quad \text{for } 0 \leq x \leq L, \quad t = 0$$

where  $T(x,t)$  is the temperature,  $x$  and  $t$  are the spatial coordinate and time,  $k$  and  $\alpha$  are the thermal conductivity and diffusivity, respectively, and  $h$  is the coefficient of convective heat transfer. We assume that the slab is initially at a temperature  $F(x)$  and, for times  $t > 0$ , it dissipates heat into the environment with constant temperature  $T_0$ .

The heat generation term due to friction is given by

$$g(x,t) = PV\mu \quad (2)$$

where  $P$  is pressure,  $V$  is sliding velocity, and  $\mu$  is the coefficient of friction which we assume to be linearly temperature dependent:

$$\mu = \mu_0[1 + \lambda(T - T_0)] \quad (3)$$

where  $\lambda$  is the constant of proportionality.

It is convenient to consider Eq. (1) in the nondimensional form by defining dimensionless parameters

$$\theta(X, \tau) = \frac{T - T_0}{T_0}, \quad \tau = \frac{t\alpha}{L^2}, \quad X = \frac{x}{L},$$

$$G = \left( \frac{PVL^2}{wk} \right) \left( \frac{\mu_0}{T_0} \right), \quad H = \frac{Lh}{k}, \quad \epsilon = \lambda T_0 \quad (4)$$

Equation (1) becomes

$$\frac{\partial^2 \theta}{\partial X^2} + G(1 + \epsilon\theta) = \frac{\partial \theta}{\partial \tau} \quad \text{in } 0 < X < 1, \quad \tau > 0 \quad (5)$$

$$-\frac{\partial \theta}{\partial X} + H\theta = 0 \quad \text{at } X = 0, \quad \tau > 0$$

$$\frac{\partial \theta}{\partial X} + H\theta = 0 \quad \text{at } X = 1, \quad \tau > 0$$

$$\theta(X,0) = f(X) \quad \text{for } 0 \leq X \leq 1, \quad \tau = 0$$

The nonhomogeneous term  $G\epsilon\theta$  in Eq. (5) is due to the nondimensional parameter  $\epsilon$ , which defines the temperature dependency of the coefficient of friction. Equation (5) has a steady solution  $\theta_s(x)$ . We investigate the stability of this steady state by considering the possibility that a small perturbation in the temperature can grow with time. Thus we write

$$\theta(X, \tau) = \theta_s(X) + \tilde{\theta}(X, \tau) \quad (6)$$

where  $\tilde{\theta}(X, \tau)$  is the perturbation. Now substituting Eq. (6) into Eq. (5):

$$\frac{\partial^2 \tilde{\theta}}{\partial X^2} + G\epsilon \tilde{\theta} = \frac{\partial \tilde{\theta}}{\partial \tau} \quad \text{in } 0 < X < 1, \quad \tau > 0 \quad (7)$$

$$\begin{aligned} -\frac{\partial \tilde{\theta}}{\partial X} + H\tilde{\theta} &= 0 \quad \text{at } X=0, \quad \tau > 0 \\ \frac{\partial \tilde{\theta}}{\partial X} + H\tilde{\theta} &= 0 \quad \text{at } X=1, \quad \tau > 0 \\ \tilde{\theta}(X, \tau) &= \tilde{f}(X) \quad \text{for } 0 \leq X \leq 1, \quad \tau = 0 \end{aligned}$$

We express the solution of Eq. (7) as a sum of normal modes, each of which has the general form

$$\tilde{\theta}(X, \tau) = \psi(X)e^{S\tau} \quad (8)$$

where  $S$  is a constant. Substituting Eq. (8) into Eq. (7) we get

$$\begin{aligned} \frac{d^2\psi}{dX^2} + (G\epsilon - S)\psi &= 0 \quad \text{in } 0 < X < 1 \\ -\frac{d\psi}{dX} + H\psi &= 0 \quad \text{at } X=0 \\ \frac{d\psi}{dX} + H\psi &= 0 \quad \text{at } X=1 \end{aligned} \quad (9)$$

Solution of Eq. (9) is in the form of

$$\psi(X) = C_1 \sin(m\pi X) + C_2 \cos(m\pi X) \quad (10)$$

where  $C_1$  and  $C_2$  are two constants which can be obtained from the boundary conditions and  $m$  is a natural number. Substituting Eq. (10) into Eq. (9) we get

$$S = G\epsilon - (m\pi)^2 \quad (11)$$

We conclude from Eq. (8) that the stability of a small perturbation depends on whether the exponential term grows or decays with time  $\tau$ , which, in turn, depends upon the sign of  $S$  for any  $m$ . Thus, from Eq. (11), decaying in any  $m$ th mode is observed for  $G\epsilon < (m\pi)^2$ . Since  $m$  is increasing, the stability condition should be satisfied for the first term  $G\epsilon < \pi^2$ . In the case of  $\epsilon = 0$ , i.e., when there is no temperature dependency of the coefficient of friction, and, with increasing time,  $\theta(X, \tau)$  will approach the steady temperature distribution  $\theta_s(X)$ . However, in the case of  $\epsilon \neq 0$ , the stability condition requires

$$\epsilon < \pi^2 \left( \frac{wk}{pVL^2} \right) \left( \frac{T_0}{\mu_0} \right) \quad (12)$$

Let us call  $\epsilon_{\text{critical}} = \pi^2 \left( \frac{wk}{pVL^2} \right) \left( \frac{T_0}{\mu_0} \right)$  and we conclude that if  $\epsilon > \epsilon_{\text{critical}}$ , the solution is unstable. Note that  $\epsilon_{\text{critical}}$  depends on the geometry ( $L$ ), pressure ( $P$ ), and sliding velocity ( $V$ ) in our system. This is in agreement with the experimental report by Venkataraman and Sundararajan [16], who found that there were different regimes of the temperature dependency of the coefficient of friction and that these three parameters had considerable influence on the regimes.

Note also that for the negative  $\epsilon < 0$ , the stability condition is always satisfied. That means that if the coefficient of friction decreases with temperature, there is no unstable behavior. This is because the instability is caused by the positive feedback between the coefficient of friction and temperature, i.e., a small positive local fluctuation of temperature would cause a local increase of the coefficient of friction, which, in turn, would cause further growth of temperature. When the feedback is negative ( $\epsilon < 0$  and; therefore  $\lambda < 0$ ) this type of unstable behavior does not occur.

To find the exact solution of Eq. (5), one can use the transformation

$$\theta(X, \tau) = \psi(X, \tau)e^{G\epsilon\tau} \quad (13)$$

which yields, on substituting

$$\begin{aligned} \frac{\partial^2\psi}{\partial X^2} + Ge^{G\epsilon\tau} \frac{\partial\psi}{\partial\tau} &= \frac{\partial\psi}{\partial\tau} \quad \text{in } 0 < X < 1, \quad \tau > 0 \\ -\frac{\partial\psi}{\partial X} + H\psi &= 0 \quad \text{at } X=0, \quad \tau > 0 \\ \frac{\partial\psi}{\partial X} + H\psi &= 0 \quad \text{at } X=1, \quad \tau > 0 \\ \psi(X, 0) &= f(X) \quad \text{for } 0 \leq X \leq 1, \quad \tau = 0 \end{aligned} \quad (14)$$

Equation (14) can be solved analytically, using first the method of the separation of variables and then the Green's function [27]. First we find the solution of the homogeneous equation without the term  $G \exp(G\epsilon\tau)$ :

$$\psi'(X, \tau) = 2 \sum_{m=1}^{\infty} e^{-(m\pi)^2\tau} \sin(m\pi X) \left( \int_{X'=0}^1 \sin(m\pi X') f(X') dX' \right) \quad (15)$$

Here we assumed that the both sides of the slab are kept at the constant temperature  $T_0$ , i.e.,  $\psi(0, \tau) = \psi(1, \tau) = 0$  and  $H \rightarrow \infty$ . This assumption simplifies the solution without affecting the stability analysis since the stability is governed by the exponential term of the solution which is not affected by the boundary conditions

The solution of the homogeneous equation can be written in terms of Green's function [27]

$$\psi'(X, \tau) = \int_{X'=0}^1 \gamma(X, \tau; X', \tau' = 0) f(X') dX' \quad (16)$$

Comparing Eq. (16) with Eq. (15) we construct the Green's function by replacing  $\tau$  by  $\tau - \tau'$ :

$$\gamma(X, \tau; X', \tau') = 2 \sum_{m=1}^{\infty} e^{-(m\pi)^2(\tau-\tau')} \sin(m\pi X) \sin(m\pi X') \quad (17)$$

where  $X'$  and  $\tau'$  are integration parameters. Then the solution of the nonhomogeneous problem is given in terms of the Green's function

$$\psi(X, \tau) = \psi'(X, \tau) + \int_{\tau'=0}^{\tau} d\tau' \int_{X'=0}^1 G \exp^{G\epsilon\tau} \gamma(X, \tau - \tau', X') dX' \quad (18)$$

Substituting Eq. (17) into Eq. (18) yields the solution of Eq. (14):

$$\begin{aligned} \psi(X, \tau) &= 2 \sum_{m=1}^{\infty} e^{-(m\pi)^2\tau} \sin(m\pi X) \left( \int_{X'=0}^1 \sin(m\pi X') f(X') dX' \right) \\ &+ 2 \sum_{m=1}^{\infty} \frac{G}{(m\pi)^2 - G\epsilon} e^{-G\epsilon\tau} \sin(m\pi X) X (1 - \cos(m\pi)) \\ &\text{for } 0 < X < 1 \end{aligned} \quad (19)$$

Finally, substituting Eq. (18) into Eq. (13), we find the solution of Eq. (5):

$$\begin{aligned} \theta(X, \tau) &= 2 \sum_{m=1}^{\infty} e^{(G\epsilon - (m\pi)^2)\tau} \sin(m\pi X) \left( \int_{X'=0}^1 \sin(m\pi X') f(X') dX' \right) \\ &+ 2 \sum_{m=1}^{\infty} \frac{G}{(m\pi)^2 - G\epsilon} \sin(m\pi X) (1 - \cos(m\pi)) \\ &\text{for } 0 < X < 1 \end{aligned} \quad (20)$$

The stability of a small perturbation of the solution given by Eq. (19) is governed by Eq. (12). This can be observed directly

from Eq. (19) noting that the stability depends on whether the exponential term grows or decays with time  $\tau$ , which, in turn, depends upon whether  $G\epsilon - (m\pi)^2 < 0$  for any  $m$  and should be satisfied for the first term  $G\epsilon < \pi^2$  as it was explained above.

To examine stability condition [Eq. (12)], numerical results for solution of Eq. (7) are presented in Fig. 2. The simulation was performed using the Matlab software package. In order to investigate the evolution of a localized hot/cold spot, we introduced a small random perturbation imposed over a constant temperature field and confined between  $0.49 < X < 0.51$ , i.e., at the center of the domain. The perturbation was a random function built by assigning random numbers with the amplitude between 0 and 2 to the values in  $0.49 < X < 0.51$  and zero otherwise. Spatial and temporal step size for the numerical simulation was 0.01 and 0.005, respectively. Figure 2 shows the response of the system to small perturbation for different values of  $\epsilon$  (0.0001, 0.00005, and 0.00001). Transient temperature is presented for four different values of the dimensionless time (0, 0.00125, 0.0025, 0.05). The parameters of Eq. (12) were chosen according to the experimental values reported in the literature on CC composites in disk brakes (for example, Zhao et al. [28]):  $P = 1$  MPa,  $k = 50$  W/mK,  $L = 0.5$  m,  $V = 1000$  m/s. These values correspond to  $\epsilon_{critical}$

$= 3.88 \times 10^{-5}$ . It is observed from Fig. 2 that the solution grew unboundedly in the first two cases corresponding to  $\epsilon > \epsilon_{critical}$  and decayed in the third case  $< \epsilon_{critical}$  in agreement with the stability criterion of Eq. (12).

We found in this section that when the temperature dependency of the coefficient of friction is introduced, the solution for the temperature field is unstable for  $\epsilon > \epsilon_{critical}$ . The comparison with experimental data shows that in practical cases the value of  $\epsilon$  is comparable with  $\epsilon_{critical}$  [28]. Therefore, the effect of the temperature dependency of the coefficient of friction should be taken into account when the stability of the frictional sliding with heat generation is analyzed. Although a rectangular slab was studied in this section, a similar effect is expected with a circular disk, as it will be discussed below.

### 3 Reproducibility of Disk Brake Test Results

In this section we present a model for an aircraft or car disk brake in contact with a pad with a temperature-dependent coefficient of friction between them and show that frictional instabilities affect the reproducibility of brake test results, e.g., the time needed to stop the car or aircraft.

**3.1 Numerical Model.** Let us consider a rigid brake disk with the outer and inner radii of  $R_{out}$  and  $R_{in}$  in contact with a rigid pad pressed together by the pressure  $P$  (Fig. 3). The torque created by the disk is

$$M = \int_{r=R_{in}}^{R_{out}} 2\pi r^2 \mu_m P dr = \frac{2}{3} \pi \mu_m P (R_{out}^3 - R_{in}^3) \quad (21)$$

where  $\mu_m$  is the mean coefficient of friction throughout the entire disk surface. The aircraft or car brake is usually equipped with  $n$  disks, so the force that decelerates the aircraft or car is given by

$$F = nC \frac{M}{R_{out}} = \frac{2}{3R_{out}} nC \pi \mu_m P (R_{out}^3 - R_{in}^3) \quad (22)$$

where  $C$  is a nondimensional coefficient dependent on the radii of the disks and pads. In Eq. (22) it is assumed that the pad radius and brake radius are identical. For an aircraft or vehicle of mass  $m$  and initial velocity  $V$ , the time required to stop is given by

$$t = \frac{Vm}{F} = \frac{3VmR_{out}}{2nC\pi\mu_m P (R_{out}^3 - R_{in}^3)} \quad (23)$$

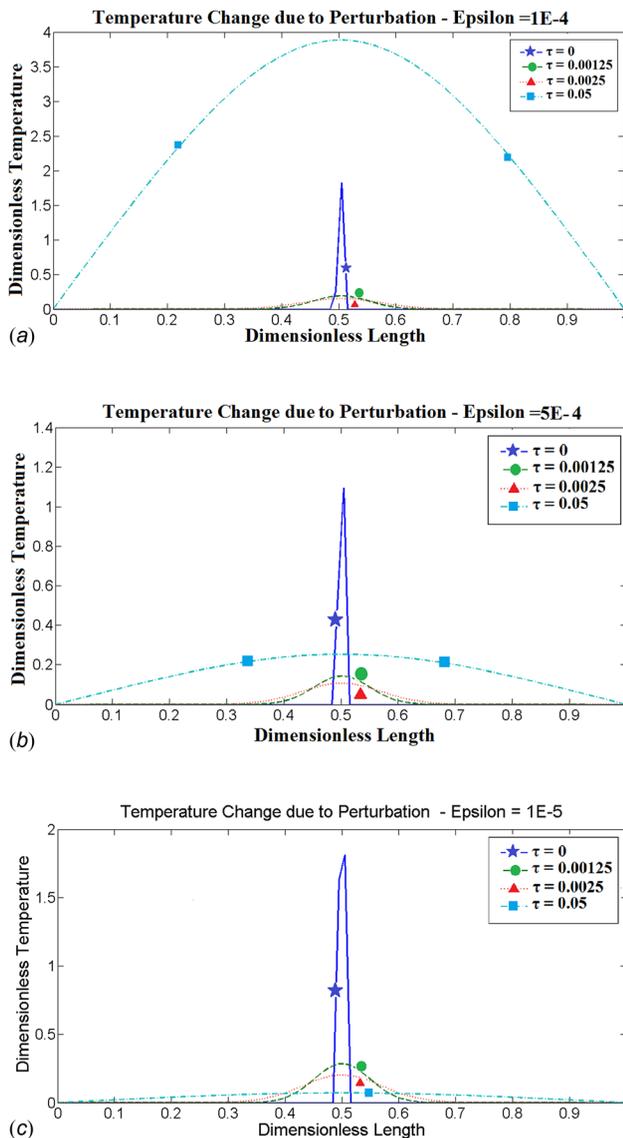


Fig. 2 Response of the system to different values of values of  $\epsilon$ : (a) 0.0001, (b) 0.00005, and (c) 0.00001.  $\tau$  is the dimensionless time.

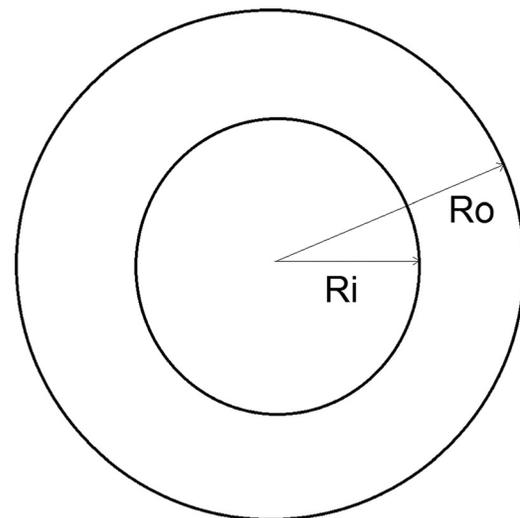


Fig. 3 Schematic of a brake disk

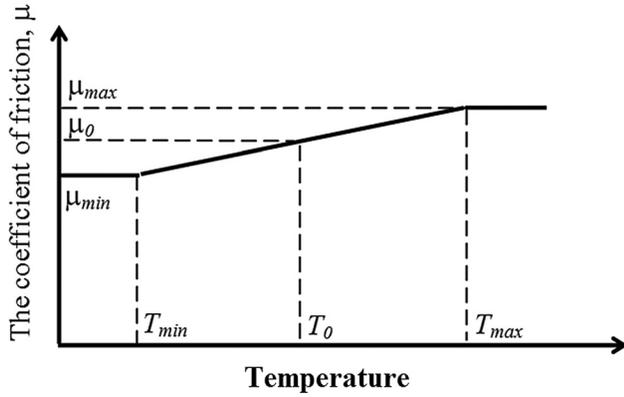


Fig. 4 Temperature dependence of the coefficient of friction

Let us assume now that the coefficient of friction depends on temperature as described by Eq. (3), and  $T_m$  is mean temperature at a certain time during transient stage so that  $\mu_m = \mu(T_m)$ . We assume a simple linear dependency given by Eq. (3) within a certain domain between the minimum and the maximum temperatures, so that the range of the coefficient of friction is  $\mu_{\min} < \mu(T) < \mu_{\max}$  (Fig. 4).

The time required to stop the aircraft is now in the range  $t_{\min} < t < t_{\max}$ , with corresponding values calculated from Eq. (23):

$$t_{\min} = \frac{3VmR_{\text{out}}}{2nC\pi\mu_{\max}P(R_{\text{out}}^3 - R_{\text{in}}^3)}$$

$$t_{\max} = \frac{3VmR_{\text{out}}}{2nC\pi\mu_{\min}P(R_{\text{out}}^3 - R_{\text{in}}^3)} \quad (24)$$

Most of the mechanical energy is converted into heat. The temperature field at the interface should satisfy the heat conduction equation, which is written here in polar coordinates:

$$\frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial T}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 T}{\partial \phi^2} + \frac{1}{wk} g(T) = \frac{1}{\alpha} \frac{\partial T}{\partial t}$$

$$g(T) = \mu \text{Pr} \omega \quad (25)$$

Stability of the solution of Eq. (17) should be analyzed now.

**3.2 Stability Analysis.** As discussed in the preceding sections, a small perturbation in the temperature field distribution (an elevated or reduced temperature) can grow unboundedly if the solution is unstable. The stability can depend on the sign of  $\lambda$  in Eq. (3). For  $\lambda > 0$ , the coefficient of friction will grow with temperature, and additional heat will be generated leading to a further increase of  $\mu$ . The heat will also be conducted to neighboring points, so the coefficient of friction at those points will grow as well, and the unstable behavior with a growing size of the “hot spot” will be observed. For  $\lambda < 0$ , quite oppositely, friction will decrease and the temperature will eventually drop to the steady-state level.

In practical conditions, the coefficient of friction can either grow or decrease with temperature. Thus, Roubicek et al. [26] observed that the coefficient of friction decreased with increasing temperature during the SAE J2430 friction test which is frequently used in the USA for evaluating the brake performance. This is attributed to physical and chemical changes in the friction layer which forms on the friction surface [25]. As it was mentioned earlier, Venkataraman and Sundararajan [16] found that the dependency of the coefficient of friction varied within different working

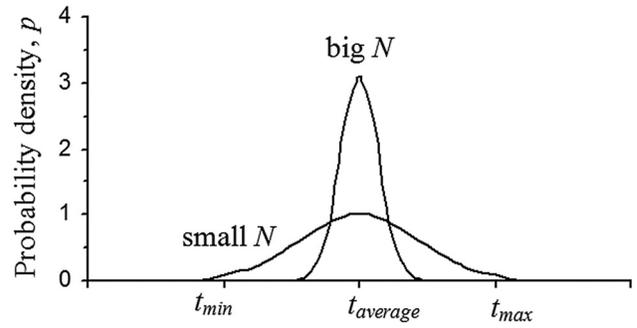


Fig. 5 Effect of the number of domains  $N$  on the reproducibility of the brake test

ranges of the sliding velocity and load. We will concentrate on the potentially unstable case of  $\lambda > 0$ .

In the case of unstable behavior, any small perturbation grows and the maximum value of the coefficient of friction  $\mu_{\max}$  (in the case of a positive perturbation) or its minimum value  $\mu_{\min}$  (in the case of a negative perturbation) will be reached within a short time. Furthermore, the hot spot will spread to neighboring regions of the surface. The area of the hot spot can be estimated as the thermal diffusivity times time  $t\alpha$ . Since the perturbation can be either positive or negative, the disk area will be divided into  $N$  domains of either the maximum or minimum coefficient of friction. The number of domains can be estimated by dividing the total area by the size of the region of perturbation:

$$N = \frac{n\pi(R_{\text{out}}^2 - R_{\text{in}}^2)}{t\alpha} \quad (26)$$

The probability distribution function for a large number of trials of equal probability (e.g., coin flips) is given by the normal distribution

$$p(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{(x - \mu_{\text{mean}})^2}{2\sigma^2}\right) \quad (27)$$

where  $\sigma$  is the standard deviation and  $\mu_{\text{mean}} = (\mu_{\max} + \mu_{\min})/2$  is the mean value. The value of  $\sigma$  is given by

$$\sigma = \frac{(t_{\max} - t_{\min})}{N} \sqrt{\frac{N}{4}} = (t_{\max} - t_{\min}) \sqrt{\frac{t\alpha}{4n\pi(R_{\text{out}}^2 - R_{\text{in}}^2)}} \quad (28)$$

in which  $t$  is defined by Eq. (23):

$$\sigma = \frac{(t_{\max} - t_{\min})}{2n\pi} \sqrt{\frac{3Vm\alpha R_{\text{out}}}{2C\mu_m P(R_{\text{out}}^2 - R_{\text{in}}^2)(R_{\text{out}}^3 - R_{\text{in}}^3)}} \quad (29)$$

For the reproducibility of the results, it is desirable that the standard deviation is as small as possible (Fig. 5). One possible way to decrease  $\sigma$  is to decrease the thermal diffusivity  $\alpha$  or to increase the total working disk area  $n\pi(R_{\text{out}}^2 - R_{\text{in}}^2)$  and the frictional traction  $\mu_m P$ . Another approach to increase the reproducibility may be texturing the disk surface so that it is artificially divided into a significant number  $N$  of domains.

**2.3 Results and Discussion.** The heat conduction equation in polar coordinates [Eq. (16)] is written in the dimensionless form

$$\frac{1}{r^*} \frac{\partial}{\partial r^*} \left( r^* \frac{\partial \theta}{\partial r^*} \right) + \frac{1}{r^{*2}} \frac{\partial^2 \theta}{\partial \phi^{*2}} + G(1 + \epsilon\theta) = \frac{\partial \theta}{\partial \tau} \quad (30)$$

where  $r^*$  and  $\phi^*$  are the dimensionless radius and angle, respectively. Moreover, we assume that the coefficient of friction

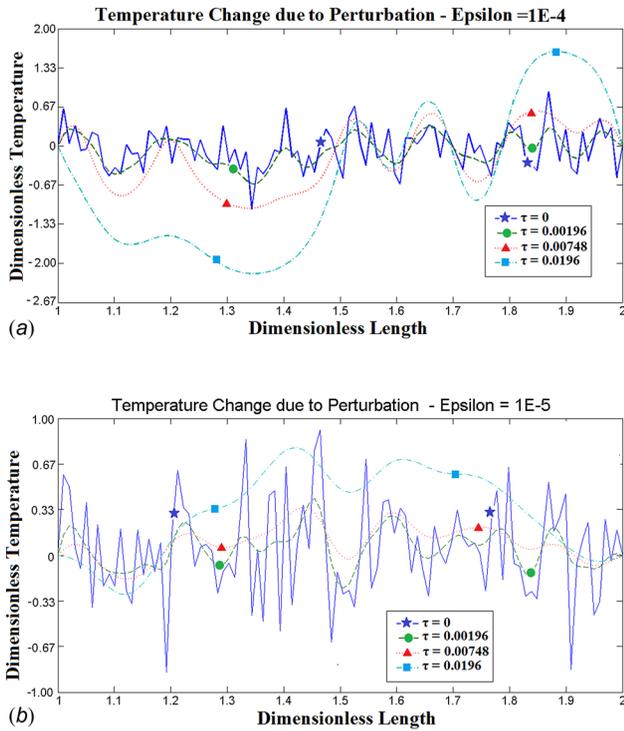


Fig. 6 Response of system to perturbation in whole domain for different values of  $\epsilon$ : (a) 0.0001 and (b) 0.00001.  $\tau$  is the dimensionless time.

depends linearly on temperature, and that the values of temperature are limited by  $T_{\min} < T < T_{\max}$ . A small perturbation of the steady solution  $\theta_s(r^*, \phi^*)$  of Eq. (30) is given by

$$\theta(r^*, \phi^*, \tau) = \theta_s(r^*, \phi^*) + \tilde{\theta}(r^*, \phi^*, \tau) \quad (31)$$

where  $\tilde{\theta}(r^*, \phi^*, \tau)$  is the small perturbation of the dimensionless temperature. Substituting Eq. (23) in Eq. (22) yields

$$\frac{1}{r^*} \frac{\partial}{\partial r^*} \left( r^* \frac{\partial \tilde{\theta}}{\partial r^*} \right) + \frac{1}{r^{*2}} \frac{\partial^2 \tilde{\theta}}{\partial \phi^{*2}} + G\epsilon \tilde{\theta} = \frac{\partial \tilde{\theta}}{\partial \tau} \quad (32)$$

The feedback loop is created due to the coupling of the temperature and the coefficient of friction, and response of system to onset of perturbation in whole domain is shown in Fig. 6. The results are presented for following parameters:  $P = 1$  MPa,  $k = 50$  W/mK,  $R_{\text{in}} = 1$  m,  $R_{\text{out}} = 2$  m,  $n = 2000$  rpm. Figure 6 clearly shows how value of  $\epsilon$  affects growing or decaying instabilities caused by perturbation in whole domain.

Then we focus our attention to reproducibility of test results. Figures 7 and 8 compare the results of response of system in two different cases, respectively: first for a rigid brake, and then for a brake divided into different sectors. For simplicity we present data in these two figures assuming  $\theta = \theta(\phi^*, \tau)$ . A random initial perturbation at every point of the disk results in initially positive perturbations, which tend to propagate and grow into the positive area, while initially negative perturbation tend to propagate and grow into the negative area (Fig. 7(a)). As a result, the disk surface after 1000 time steps of simulation was divided into several domains with the maximum and minimum values of temperature. The average value of the friction force was calculated by averaging the coefficient of friction of the entire disk and then by all time steps. The simulation was run 100 times and a histogram showing a probability distribution of the average  $\mu$  was produced (Fig. 7(b)).

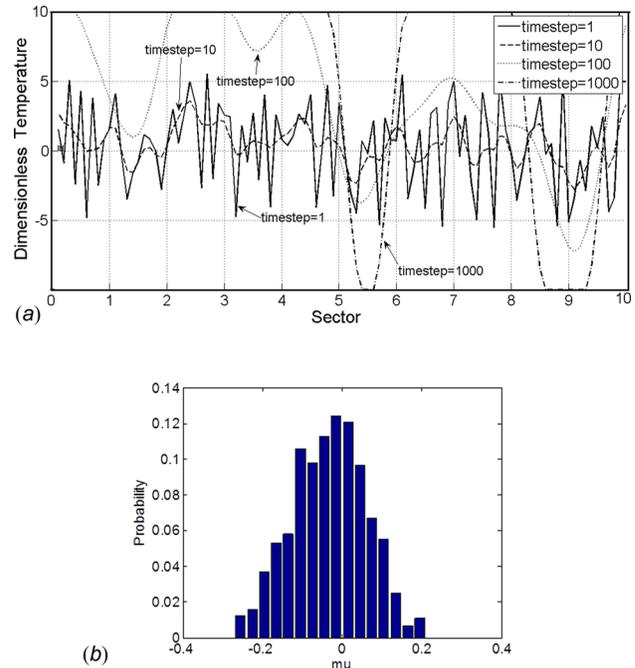


Fig. 7 Simulation results for  $T$  at 1, 10, 100, and 1000 time steps: (a) random fluctuation and (b) distribution of the average value of the coefficient of friction after 100 runs

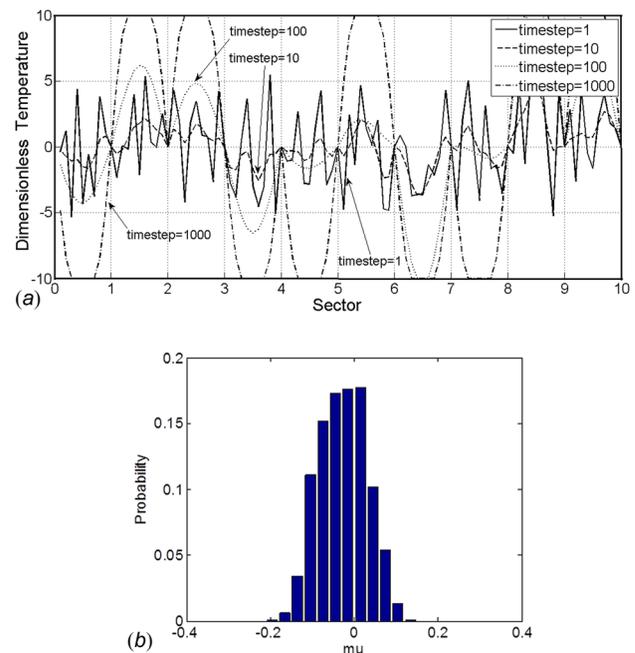


Fig. 8 Simulation results for  $T$  at 1, 10, 100, and 1000 time steps: (a) random fluctuation and (b) distribution of the average value of the coefficient of friction after 100 runs

After that, it was assumed that the disk was divided into 10 sectors in the  $\phi$  direction with zero thermal conductivity between the sectors. The same simulations were run and the results are shown in Fig. 8. It is observed that a random initial perturbation at any point results in the formation of a number of domains (identical with the sectors) with maximum or minimum temperature (Fig. 8(a)). Again the histogram showing the average value of the average  $\mu$  was produced on the basis of 100 simulation (Fig. 8(b)).

The results show that the deviation of the average  $\mu$  is much lower in this case (namely, the variance  $\sigma^2 = 0.0027$ ) than in the first case ( $\sigma^2 = 0.0086$ ), which is understandable, since due to the insulation of the sectors the instability cannot propagate throughout the entire area of the disk. It is therefore suggested that texturing the disk or dividing it into sectors can increase the reproducibility of the results.

#### 4 Conclusions

We studied the stability of frictional sliding with the temperature-dependent coefficient of friction. We presented a mathematical model without including effects of transforming layer and chemical/physical properties change with temperature, and formulated the stability condition governing whether the perturbations imposed on the surface temperature in the frictional sliding can grow or decay depending upon the working conditions such as pressure, sliding velocity, and geometry. Although it is usually ignored in most disk-pad contact models, this temperature dependency can have a significant effect on the stability. The temperature dependency of the coefficient of friction leads to the formation of hot and cold spots on the brake disks. The number of these spots or domains depends upon the thermal diffusivity of the disk material and affects the reproducibility of the brake test results. A larger number of spots is desirable for better reproducibility. It can be achieved either by decreasing the thermal diffusivity (however this approach comes in conflict with the need of high dissipation rates), by increasing the disk area, or, alternatively, by texturing the surface and dividing it artificially into domains.

#### Acknowledgment

The authors acknowledge the support of the University of Wisconsin-Milwaukee (UWM) Research Growth Initiative (RGI) grant.

#### References

- [1] Adams, G. G., and Nosonovsky, M., 2000, "Contact Modeling—Forces," *Tribol. Int.*, **33**, pp. 441–442.
- [2] Rice, J. R., Lapusta, N., and Ranjith, K., 2001, "Rate and State Dependent Friction and the Stability of Sliding Between Elastically Deformable Solids," *J. Mech. Phys. Solids*, **49**, pp. 1865–1898.
- [3] Adams, G. G., 1995, "Self-Excited Oscillations of Two Elastic Half-Spaces Sliding With a Constant Coefficient Of Friction," *ASME J. Appl. Mech.*, **62**, pp. 867–872.
- [4] Nosonovsky, M., and Adams, G. G., 2004, "Vibration and Stability of Frictional Sliding of Two Elastic Bodies With a Wavy Contact Interface," *ASME J. Appl. Mech.*, **71**, pp. 154–161.
- [5] Nosonovsky, M., and Adams, G. G., 2000, "Steady State Frictional Sliding of Two Elastic Bodies With a Wavy Contact Interface," *ASME J. Tribol.*, **122**, pp. 490–495.
- [6] Nosonovsky, M., and Bhushan, B., 2009, "Thermodynamics of Surface Degradation, Self-Organization and Self-Healing for Biomimetic Surfaces," *Philos. Trans. R. Soc. London Ser. A*, **367**, pp. 1607–1727.
- [7] Nosonovsky, M., 2010, "Self-Organization At the Frictional Interface for Green Tribology," *Philos. Trans. R. Soc. London Ser. A*, **368**, pp. 4755–4774.
- [8] Nosonovsky, M., 2010, "Entropy in Tribology: In Search of Applications," *Entropy*, **12**, pp. 1345–1390.
- [9] Mortazavi, V., and Nosonovsky, M., 2011, "Friction-Induced Pattern-Formation and Turing Systems," *Langmuir*, **27**, pp. 4772–4779.
- [10] Barber, J. R., 1969, "Thermoelastic Instabilities in the Sliding of Conforming Solids," *Proc. R. Soc. London Ser. A*, **312**, pp. 381–394.
- [11] Aferrante, L., Ciavarella, M., and Barber, J. R., 2006, "Sliding Thermoelastodynamic Instability," *Proc. R. Soc. London Ser. A*, **462**, pp. 2161–2176.
- [12] Fox-Rabinovich, G. S., and Totten, G. E. (Eds.), 2006, *Self-Organization During Friction*, CRC, Boca Raton, FL.
- [13] Lee, K., and Barber, J. L., 1993, "Frictionally Excited Thermoelastic Instability in Automotive Disk Brakes," *ASME J. Tribol.*, **115**, pp. 607–614.
- [14] Kinkaïd, N. M., O'Reilly, O. M., and Papadopoulos, P., 2003, "Automotive Disc Brake Squeal," *J. Sound Vib.*, **267**, pp. 105–166.
- [15] Paliwal, M., Mahajan, A., and Filip, P., 2003, "Analysis of High Frequency Squeal in a Disc-Brake System Using a Stick-Slip Friction Model," ASME 2003 International Mechanical Engineering Congress and Exposition, Dynamic Systems and Control, Vols. 1 and 2.
- [16] Venkataraman, B., and Sundararajan, G., 2002, "The Influence of Sample Geometry on the Friction Behaviour of Carbon-Carbon Composites," *Acta Mater.*, **50**, pp. 1153–1163.
- [17] Stimson, I. L., and Fisher, R., 1980, "Design and Engineering of Carbon Brakes," *Philos. Trans. R. Soc. London Ser. A*, **294**, pp. 583–590.
- [18] Byrne, C., 2004, "Modern Carbon Composite Brake Materials," *J. Compos. Mater.*, **38**, pp. 1837–1850.
- [19] Gomes, J. R., Silva, O. M., Silva, C. M., Pardini, L. C., and Silva, R. S., 2001, "The Effect of Sliding Speed and Temperature on the Tribological Behaviour of Carbon-Carbon Composites," *Wear*, **249**, pp. 240–245.
- [20] Luo, R., Liu, T., Li, J., Zhang, H., Chen, Z., and Tian, G., 2004, "Thermophysical Properties of Carbon/Carbon Composites and Physical Mechanism of Thermal Expansion and Thermal Conductivity," *Carbon*, **42**, pp. 2887–2895.
- [21] Yuan, Y., Luo, R., Zhang, F., Li, J., Liu, T., and Zhou, W., 2005, "Influence of High Temperature Heat Treatment on the Friction Properties of Carbon/Carbon Composites Under Wet Conditions," *Mater. Sci. Eng. A*, **402**, pp. 203–207.
- [22] Kasem, H., Bonnamy, S., Rousseau, B., Estrade-Szwarckopf, H., Berthier, Y., and Jacquemard, P., 2007, "Interdependence Between Wear Process, Size of Detached Particles and CO<sub>2</sub> Production During Carbon/Carbon Composite Friction," *Wear*, **263**, pp. 1220–1229.
- [23] Yen, B. K., and Ishihara, T., 1994, "The Surface Morphology and Structure of Carbon-Carbon Composites in High-Energy Sliding Contact," *Wear*, **174**, pp. 111–117.
- [24] Nosonovsky, M., and Bhushan, B., 2005, "Scale Effect in Dry Friction During Multiple-Asperity Contact," *ASME J. Tribol.*, **127**, pp. 37–46.
- [25] Filip, P., Weiss, Z., and Rafaja, D., 2002, "On Friction Layer Formation in Polymer Matrix Composite Materials for Brake Applications," *Wear*, **252**, pp. 189–198.
- [26] Roubicek, V., Raclavska, H., Juchelkova, D., and Filip, P., 2008, "Wear and Environmental Aspects of Composite Materials for Automotive Braking Industry," *Wear*, **265**, pp. 167–175.
- [27] ÖzisMik, M. N., 1993, *Heat Conduction*, Wiley-IEEE, New York.
- [28] Zhaoa, S., Hilmas, G. E., and Dharani, L. R., 2008, "Behavior of a Composite Multidisk Clutch Subjected to Mechanical and Frictionally Excited Thermal Load," *Wear*, **264**, pp. 1059–1068.