Vibration and Stability of Frictional Sliding of Two Elastic Bodies With a Wavy Contact Interface

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The stability of steady sliding, with Amontons-Coulomb friction, of two elastic bodies with a rough contact interface is analyzed. The bodies are modeled as elastic half-spaces, one of which has a periodic wavy surface. The steady-state solution yields a periodic set of contact and separation zones, but the stability analysis requires consideration of dynamic effects. By considering a spatial Fourier decomposition of the vibration modes, the dynamic problem is reduced to a singular integral equation for determining the eigenvectors (modes) and eigenvalues (frequencies). A pure imaginary root for an eigenvalue corresponds to a standing wave confined to the interface, while a positive/negative real part of the eigenvalue indicates instability/dissipation. A complex eigenvector indicates a complex mode of vibration. Two types of modes are considered—periodic symmetric modes with period equal to the surface waviness period and periodic antisymmetric modes with the period equal to twice the surface waviness. The singular integral equation is solved by reducing it to a system of linear algebraic equations using a Jacobi polynomial series and a collocation method. For the limit of zero friction it can be demonstrated analytically that the problem is self-adjoint and the eigenvalues, if they exist, are pure imaginary (no energy dissipation). These roots are found for a wide range of material properties and ratios of separation to contact zones lengths. For the limiting case of complete contact, the solution found corresponds to a superposition of two slip waves (generalized Rayleigh waves) traveling in opposite directions and forming a standing wave. With increasing separation zone length, the vibration frequency decreases from the slip wave frequency to the smaller surface wave frequency of the two bodies. With a nonzero separation zone, solutions can exist for material combinations which do not allow slip waves. For nonzero friction and sliding velocities, unstable solutions are found. The degree of instability is proportional to the product of the friction coefficient and the sliding velocity. These instabilities may contribute to the formation of friction-induced vibrations at high sliding speeds. [DOI: 10.1115/1.1653684]

1 Introduction

The vibration and stability of dry frictional sliding of two elastic bodies with a wavy contact interface is the subject of this investigation. It is well known that vibrations, which are usually undesirable in technical applications, can occur during frictional sliding. These friction-induced vibrations are a result of the instability of steady-state sliding, which is usually attributed to either the difference between the static and kinetic friction coefficients, to a decrease of the kinetic friction coefficient with increasing velocity, or to a spatial variation of the friction coefficient along the interface.

In the past years the influence of elastodynamic phenomena on friction has been intensely investigated. Adams [1] showed that the sliding of two flat elastic half-spaces with a constant coefficient of friction has dynamically unstable solutions for a wide range of material parameter combinations, coefficients of friction, and sliding velocity (including very small speeds). These self-excited vibrations are in the form of interface waves, which are confined to the interface region and have amplitudes which decrease exponentially with distance from the interface, while increasing exponentially with time. These waves can contribute to the formation of friction-induced vibrations. The rate of amplitude increase is proportional to the wave number, which causes infinitesimal wavelengths to increase at an unlimited rate. Thus the propagation of an arbitrary pulse is ill-posed as was found by Renardy [2] for sliding of an elastic solid against a rigid substrate with a sufficiently high friction coefficient. Simões and Martins [3] investigated the effect of introducing an intrinsic length scale into the problem and found a method of regularization of the ill-posedness by using a nonlocal friction law.

Ranjith and Rice [4] analyzed frictional sliding of two elastic half-spaces in the small velocity limit. If a generalized Rayleigh wave exists at the interface of the two bodies for frictionless contact, steady sliding with arbitrary small friction becomes unstable. Generalized Rayleigh waves, also known as slip waves or smooth contact Stoneley waves, exist when the material mismatch is not very high and they become equivalent to Rayleigh surface waves in the limiting case of identical materials (Achenbach and Epstein [5]). Ranjith and Rice [4] showed that if a memory-dependent rate-and-state friction law is considered instead of the instantaneous Coulomb's law, the pulse-propagation problem becomes well-posed, although the solution is still unstable.

The effect of surface roughness on the sliding of two elastic bodies was considered by Adams [6]. He analyzed a simple model consisting of a beam on elastic foundation acted upon by a series

of moving elastic springs, which represented the asperities. The solution was found to be dynamically unstable for any finite speed with the degree of instability increasing with increasing speed.

It is well known that during dry frictional contact, the true contact area is significantly less than the apparent contact area because a nominally flat surface is rough and has asperities. The contact takes place only on the peaks of the asperities. Many models have been developed starting in the early 1960s, which consider different laws of asperity distributions, either statistical or fractal, and provide theoretical results which justify the linear Coulomb friction law (for a review see Adams and Nosonovsky [7]). These statistical models are usually uncoupled in the sense that the contribution of each individual asperity is calculated separately, with no asperity interaction through the bulk of the body. On the other hand, coupled models deal with elastic deformation of the body as a whole, but due to the complexity of the problems only simple contact profiles have been considered. One of the simplest models of this type is a two-dimensional problem with a sinusoidal periodic profile; the peaks of the sinusoid represent asperities.

The static frictionless two-dimensional contact problem with a harmonic contact profile is well known in the literature. It was considered for the first time by Westergaard [8] who used the complex stress function method. Independently the same problem was solved by other researchers who used different mathematical techniques, including the Green's function and a singular integral equation formulation (Shtaerman [9]), the stress function approach and Fourier series analysis (Dundurs, Tsai, and Keer [10]), and the complex potential method of Muskhelishvili (Kuznetsov [11]). The frictional quasistatic sliding problem with one rigid body was studied by Kuznetsov [12]. Nosonovsky and Adams [13] analyzed the frictional problem for the general case of two elastic materials and any subsonic sliding velocity. They obtained the dependencies of the contact area on the remotely applied pressure for different friction coefficients and sliding velocities. The authors also indicated that their work formed the foundation for a future stability analysis.

The present paper analyzes the stability of the steady sliding solution obtained by Nosonovsky and Adams [13]. For the limiting case of complete contact with friction the problem reduces to the one investigated by Adams [1], for which the instability is in the form of traveling interface waves, growing in amplitude with time. For incomplete contact a solution in the form of complex modes of vibration is sought. In the frictionless quasistatic limit this solution, if it exists, describes normal modes of vibration which are localized near the interface. With increasing sliding velocity and friction coefficient, these vibration modes become complex and the solution can become unstable. The algebraic calculations are quite complicated and so a symbolic interpreter language (Mathematica, [14]) has been used extensively throughout this investigation.

These results have relevance in furthering our understanding of sliding friction as well as in the design and analysis of lip seals, such as the synthetic rubber seals used extensively in automobiles and other devices. Some models of lip seals include sinusoidal micro-undulations and micro-asperities (Salant and Flaherty [15]). A typical value of Young's modulus is 10 MPa, [15], which gives a Rayleigh wave speed of about 100 m/s. Hence the sliding velocity need not be extremely high in order to be a significant fraction of the Rayleigh wave speed.

2 Formulation of the Problem

Consider the plane-strain sliding frictional contact of two semi-infinite elastic bodies, one of which is flat and the other of which has a slightly wavy surface. The bodies are pressed together by a uniform normal traction $\hat{p}$ and slide under a uniform tangential traction $\mu\hat{p}$ applied at infinity (Fig. 1). The contact regions are determined from the solution of the steady-state problem, [13], whereas small vibrations near the steady-state solution will be
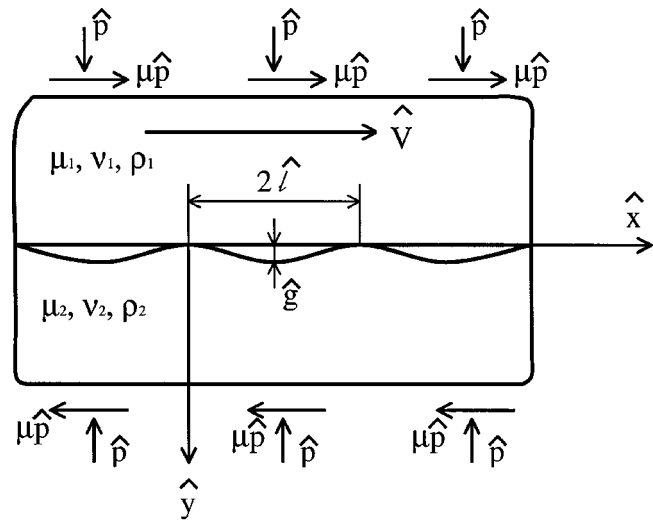

Fig. 1 Sliding of a flat elastic half-space upon a wavy elastic half-space

considered here. The coefficient of kinetic friction between the two bodies is μ. The lower surface is assumed to have a purely sinusoidal boundary at $\hat{y}=0$ with a period of $2\hat{l}$ and peak-to-valley amplitude $\hat{g}$, i.e.,

$$\hat{y}(\hat{x}) = \frac{\hat{g}}{2}\left(1 - \cos\frac{\pi(\hat{x}+\hat{E})}{\hat{l}}\right), \tag{1}$$

where $\hat{x}$ and $\hat{y}$ are the coordinates attached to the lower body and $\hat{E}$ will be defined later. The upper body is moving to the right with constant velocity $\hat{V}$. Now the dimensionless coordinates and parameters

$$x = \frac{\pi\hat{x}}{\hat{l}}, \quad y = \frac{\pi\hat{y}}{\hat{l}}, \quad g = \frac{\pi\hat{g}}{\hat{l}}, \quad E = \frac{\pi\hat{E}}{\hat{l}},$$

$$V = \hat{V}\sqrt{\frac{\rho_1}{\mu_1}}, \quad t = \frac{\pi\hat{t}}{\hat{l}}\sqrt{\frac{\mu_1}{\rho_1}} \tag{2}$$

are defined, where ρ_1 and ρ_2 are the densities of the two bodies, and μ_1 and μ_2 are their elastic shear moduli.

The Navier equations for the dimensionless displacements $u(x,y,t)$, $v(x,y,t)$ in the x and y-directions, respectively, are given by

$$(\beta_1^2-V^2)u_{,xx}^{(1)}+u_{,yy}^{(1)}+(\beta_1^2-1)v_{,xy}^{(1)}=\frac{\partial^2 u^{(1)}}{\partial t^2}-2V\frac{\partial^2 u^{(1)}}{\partial x\partial t}$$

$$\beta_1^2 v_{,yy}^{(1)}+(1-V^2)v_{,xx}^{(1)}+(\beta_1^2-1)u_{,xy}^{(1)}=\frac{\partial^2 v^{(1)}}{\partial t^2}-2V\frac{\partial^2 v^{(1)}}{\partial x\partial t}$$

$$\beta_2^2 u_{,xx}^{(2)}+u_{,yy}^{(2)}+(\beta_2^2-1)v_{,xy}^{(2)}=\kappa^2\frac{\partial^2 u^{(2)}}{\partial t^2}$$

$$\beta_2^2 v_{,yy}^{(2)}+v_{,xx}^{(2)}+(\beta_2^2-1)u_{,xy}^{(2)}=\kappa^2\frac{\partial^2 v^{(2)}}{\partial t^2} \tag{3}$$

where the shear wave speed ratio is κ and the ratios of longitudinal (C_L) and shear (C_S) wave speeds are related to the Poisson's ratios of the two materials ν_1 and ν_2 according to

$$\beta_k=\left(\frac{C_L}{C_S}\right)_k=\sqrt{\frac{2(1-\nu_k)}{1-2\nu_k}}, \quad \kappa^2=\left(\frac{C_S^{(1)}}{C_S^{(2)}}\right)^2=\frac{\mu_1\rho_2}{\mu_2\rho_1}, \tag{4}$$

in which $k=1$ for the upper body and $k=2$ for the lower body. Also note that indices $^{(1)}$ and $^{(2)}$ are related to the upper and the lower bodies, respectively.

The deformed state of the bodies is considered as the superposition of uniform stresses (caused by the dimensionless applied pressure $p=\hat{p}/\mu_1$ and dimensionless tangential traction μp) and the dimensionless residual stresses ($\sigma_{xx},\sigma_{xy},\sigma_{yy}$). The dimensionless residual stresses are related to the dimensional residual stresses according to

$$\sigma_{ij}^{(k)}=\frac{\hat{\sigma}_{ij}^{(k)}}{\mu_k} \tag{5}$$

and vanish at infinity. The dimensionless stresses are related to the dimensionless deformations by

$$\sigma_{yy}^{(k)}=(\beta_k^2-2)u_{,x}^{(k)}+\beta_k^2 v_{,y}^{(k)}$$
$$\sigma_{xy}^{(k)}=u_{,y}^{(k)}+v_{,x}^{(k)} . \tag{6}$$

The boundary conditions at infinity ($|y|=\infty$) state that stresses should be equal to the applied tractions. On the contact surface ($y=0$) there are two kinds of boundary conditions—nonmixed and mixed boundary conditions. The nonmixed conditions are valid on the entire surface:

$$\sigma_{xy}^{(1)}=\mu\sigma_{yy}^{(1)}, \quad -\infty<x<\infty, \tag{7}$$

$$\sigma_{xy}^{(2)}=\mu\sigma_{yy}^{(2)}, \quad -\infty<x<\infty, \tag{8}$$

$$\mu_1\sigma_{yy}^{(1)}=\mu_2\sigma_{yy}^{(2)}, \quad -\infty<x<\infty. \tag{9}$$

The shear moduli appear in (9) due to the manner in which the stresses were nondimensionalized (5). The mixed boundary conditions are satisfied only in the separation zone:

$$\sigma_{yy}^{(1)}=p, \quad c<|x|<\pi \tag{10}$$

and in the contact zone:

$$\frac{\partial v^{(1)}}{\partial x}-\frac{\partial v^{(2)}}{\partial x}=\frac{g}{2}\sin(x+E), \quad -c<x<c. \tag{11}$$

Here x is equal to zero at the center of the contact zone. Thus the parameter E is the coordinate distance representing the peak of the surface waviness relative to the center of the contact zone.

Suppose that the steady state solution is given by $u_0^{(1)}(x,y)$, $v_0^{(1)}(x,y)$, $u_0^{(2)}(x,y)$, $v_0^{(2)}(x,y)$ with the length of the contact zone $2c$ and the eccentricity E. In order to analyze the stability of the steady-state solution, small vibrations near the steady-state solution are considered. The complete solution $\bar{u}^{(k)}$, $\bar{v}^{(k)}$ of (3) is a superposition of the steady-state solution and the small vibrations, i.e.,

$$\bar{u}^{(k)}(x,y,t)=u_0^{(k)}(x,y)+u^{(k)}(x,y,t),$$
$$\bar{v}^{(k)}(x,y,t)=v_0^{(k)}(x,y)+v^{(k)}(x,y,t).$$

In order to satisfy the Coulomb friction inequalities, it is required that the sliding velocity V always be greater than the local x-component of the small vibration relative velocity at the interface, i.e.,

$$\frac{\partial u^{(2)}(x,0,t)}{\partial t}-\frac{\partial u^{(1)}(x,0,t)}{\partial t}<V. \tag{12}$$

For small vibrations the nonmixed boundary conditions (7)–(9) remain in the same form, whereas the mixed boundary conditions (10), (11) are reduced to

$$\sigma_{yy}^{(1)}=0, \quad c<|x|<\pi \tag{13}$$

$$\frac{\partial v^{(1)}}{\partial x}-\frac{\partial v^{(2)}}{\partial x}=0, \quad |x|<c. \tag{14}$$

3 Formulation in Integral Equation Form

Any initial small perturbation can be represented as a superposition of the modes $u_k(x,y)$, $v_k(x,y)$ which correspond to the eigenvalues Λ_k, i.e.,

$$u(x,y,t)=\text{Re}\sum_{k=1}^{\infty} C_k u_k(x,y)e^{\Lambda_k t}$$

$$v(x,y,t)=\text{Re}\sum_{k=1}^{\infty} D_k v_k(x,y)e^{\Lambda_k t}. \tag{15}$$

A stable solution must have the real parts of all Λ_k negative or zero.

The modes are sought in the form

$$u^{(1)}(x,y)=\sum_{n=-\infty}^{\infty} A_n(e^{s_1 y}+\delta_3 e^{s_2 y})e^{inx/m}$$

$$v^{(1)}(x,y)=\sum_{n=-\infty}^{\infty} A_n(\delta_1 e^{s_1 y}+\delta_2\delta_3 e^{s_2 y})e^{inx/m} \tag{16}$$

for the upper body and

$$u^{(2)}(x,y)=\sum_{n=-\infty}^{\infty} B_n(e^{s_1' y}+\delta_3' e^{s_2' y})e^{inx/m}$$

$$v^{(2)}(x,y)=\sum_{n=-\infty}^{\infty} B_n(\delta_1' e^{s_1' y}+\delta_2'\delta_3' e^{s_2' y})e^{inx/m} \tag{17}$$

for the lower body, where $m=1$ for a mode with a period of 2π and $m=2$ for an antisymmetric mode $v(x+2\pi,y)=-v(x,y)$ with a period of 4π. The terms which represent the translation of the system as a rigid body can be set to zero ($A_0=0$ and $B_0=0$) without loss of generality. Note that the subscript k has been omitted for conciseness.

In order to satisfy the Navier Eqs. (3), s_1 and s_2 in (16), (17) must be roots of the characteristic equation

$$\beta_1^2 s^4+[Q_1\beta_1^2+Q_2+(n/m)^2(\beta_1^2-1)^2]s^2+Q_1 Q_2=0. \tag{18}$$

The solution of (18) is

$$s_1=\pm\sqrt{-Q_2}, \quad s_2=\pm\sqrt{-Q_1/\beta_1^2}, \tag{19}$$

where

$$Q_1=2(n/m)iV\Lambda-\Lambda^2-(n/m)^2(\beta_1^2-V^2),$$

$$Q_2=2(n/m)iV\Lambda-\Lambda^2-(n/m)^2(1-V^2). \tag{20}$$

Since the solution must be bounded at infinity, among the four roots (19) only the two roots with positive real parts are considered for the upper body. The Navier equations further require

$$\delta_1=i\frac{s_1^2+Q_1}{(n/m)(\beta_1^2-1)s_1}=-i\frac{n}{ms_1}, \tag{21}$$

$$\delta_2=i\frac{s_2^2+Q_1}{(n/m)(\beta_1^2-1)s_2}=-i\frac{ms_2}{n}, \tag{22}$$

while A_n and δ_3 are unknown coefficients which will be determined from the boundary conditions.

The displacement field (16) produces stresses on the contact surface given by

$$\sigma_{yy}^{(1)}=\sum_{n=-\infty}^{\infty} A_n[(1+\delta_3)i(n/m)(\beta_1^2-2)$$

$$+\beta_1^2(s_2\delta_1+s_2\delta_3\delta_2)]e^{inx/m}, \quad -\infty<x<\infty,$$

$$\sigma_{xy}^{(1)} = \sum_{n=-\infty}^{\infty} A_n[(s_1 + \delta_1 i(n/m))$$

$$+ (\delta_3 s_2 + \delta_3 \delta_2 i(n/m))]e^{inx/m}, \quad -\infty < x < \infty. \quad (23)$$

Coulomb's friction law (7) on the surface requires

$$\delta_3 = -\frac{\mu i(n/m)(\beta_1^2 - 2) + \mu \beta_1^2 s_1 \delta_1 - (s_1 + \delta_1 i(n/m))}{\mu i(n/m)(\beta_1^2 - 2) + \mu \beta_1^2 s_2 \delta_2 - (s_2 + \delta_2 i(n/m))}. \quad (24)$$

Similar relations can also be written for the lower body; the corresponding variables are marked with a prime (′), i.e.,

$$Q_1' = -\Lambda^2/\kappa^2 - (n/m)^2 \beta_2^2, \quad Q_2' = -\Lambda^2/\kappa^2 - (n/m)^2. \quad (25)$$

$$\delta_1' = i \frac{s_1'^2 + Q_1'}{(n/m)(\beta_2^2 - 1)s_1'} \quad (26)$$

$$\delta_2' = i \frac{s_2'^2 + Q_1'}{(n/m)(\beta_2^2 - 1)s_2'}. \quad (27)$$

$$\delta_3' = -\frac{\mu i(n/m)(\beta_2^2 - 2) + \mu \beta_2^2 s_1' \delta_1' - (s_1' + \delta_1' i(n/m))}{\mu i(n/m)(\beta_2^2 - 2) + \mu \beta_2^2 s_2' \delta_2' - (s_2' + \delta_2' i(n/m))}. \quad (28)$$

The real part of s_1' and s_2' must be negative for the solution to be bounded at infinity.

The condition of continuity (9) of σ_{yy} on the surface gives

$$B_n = \delta_4 A_n,$$

$$\delta_4 = \left(\frac{\mu_1}{\mu_2}\right) \frac{(1 + \delta_3)(\beta_1^2 - 2)i(n/m) + \beta_1^2(s_1 \delta_1 + s_2 \delta_3 \delta_2)}{(1 + \delta_3')(\beta_2^2 - 2)i(n/m) + \beta_2^2(s_1' \delta_1' + s_2' \delta_3' \delta_2')}. \quad (29)$$

Summarizing, the nonmixed boundary conditions have been used in order to determine all the unknown coefficients except for A_n.

The two mixed boundary conditions (13) and (14) now yield

$$\sum_{n=-\infty}^{\infty} [\delta_1 + \delta_2 \delta_3 - \delta_4(\delta_1' + \delta_2' \delta_3')]i(n/m)A_n e^{inx/m} = 0, \quad |x| < c \quad (30)$$

$$\sum_{n=-\infty}^{\infty} [(1 + \delta_3)(\beta_1^2 - 2)i(n/m) + \beta_1^2(s_1 \delta_1 + s_2 \delta_2 \delta_3)]A_n e^{inx/m}$$

$$= 0, \quad c < |x| < \pi. \quad (31)$$

Note that for $n = 1$ the vanishing of the term in brackets of Eq. (30) constitutes the slip wave equation for a wavelength of $2m\pi$, while the vanishing of the term in brackets of Eq. (31) constitutes the Rayleigh wave equation for the upper body. The parameter δ_4 is undefined at the Rayleigh wave speed of the upper body. Now in order to satisfy (30) and (31), the unknown function $\Phi(\xi)$ is introduced such that

$$[(1 + \delta_3)(\beta_1^2 - 2)i(n/m) + \beta_1^2(s_1 \delta_1 + s_2 \delta_2 \delta_3)]A_n$$

$$= \int_C \Phi(\xi)e^{-in\xi}d\xi. \quad (32)$$

The integration is performed in the contact zone C ($-c < \xi < c$, for $m = 1$), ($-c < \xi < c$, $-c < \xi + 2\pi < c$ for $m = 2$). Then with the use of the identities for generalized functions (Gel'fand and Shilov [16])

$$\sum_{n=1}^{\infty} \cos(nx) = -\frac{1}{2} + \pi \delta(x), \quad |x| < \pi, \quad (33)$$

$$\sum_{n=1}^{\infty} \sin(nx) = \frac{1}{2} \cot \frac{x}{2} \quad (34)$$

Eq. (31) becomes

$$\int_C \Phi(\xi) \sum_{n=-\infty}^{\infty} e^{in(x-\xi)/m}d\xi = \int_C \Phi(\xi)2m\pi\delta(x-\xi)d\xi$$

$$= 0, \quad x \notin C, \quad (35)$$

which is satisfied automatically in the separation zone. Furthermore (30) yields a homogeneous integral equation for $\Phi(\xi)$ given by

$$\int_C \Phi(\xi) \sum_{n=-\infty}^{\infty} K_n e^{in(x-\xi)/m}d\xi = 0, \quad x \in C, \quad (36)$$

where

$$K_n = \frac{\delta_1 + \delta_2 \delta_3 - \delta_4(\delta_1' + \delta_2' \delta_3')}{(1 + \delta_3)(\beta_1^2 - 2) + \beta_1^2(s_1 \delta_1 + s_2 \delta_2 \delta_3)\dfrac{m}{in}}. \quad (37)$$

Integral Eq. (36) is singular because the summation in its kernel diverges as $x \to \xi$. In order to solve Eq. (36), K_n is decomposed into two terms:

$$K_n = K^{\infty} + k_n^{rem}, \quad n > 0,$$

$$K_n = K^{-\infty} + k_n^{rem}, \quad n < 0,$$

$$K_0 = k_0^{rem} = 0, \quad (38)$$

where

$$K^{\infty} = K_r^{\infty} + iK_i^{\infty} = \lim_{n \to \infty} K_n,$$

$$K^{-\infty} = K_r^{\infty} - iK_i^{\infty} = \lim_{n \to -\infty} K_n,$$

in which K_r^{∞} and K_i^{∞} are the real and imaginary parts of K^{∞}, and k^{rem} is the remaining part.

Equation (36) now can be written for $m = 1$ as

$$2\pi\Phi(x)K_r^{\infty} + \int_{-c}^{c} \Phi(\xi)\left\{-K_r^{\infty} - K_i^{\infty}\cot\frac{x-\xi}{2} + k(x-\xi)\right\}d\xi$$

$$= 0, \quad |x| < c, \quad (39)$$

where

$$k(x-\xi) = \sum_{n=-\infty}^{\infty} k_n^{rem}e^{in(x-\xi)}, \quad m = 1. \quad (40)$$

For $m = 2$ Eq. (36) yields

$$4\pi\Phi(x)K_r^{\infty} + \int_{-c}^{c} \Phi(\xi)\left\{-K_i^{\infty}\cot\frac{x-\xi}{4} - K_i^{\infty}\tan\frac{x-\xi}{4}\right.$$

$$\left. + k(x-\xi)\right\}d\xi = 0, \quad |x| < c, \quad (41)$$

where

$$k(x-\xi)=\sum_{n=-\infty}^{\infty}k_n^{rem}e^{in(x-\xi)/2}(1-(-1)^n), \quad m=2 \quad (42)$$

In the derivation of (39)–(42) the properties of $\Phi(x)$, namely, the antisymmetry $\Phi(x+2\pi)=-\Phi(x)$ for $m=2$, and the periodicity $\Phi(x+2\pi)=\Phi(x)$ for $m=1$ have been utilized, as well as double angle formulas for the trigonometric functions. Note that the natural frequencies of vibration for complete contact are given by the roots of

$$K_n=0, \quad m=1,2, \quad n=1,2,3..., \quad (43)$$

where $m=1$ corresponds to the symmetric modes and $m=2$ corresponds to the antisymmetric modes.

Making the change of variables

$$x\rightarrow cx, \quad \xi\rightarrow c\xi, \quad \Phi(cx)=\phi(x) \quad (44)$$

yields for $m=1$

$$\frac{2\pi}{c}\phi(x)K_r^\infty + \int_{-1}^{1}\phi(\xi)\left\{-K_r^\infty - K_i^\infty\cot\frac{c(x-\xi)}{2}\right.$$
$$\left. + k(c(x-\xi))\right\}d\xi=0, \quad |x|<1 \quad (45)$$

while for $m=2$

$$\frac{4\pi}{c}\phi(x)K_r^\infty + \int_{-1}^{1}\phi(\xi)\left\{-K_i^\infty\cot\frac{c(x-\xi)}{4} - K_i^\infty\tan\frac{c(x-\xi)}{4}\right.$$
$$\left. + k(c(x-\xi))\right\}d\xi=0, \quad |x|<1. \quad (46)$$

Note that Eqs. (45), (46) govern the *stability* of the steady-state solution which, in turn, was obtained from the equation

$$\frac{2\pi}{c}\phi(x)K_r^\infty + \int_{-1}^{1}\phi(\xi)\left\{-K_i^\infty\cot\frac{c(x-\xi)}{2}\right\}d\xi=f(x), \quad |x|<1,$$
$$(47)$$

where $f(x)$ describes the slope of the interface profile, which is sinusoidal for this problem, [13]. The stability analysis is independent of the particular form of the profile $f(x)$ and uses only the length of the contact zone c obtained from the steady-state solution of (47).

The Eq. (47), which was solved by Kuznetsov [12] for the particular case of one rigid and one elastic body and low sliding velocities, governs several important problems of contact elasticity, such as propagating intersonic stick-slip regions (Adams [17]) and ultrasonic motors (Zhari [18]). The algorithm of stability analysis which is considered in this work can be applied to the stability analysis of steady-state problems governed by Eq. (47), whereas the particular form of the function $k(c(x-\xi))$ is different for each problem.

4 Special Solutions for Limiting and Resonance Cases

4.1 Frictionless Case. Simplifications can be made for a number of special cases. Let us first consider the frictionless static case ($\mu=0$, $V=0$). It is possible to show that the roots are pure imaginary (no energy is dissipated, $\Lambda=i\lambda$ where λ is real) for this case and that the eigenvalue problem is self-adjoint.

For pure imaginary roots Eqs. (19)–(29) simplify to

$$Q_1=\lambda^2-(n/m)^2\beta_1^2, \quad Q_2=\lambda^2-(n/m)^2, \quad (48)$$

$$s_1'=-\sqrt{(n/m)^2-\lambda^2/\kappa^2}, \quad s_2'=-\sqrt{(n/m)^2-\lambda^2/(\kappa^2\beta_1^2)}$$

$$s_1=\sqrt{(n/m)^2-\lambda^2}, \quad s_2=\sqrt{(n/m)^2-\lambda^2/\beta_1^2}, \quad (49)$$

where

$$\Lambda=\pm i\lambda. \quad (50)$$

Equations (24) and (28) yield

$$\delta_3=\frac{s_1^2+(n/m)^2}{2s_1s_2}, \quad (51)$$

$$\delta_3'=\frac{s_1'^2+(n/m)^2}{2s_1's_2'}, \quad (52)$$

For the limit of $n\rightarrow\infty$

$$K_\infty=K_i^\infty, \quad K_r=0, \quad (53)$$

and

$$k(x-\xi)=2i\sum_{n=1}^{\infty}(K_n-K_i^\infty)\sin[n(x-\xi)]. \quad (54)$$

Note that Q_1, Q_2, Q_1', Q_2', s_1, s_2, s_1', s_2', δ_3, δ_3', δ_5 are pure real, while δ_1, δ_1', δ_2, δ_2' are pure imaginary. All the quantities remain the same when the sign of λ is reversed.

4.2 Small Separation Zone. For the case of a vanishingly small separation zone ($c\rightarrow\pi$) Eq. (31) must be satisfied only at a single separation point $|x|=\pi$ whereas Eq. (30) has a solution in the form of

$$A_{\pm1}\neq0,$$

$$A_{\pm n}=0, \quad n>1. \quad (55)$$

This is the solution for slip waves (generalized Rayleigh waves) with two waves of the same amplitude traveling in opposite directions and forming a standing slip wave. To satisfy (31) at $|x|=\pi$ the phase of the standing slip wave must be chosen that gives zero normal stress at the separation points.

4.3 Small Contact Zone. For a small contact zone ($c\rightarrow0$) Eq. (30) must be satisfied only for the contact points $x=2\pi n$. A resonance type of solution (55) in the form of standing Rayleigh waves in each body, which are independent of each other, exists and satisfies (31) only if the Rayleigh wave speeds of the two bodies $C_R^{(1)}$ and $C_R^{(2)}$ are related in the following manner:

$$pC_R^{(1)}=qC_R^{(2)}, \quad p,q=1,2,3,\ldots. \quad (56)$$

In this case standing Rayleigh waves with wavelengths of $2\pi/p$ and $2\pi/q$ can exist in the two bodies and their nodes will coincide with each other and with the contact points.

In the general case when the Rayleigh wave speed ratio does not permit such a resonance, a standing Rayleigh wave can exist in only one body with node points at the points of contact. It must be stressed that although mathematically it is possible to consider any small value of c, physically the displacements caused by the small vibrations must by much smaller than those of the steady-state solution and should not cause any sufficient change of the contact zone length c. In the limiting case of $c\rightarrow0$ this condition cannot be satisfied.

4.4 Equal Rayleigh Wave Speeds. In the case of equal Rayleigh wave speeds for the two materials in contact, the bracketed term on the left-hand side of (32) vanishes and thus the trivial solution of Eqs. (39) and (41) given by

$$\Phi(x)=0 \quad (57)$$

does not correspond to each A_n vanishing. Note that in this case Eqs. (30) and (31) have a solution simultaneously as the slip wave speed and Rayleigh wave speeds coincide. Therefore a solution of

the form (55) exists. This solution implies that the contact pressure is equal to zero, while the normal displacements are continuous but not zero. It corresponds to the case of two Rayleigh waves (traveling or standing) of the same amplitude, wavelength, and phase, which exist simultaneously in the two bodies without interaction. This solution exists for any value of the friction coefficient. It exists also for the case of high sliding velocity if the Rayleigh wave speeds are related to the sliding velocity by

$$C_R^{(1)} - C_R^{(2)} = V. \tag{58}$$

5 Numerical Analysis of the Integral Equation

5.1 Reduction to a System of Linear Algebraic Equation.
The lowest frequency of vibration, which corresponds to $m=2$, $n=1$ in Eq. (43), is the most important and so we concentrate on a numerical investigation of this case. Applying the method developed by Erdogan and Gupta [19], we write Eq. (46) in the form

$$
4\pi\phi(x)K_r^\infty - 4K_i^\infty \int_{-1}^{1} \frac{\phi(\xi)d\xi}{x-\xi} + \int_{-1}^{1} \phi(\xi)\left\{ K_i^\infty \frac{4\phi(\xi)}{x-\xi} \right.
$$

$$
\left. - cK_i^\infty \cot\frac{c(x-\xi)}{4} - cK_i^\infty \tan\frac{c(x-\xi)}{4} + ck(c(x-\xi)) \right\} d\xi
$$

$$
= 0, \quad |x| < 1. \tag{59}
$$

The singular term is in the first integral whereas the second integrand is bounded. Following Erdogan and Gupta [19] we define the degree of singularity at the leading edge as

$$
\alpha = \frac{1}{2\pi i} \log\left(\frac{K_r^\infty - iK_i^\infty}{K_r^\infty + iK_i^\infty} \right) = -\frac{1}{2\pi}\arctan\frac{K_i^\infty}{K_r^\infty}, \quad 0 < \alpha < 1 \tag{60}
$$

and the weighting function as

$$
w(x) = (1-x)^\alpha (1+x)^{1-\alpha}. \tag{61}
$$

Note that the stresses are bounded at the transitions between separation and contact zones.

The unknown function $\phi(x)$ is sought in the form

$$
\phi(x) = \sum_{n=0}^{\infty} c_n w(x) P_n^{(\alpha, 1-\alpha)}(x), \tag{62}
$$

where $P_n^{(\alpha, 1-\alpha)}(x)$ is a Jacobi polynomial of order n. The coefficients c_n can be found from the system of linear equations

$$
D(x_i) = \sum_{n=0}^{\infty} c_n D_{ni}
$$

$$
= \sum_{n=0}^{\infty} c_n \left\{ -8K_i^\infty \pi P_{n+1}^{-\alpha,-\alpha}(x_i) \right.
$$

$$
+ \int_{-1}^{1} w(\xi) P_n^{\alpha,\alpha}(\xi) \left(\frac{4K_i^\infty}{x-\xi} - cK_i^\infty \cot\frac{c(x_i-\xi)}{4} \right.
$$

$$
\left. \left. - cK_i^\infty \tan\frac{c(x_i-\xi)}{4} + ck(c(x_i-\xi)) \right) d\xi \right\}
$$

$$
= 0, \tag{63}
$$

where D_{ni} denotes the term in figure brackets.

In order to solve this system of equations, the summation can be truncated at $n=N$ with a finite value of N. A collocation method can be applied to Eq. (63). The collocation points in this bounded integrand are taken as the evenly spaced in the interval $(-1,1)$, i.e., x_i $(i=0,1,2,\ldots N)$, and the integration points are defined according to the Jacobi-Gauss method

$$
\xi_j = \cos\frac{\pi(2j+1)}{2(N+1)}, \quad j=0,1,\ldots N. \tag{64}
$$

In order for nontrivial solutions of (63) to exist, the determinant of the $N+1$ by $N+1$ matrix

$$
D(\Lambda,\mu) = |D_{p-1}, q-1| = 0 \tag{65}
$$

must vanish.

5.2 Small Friction and Sliding Velocity.
Let us now consider the case of small friction $\mu \to 0$; the roots are expected to be close to those of the frictionless case. It was found in the analysis of the complete contact case, [4], that the real part of the root depends linearly on μ for small μ, so let us also assume such a linear dependence. For a small friction coefficient the root Λ is localized near the root for the frictionless case which lies on the imaginary axis, i.e.,

$$
\Lambda = \pm i\lambda + \mu\Lambda_1 \tag{66}
$$

where $i\lambda$ is pure imaginary. The determinant (65), which is a function of both μ and Λ, can be represented as

$$
D(\Lambda,\mu) = \frac{\partial D}{\partial \mu}\mu + \frac{\partial D}{\partial \Lambda}\mu\Lambda_1 = 0, \tag{67}
$$

which yields

$$
\Lambda_1 = \frac{\partial D/\partial \mu}{\partial D/\partial \Lambda}. \tag{68}
$$

The determinant $D(i\lambda,\mu)$ is a pure real function when its first argument is pure imaginary, which results in a pure imaginary derivative $\partial D/\partial \Lambda$. For the case of zero sliding velocity ($V=0$) it can be shown that the function $k(c(x-\xi))$ given by Eqs. (40) and (42) is pure real for nonzero μ, which results in a pure imaginary $\partial D/\partial \mu$ and yields a pure imaginary Λ_1. Thus for zero sliding velocity and small friction, the roots of $D(\Lambda)$ remain pure imaginary and no energy dissipation or instability occurs.

The numerical investigation shows that for a small nonzero V the imaginary part of $\partial D/\partial \mu$ is proportional to V, i.e.,

$$
\text{Im}\left(\frac{\partial D}{\partial \mu} \right) = \gamma V. \tag{69}
$$

The degree of stability η can be defined as

$$
\eta = \Lambda_1/V. \tag{70}
$$

5.3 Numerical Results and Discussion.
Numerical results were obtained using the collocation method previously described. All results and discussion are for the shear wave speed of the upper body less than that of the lower body ($\kappa < 1$). Consider first the frictionless case with zero sliding velocity. The dependence of the frequency of vibration on the contact zone half-length for material combinations which allow generalized Rayleigh waves is presented in Fig. 2. The frequency corresponds to the first antisymmetric mode ($n=1, m=2$) and is normalized in such a manner that $\lambda = 1$ corresponds to the frequency of a shear wave with a wavelength of 4π propagating in the upper body. In the limiting case of complete contact ($c=\pi$) the frequency multiplied by the wavelength is equal to the generalized Rayleigh wave velocity. Physically this corresponds to the superposition of two generalized Rayleigh waves which travel in opposite directions and form a standing wave. With decreasing contact zone length the frequency decreases and in the limiting case of zero separation zone approaches the Rayleigh wave frequency of the upper body. Physically this corresponds to a standing Rayleigh wave in the upper body which contacts the lower body at its nodes and hence produces no motion of the lower body. The data presented on this and other figures are for equal Poisson's ratios of the two bodies ($\nu_1 = \nu_2 = 0.25$) and for different values of μ_1/μ_2 and κ^2.
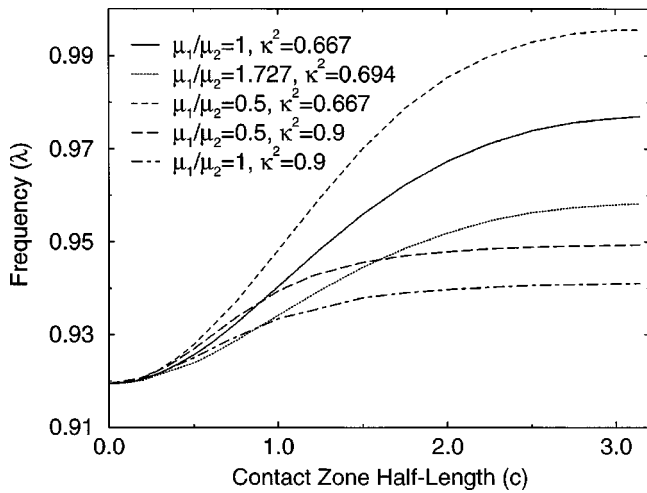
Fig. 2 Frequency of vibration divided by 4π versus the contact zone half-length (generalized Rayleigh waves possible)

Fig. 4 Contact pressure for the frictionless case, $\mu_1/\mu_2=1$, $\nu_1=\nu_2=0.25$, $\kappa^2=0.667$

The dependence of the frequency of vibration on the contact zone length for material combinations for which generalized Rayleigh waves do not exist is presented on Fig. 3. It can be seen that, for a sufficiently large contact zone, no solution exists. With the contact zone length below some critical value, a solution exists with the frequency in the range between the upper body Rayleigh wave and shear wave speed frequencies. There are no solutions for the frequency higher than unity (shear wave speed in the upper body) since this would contradict the assumption of s being pure real in Eq. (19) and therefore the radiation condition would be violated. It was found from the numerical analysis that the critical value of c does not decrease below the value of $c=1.5$ for the given values of the Poisson's ratio $\nu_1=\nu_2=0.25$.

Figure 4 presents the contact pressure distribution for the frictionless case with $\kappa^2=0.667$ and $\mu_1/\mu_2=1$. For the complete contact limit ($c \to \pi$) the pressure amplitude distribution is proportional to $\cos x$, which corresponds to a standing slip wave. For a small contact zone the distribution is elliptical as in a Hertz contact, since the first (constant) term of the series (62) multiplied by the weighting function (61) dominates.

Figure 5 presents the dependence of the degree of instability divided by the product of the frictional coefficient and sliding velocity, i.e., $\text{Re}(\Lambda)/(\mu V)$ vs. the contact zone half-length. The data presented is for three cases for which the generalized Ray-
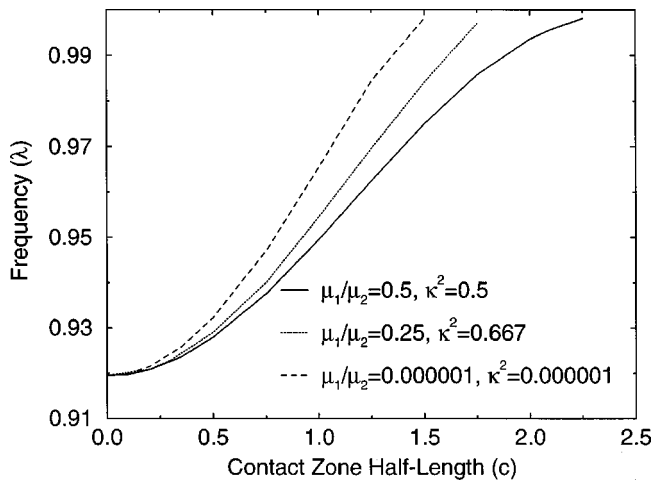
leigh waves exist ($\mu_1/\mu_2=1$, $\kappa^2=0.667$; $\mu_1/\mu_2=1.727$, $\kappa^2=0.694$; and $\mu_1/\mu_2=0.5$, $\kappa^2=0.667$) and one case for which a vibration solution exists only for the contact zone $c<2.25$ ($\mu_1/\mu_2=0.5, \kappa^2=0.5$). The degree of instability was found to be proportional to μV for reasonably small $\mu=0.1$ and $V=0.01$ for incomplete contact. Thus the solution is unstable for any finite nonzero sliding velocity and friction coefficient, which is consistent with the results of Adams [6] for a simple model of a beam on elastic foundation. For small separation zones ($c \to \pi$) the proportionality of $\text{Re}(\Lambda)$ to μV becomes unbounded. This is because for complete contact the degree of instability is finite for vanishing small V, [1,4]. For small contact zone ($c \to 0$) the degree of instability approaches zero, because the derivative $\partial D/\partial \Lambda$ becomes unbounded, although there were numerical difficulties in obtaining results for this limiting case.

Note that for the case of zero sliding velocity ($V=0$) there are always two pure imaginary roots of opposite sign

$$\Lambda = \pm i\lambda \qquad (71)$$

due to the fact that there are only terms with Λ^2 in Eqs. (20)–(25). Let us investigate the behavior of the degree of instability according to Eq. (68) when the sign of the root is reversed for a small

Fig. 3 Frequency of vibration divided by 4π vs. the contact zone half-length (generalized Rayleigh waves do not exist)
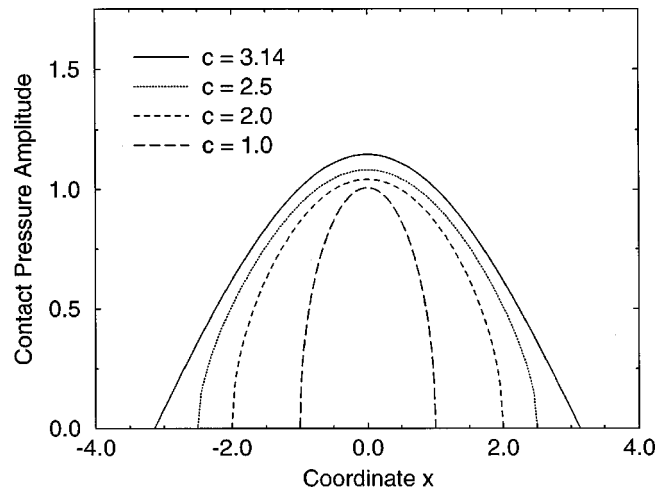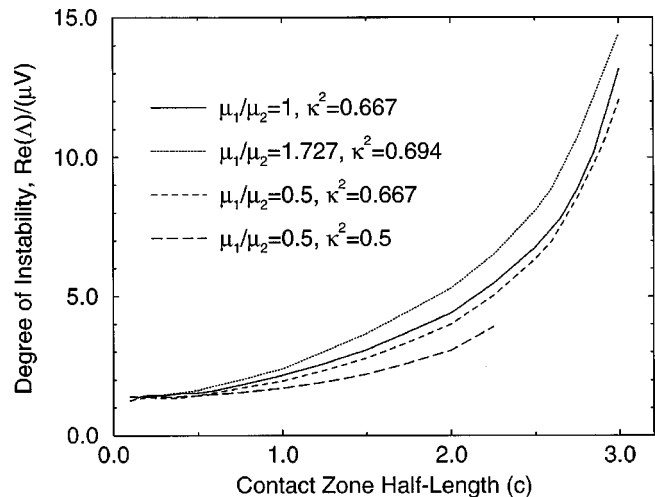

Fig. 5 Degree of instability divided by the friction coefficient and sliding velocity versus the contact zone length for small friction and sliding velocity

finite value of V. The derivative $\partial D/\partial \Lambda$ changes its sign since $D(i\lambda)$ is an even function of λ. In the case of small non-zero V a change of sign in Λ is equivalent to a change of sign in V because only terms in V^2, Λ^2, and ΛV appear in Eqs. (20) and (25). A change of sign in V, in turn, is equivalent to a change of sign of μ (sliding in an opposite direction). Therefore, the derivative $\partial D/\partial \mu$ changes its sign when the opposite sign root $\Lambda = -i\lambda$ is considered, for the same small sliding velocity and small μ. As a result of this, the degree of instability $\mathrm{Re}(\Lambda)$ keeps its sign according to (68) when the opposite sign root is considered. Thus, both roots can only be simultaneously either stable or unstable for small μ and V.

In the general case the function $\phi(x)$ obtained from Eq. (62) is complex. The normal stress can be found from $(23)_1$ and (32) as

$$
\begin{aligned}
\sigma_{yy}^{(1)} &= 2\pi m \, \mathrm{Re}[\phi(x/c)\exp(\Lambda t)] \\
&= 2\pi m[\mathrm{Re}\{\phi(x/c)\}\cos(\mathrm{Im}(\Lambda)t) \\
&\quad -\mathrm{Im}\{\phi(x/c)\}\sin(\mathrm{Im}(\Lambda)t)]\exp(\mathrm{Re}(\Lambda)t). \quad (72)
\end{aligned}
$$

The shear stress $(23)_2$ and displacements (16)–(17) can be found in a similar manner. Such modes of vibration represented by a complex $\phi(x)$ are called *complex modes*, as opposed to real normal modes (pure real $\phi(x)$ for the case of zero friction or velocity). In a complex mode the displacements at all points do not become zero simultaneously, but the motion can be represented as a superposition of real and imaginary parts. In a sense a complex mode can be thought of as two modes of the same frequency which are constrained to have a fixed ratio of amplitudes and to be $90°$ out of phase with each other. From (72) it is noted that a real mode would exist if either the real part or the imaginary part of $\phi(x/c)$ were to vanish. Similarly if the real part of $\phi(x/c)$ is $A\cos(x/c)$ and the imaginary part of $\phi(x/c)$ is $A\sin(x/c)$, in which A is an arbitrary constant, then (72) would become a traveling wave. Thus a normal mode of vibration and a traveling wave can be considered to be special cases of a complex mode of vibration. A qualitatively similar result was obtained by Adams [6].

It can be seen from Fig. 5 that for a sliding velocity equal to only one-thousandth of the shear wave speed ($V = 0.001$) and for a friction coefficient of $\mu = 0.1$, the value of the positive real part of Λ is of the order of 0.001. This indicates that for sliding velocities and friction which are in the range of engineering applications, the mechanism of destabilization identified here may yield a degree of instability sufficient to overcome structural damping and lead to friction-induced vibrations.

6 Conclusions

The steady sliding of two elastic half-spaces with Coulomb friction has been investigated. Roughness of the surfaces was modeled by a periodic wavy surface on one of the bodies. The contact occurs at periodically spaced contact regions on the peaks of the wavy asperities. The length of the contact regions is known from the solution of the steady-state problem and the stability of sliding was then analyzed. The overall solution was sought in the form of a superposition of the steady-state solution and small vibrations. Using a Fourier series representation for the vibration modes, the dynamic problem was reduced to a singular integral equation for determining the eigenvectors (modes) and eigenvalues (frequencies). The singular integral equation was analyzed by decomposing the unknown function as a Jacobi polynomial series multiplied by a weighting function and applying a collocation method. This procedure allowed us to replace the singular integral equation with a system of linear algebraic equations which was solved numerically.

For the frictionless case it was found that normal vibration modes exist for a wide range of material combinations. These modes correspond, in the case of complete contact, to generalized Rayleigh waves and can be interpreted as standing generalized Rayleigh waves. In the case of incomplete contact, and for the wavelength equal to twice the waviness period, the lowest frequency of vibration lies in the range between the lowest Rayleigh wave frequency of the two bodies and the generalized Rayleigh wave frequency. For the case of a vanishing small contact zone, the vibration mode can be interpreted as a standing Rayleigh wave in one of the bodies. For the general case of incomplete contact, vibration has been found for a wider range of material parameters than that for which generalized Rayleigh waves exist.

For the case of nonzero friction, complex vibration modes and eigenvalues were found. There are eigenvalues with a positive real part, which means that the vibrations have an amplitude which grows exponentially with time and thus steady sliding is unstable. In the case of incomplete contact the degree of instability, for small friction and velocity, is proportional to both the friction coefficient and the sliding velocity. The degree of instability may be high enough to contribute to the formation of friction-induced vibrations for low damping and moderate or high sliding velocities (of the order of 0.001 of the shear wave speed or higher). In the limiting case of complete contact the results reduce to the destabilizing generalized Rayleigh wave analyzed by Adams [1] and subsequently by Ranjith and Rice [4]. This result demonstrates that the effect of the dynamic sliding instability of Coulomb friction exists not only in the case of complete contact, but also in the more realistic case when the true contact area is much less than the nominal contact area. Finally, note that this analysis considers the cases for which the wavelength of vibration is equal to one or two times the waviness period. For wavelengths much greater than the waviness period, it might be anticipated that the effect of surface roughness would be less significant than was found here and thus the results of [1] and [4] would be more directly applicable.

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