Some Lie Superalgebras Associated to the Weyl algebras

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We work throughout over an algebraically closed field k of characteristic zero. If $\mathfrak{g}$ is a simple Lie algebra different from $s\ell(n)$, Joseph shows in [J2], that there is a unique completely prime ideal, J_0 whose associated variety is the closure of the minimal nilpotent orbit in $\mathfrak{g}*$. When $\mathfrak{g}$ is the symplectic algebra $\mathfrak{g} = sp(2r)$, this ideal may be constructed as follows. It is well known that the symmetric elements of degree two in the r^{th} Weyl algebra A_r form a Lie algebra isomorphic to $sp(2r)$ [D, Lemma 4.6.9]. Hence there is an algebra map $\phi : U(\mathfrak{g}) \longrightarrow A_r$ whose kernel is clearly completely prime and primitive. Since the image of ϕ has Gel'fand Kirillov dimension $2r$, and this is the dimension of the minimal nilpotent orbit in $\mathfrak{g}*$ by [CM, Lemma 4.3.5], we have $\ker \phi = J_0$.

Now if $\mathfrak{g}$ is a classical simple Lie superalgebra, and $U(\mathfrak{g})$ contains a completely prime primitive ideal different from the augmentation ideal, then $\mathfrak{g}$ is isomorphic to an orthosymplectic algebra $osp(1, 2r)$ (Lemma 1). We observe that if $\mathfrak{g} = osp(1, 2r)$, then there is a surjective homomorphism $U(\mathfrak{g}) \longrightarrow A_r$ whose kernel J satisfies $J \cap U(\mathfrak{g}_0) = J_0$. It follows that $\mathfrak{g}$ acts via the adjoint representation on A_r, and we determine the decomposition of this representation explicitly.

This turns out to be a useful setting in which to study the Lie structure of certain associative algebras. A result of Herstein [He] states that if A is a simple algebra with center Z, then $[A, A]/[A, A] \cap Z$ is a simple Lie algebra, unless $[A : Z] = 4$, and Z has characteristic two. Additional results have been obtained for various generalized Lie structures in [BFM] and [Mo].

Let A_r be the r^{th} Weyl algebra over k with generators $x_1, \ldots, x_r, \partial_1, \ldots, \partial_r$ such that $\partial_i x_j - x_j \partial_i = \delta_{ij}$.

If A is any $\mathbb{Z}_2$-graded associative algebra, we can regard A as a Lie superalgebra by setting

$$[a, b] = ab - (-1)^{\alpha\beta} ba$$

where a, b are elements of A of degree α, β respectively. We regard A_r can be made into a $\mathbb{Z}_2$-graded algebra by setting $\deg \gamma_i = \deg \partial_i = 1$.

In [Mo] Montgomery shows that if we consider the r^{th} Weyl algebra A_r as a $\mathbb{Z}_2$-graded algebra, then $[A_r, A_r]/([A_r, A_r] \cap k)$ is a simple Lie superalgebra, and that when $r = 1, A_1 = k \oplus [A_1, A_1]$.

Using the adjoint representation of $\mathfrak{g}$ on A_r we show that $A_r = k \oplus [A_r, A_r]$ for all r. In addition if $r \neq s$, then $[A_r, A_r]$ is not isomorphic to $[A_s, A_s]$ as a Lie superalgebra. This answers a question of Montgomery.

Much is known about the enveloping algebras of the Lie superalgebras $osp(1, 2r)$ [M1], [M2]. However, we have tried to keep this paper as self contained as possible.

Lemma 1. If $\mathfrak{g}$ is a classical simple Lie superalgebra which is not isomorphic to $osp(1, 2r)$ for any r, then the only completely prime ideal of $U(\mathfrak{g})$ is the augmentation ideal.

Proof. It is shown in [B, pages 17-20], that if $\mathfrak{g} \neq osp(1, 2r)$, then $\mathfrak{g}$ contains an odd element x such that $[x, x] = 0$. Hence if P is a completely prime ideal, then $x^2 = 0 \in P$ forces $x \in P$. Since $P \cap \mathfrak{g}$ is an ideal of $\mathfrak{g}$, this implies $\mathfrak{g} \subseteq P$.

Lemma 2. If $\mathfrak{g} = osp(1, 2r)$, there is a surjective homomorphism $U(\mathfrak{g}) \longrightarrow A_r$.

Proof. Set

$$\mathfrak{g}_1 = \sum_i k x_i + \sum_i k \partial_i$$

and

$$\mathfrak{g}_0 = \sum_{i,j} kx_ix_j + \sum_{i,j} k\partial_i\partial_j + \sum_{i,j} k(x_i\partial_j + \partial_j x_i)$$

We may identify $\mathfrak{g}_0$ with the second symmetric power $S^2\mathfrak{g}_1$ of $\mathfrak{g}_1$. Then $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_r$ becomes a Lie superalgebra under the bracket

$$[a, b] = ab - (-1)^{\alpha\beta}ba$$

where $a \in \mathfrak{g}_\alpha$ and $b \in \mathfrak{g}_\beta$. It follows immediately from the description of $osp(m, n)$ given in [K, 2.1.2, supplement] that $\mathfrak{g} \cong osp(1, 2r)$.

Now let $\mathfrak{a}_r$ be the r^{th} Heisenberg Lie algebra with basis $X_1, \ldots, X_r, Y_1, \ldots, Y_r, Z$ and nonvanishing brackets given by $[X_i, Y_j] = \delta_{ij}Z$. Thus $U(\mathfrak{a}_r)/(Z-1)$ is isomorphic to A_r via the map sending X_i to x_i and Y_i to y_i. By [D, Lemma 4.6.9], $\mathfrak{g}_0 = sp(2r)$ acts by derivations on $\mathfrak{a}_r$, and hence on $U(\mathfrak{a}_r)$ and on the symmetric algebra $S(\mathfrak{a}_r)$. Therefore by [D, Proposition 2.4.9], the symmetrisation map $w : S(\mathfrak{a}_r) \longrightarrow U(\mathfrak{a}_r)$ is an isomorphism of $\mathfrak{g}_0$-modules. Set $S = S(\mathfrak{a}_r)/(Z-1)$. Clearly w induces an isomorphism $\overline{w} : S \longrightarrow A_r$. Now S is a polynomial algebra in $2r$ variables, and we let $S(n)$ be the subspace of homogeneous polynomials of degree n. Clearly $S(n)$ is a $\mathfrak{g}_0$-module. Set $A(n) = \overline{w}(S(n))$. Our main result is the following.

Theorem 3. Under the adjoint action
1) $A(n)$ is a simple $\mathfrak{g}_0$-module for all n.
2) $A(2n) \oplus A(2n-1)$ is a simple $\mathfrak{g}$-module for all n.

In order to prove the theorem, we need some notation.
For $1 \leq i \leq r-1$, consider the elements of $\mathfrak{g}$ given by

$$e_i = x_{i+1}\partial_i, \quad f_i = x_i\partial_{i+1}$$

and

$$h_i = [e_i, f_i] = x_{i+1}\partial_{i+1} - x_i\partial_i.$$

In addition, set $e_r = \partial_r, f_r = x_r$ and $h_r = -[e_r, f_r]/2 = -(x_r\partial_r + \partial_r x_r)/2$. Then $\mathfrak{h} = span\{h_i | 1 \leq i \leq r\}$ is a Cartan subalgebra of $\mathfrak{g}$. We let $\alpha_1, \ldots, \alpha_r \in \mathfrak{h}^*$ be the positive roots determined by $[h, e_i] = \alpha_i(h)e_i$ for all $h \in \mathfrak{h}$. The values $\alpha_i(h_j)$ are the entries in the (symmetrized) Cartan matrix

$$\begin{bmatrix} 2 & -1 & & & & \\ -1 & 2 & -1 & & & \\ & -1 & 2 & \ddots & & \\ & & \ddots & \ddots & -1 & \\ & & & -1 & 2 & -1 \\ & & & & -1 & 1 \end{bmatrix}$$

Let $\mathfrak{n}$ be the subalgebra of $\mathfrak{g}$ generated by $e_1, \dots, e_r$ and $\mathfrak{n}_0 = \mathfrak{n} \cap \mathfrak{g}$. If L is a $\mathfrak{g}$-module (resp. $\mathfrak{g}_0$-module) we say that $v \in L$ is a highest weight vector for $\mathfrak{g}$ (resp. for $\mathfrak{g}_0$) of weight $\lambda \in \mathfrak{h}^*$ if $hv = \lambda(h)v$ for all $h \in \mathfrak{h}$ and $\mathfrak{n}v = 0$ (resp. $\mathfrak{n}_0 v = 0$).

The bilinear form $(,)$ defined on $\mathfrak{h}^*$ by $(\alpha_i, \alpha_j) = \alpha_i(h_j)$ is invariant under the action of the Weyl group. For later computations involving $(,)$ it is convenient to use the following alternative description [K, 2.5.4]. Identify $\mathfrak{h}^*$ with k^r with standard basis $\epsilon_1, \dots, \epsilon_r$ and $(,)$ with the usual inner product. Then $\alpha_i = \epsilon_i - \epsilon_{i+1}$ for $1 \le i \le r-1$ and $\alpha_r = \epsilon_r$. Let ρ_0 (resp. ρ_1) denote the half-sum of the positive even (resp. odd) roots of $\mathfrak{g}$ and $\rho = \rho_0 - \rho_1$. Under the identification above we have $\rho_0 = \sum_{i=1}^r (r-i+1)\epsilon_i$, $\rho_1 = \frac{1}{2}\sum_{i=1}^r \epsilon_i$ and $\rho = \frac{1}{2}\sum_{i=1}^r (2r-2i+1)\epsilon_i$.

We now return to the homomorphism $\phi : U(\mathfrak{g}) \longrightarrow A_r$. Set $J = Ker\phi$. Note that $R = \mathbb{C}[x_1, \dots, x_r]$ is a simple A_r-module and hence a faithful simple $U(\mathfrak{g})/J$-module. Also $1 \in R$ is a highest weight vector of weight λ where $\lambda(h_i) = 0$ for $1 \le i \le r-1$, and $\lambda(h_r) = -1/2$. An easy computation shows that $\lambda = -\frac{1}{2}\sum_{i=1}^r i\alpha_i = -\rho_1$. Thus we have shown.

Corollary 4. *J is the annihilator of the simple highest weight module with weight $-\rho_1$.*

Lemma 5. Under the adjoint action of $\mathfrak{g}_0$ or $\mathfrak{g}$ on A_r,
1) ∂_1^n is a highest weight vector for $\mathfrak{g}_0$ of weight $n\epsilon_1$.
2) If n is even, ∂_1^n is a highest weight vector for $\mathfrak{g}$.

Proof. A simple computation.

If $\lambda \in \mathfrak{h}^*$, we denote the simple $\mathfrak{g}_0$-module with highest weight λ by $L(\lambda)$.

Lemma 6. We have $dimL(n\epsilon_1) = \begin{pmatrix} 2r+n-1 \\ n \end{pmatrix}$ for all n.

Proof. By Weyl's dimension formula

$$dimL(\lambda) = \Pi_{\alpha>0}\frac{(\lambda+\rho_0,\alpha)}{(\rho_0,\alpha)}$$

where the product is taken over all positive even roots α. The even roots α for which $(\epsilon_1,\alpha) > 0$ are listed in the first column of the table below. The other columns give the information we need.

α	(ρ_0,α)	$(n\epsilon_1,\alpha)$
$\epsilon_1-\epsilon_{i+1},\ 1\le i\le r-1$	i	n
$\epsilon_1+\epsilon_j,\ 2\le j\le r$	2r - j + 1	n
$2\epsilon_1$	$2r$	$2n$

Therefore

$$\dim L(n\epsilon_1) = \prod_{i=1}^{r}\frac{n+i}{i}\prod_{j=2}^{r}\frac{2r+n-j+1}{2r-j+1} = \begin{pmatrix} 2r+n-1 \\ n \end{pmatrix}.$$

Proof of Theorem 3. Set $A = A_r$. Part 1) of the Theorem follows from Lemmas 5 and 6, since $dimA(n) = \begin{pmatrix} 2r+n-1 \\ n \end{pmatrix}$. Thus $B(n) = A(2n) \oplus A(2n-1)$ is a direct sum of two nonisomorphic simple $\mathfrak{g}_0$-modules. Also the highest weight vectors ∂_1^{2n} and ∂_1^{2n-1} for these $\mathfrak{g}_0$-modules satisfy

$$[x_1,\partial_1^{2n}] = -2n\partial_1^{2n-1}$$

$$[\partial_1,\partial_1^{2n-1}] = 2\partial_1^{2n}$$

Let M be the ad$\mathfrak{g}$-submodule of A generated by ∂_1^{2n}. It follows that $B(n) \subseteq M$. Also M is a finite dimensional image of a Verma module (which has a unique simple quotient). On the other hand all finite dimensional simple $\mathfrak{g}$-modules are completely reducible by [DH]. It follows that M is a simple ad$\mathfrak{g}$-module. (c.f. the argument in [Jan, Lemma 5.14]).

We do not know yet that $B(n)$ is an ad$\mathfrak{g}$-module. This can be seen as follows. We define a filtration $\{B_n\}$ on A by setting $B_n = \oplus_{m \leq n} B(m)$. Note that this filtration is the image of the filtration $\{U_n\}$ of $U(\mathfrak{g})$ defined by $U_n = U_1^n$ where $U_1 = k \oplus \mathfrak{g}$. Hence the associated graded ring $\oplus_{n \geq 0} B_n/B_{n-1}$ is supercommutative. It follows that $[\mathfrak{g}, B_n] \subseteq B_n$ and so $M \subseteq B_n$. If M strictly contained $B(n)$, we would have $M \cap (B(n-1) \oplus \ldots \oplus B(1) \oplus k) \neq 0$. By induction, the $B(i)$ with $i < n$ are simple ad$\mathfrak{g}$-modules, so M would contain ∂_1^{2i} for some $i < n$. However a simple $U(\mathfrak{g})$-module cannot contain more than one highest weight vector. This contradiction shows that $M = B(n)$ and completes the proof.

Theorem 7. We have $[A_r, A_r] = \oplus_{n>0} A(n)$. In particular $A_r = k \oplus [A_r, A_r]$.

Proof. Note that if $a, b, c \in A$ have degrees α, β and γ, then as noted in [Mo, Lemma 1.4 (3)]

$$[ab, c] = [a, bc] + (-1)^{\alpha(\beta+\gamma)}[b, ca].$$

Therefore, since A_r is generated by the image of $\mathfrak{g}$, we have $[A_r, A_r] = [A_r, \mathfrak{g}]$. The result now follows from Theorem 3.

Remark. From [Mo, Theorem 4.1] it follows that $[A_r, A_r]$ is a simple Lie superalgebra for all r.

A question raised in [Mo] is whether, for different r the $[A_r, A_r]$ are all nonisomorphic. We show this is the case by finding the largest rank of a finite dimensional simple Lie superalgebra contained in $[A_r, A_r]$. Note that $sp(2r) \cong A(2) \subseteq [A_r, A_r]$. On the other hand we have

Lemma 8. If L is a finite dimensional simple Lie subalgebra of $[A_r, A_r]$, then $rank(L) \leq r$.

Proof. Note that under the stated hypothesis, L is a Lie subalgebra of A_r with the usual Lie bracket $[a, b] = ab - ba$. Now in [J1], Joseph investigates for each simple Lie algebra L, the least integer $n = n_A(L)$ such that L is isomorphic to a Lie subalgebra of A_n. (The integer $n_A(L)$ is determined to within one for all classical Lie algebras.) In particular it follows from Lemma 3.1 and Table 1 of [J1] that $n_A(L) \geq rank(L)$.

Corollary 9. If $[A_r, A_r] \cong [A_s, A_s]$ as Lie superalgebras, then $r = s$.

For the sake of completeness, we give a proof of Corollary 9 which is independent of [J1]. It is enough to show that if $\mathfrak{g}_0 = sp(2r)$ is a Lie subalgebra of a Weyl algebra A_n, then $n \geq r$. The elements $x_1 x_i, x_1 \partial_i$, with $2 \leq i \leq r$ and x_1^2 span a Heisenberg subalgebra $\mathfrak{a} = \mathfrak{a}_{r-1}$ of $\mathfrak{g}_0$ with center spanned by x_1^2. The inclusion $\mathfrak{g}_0 \subseteq A_n$ induces a homomorphism $\phi : U(\mathfrak{g}_0) \longrightarrow A_n$. If $I = \ker \phi \cap U(\mathfrak{a}) \neq 0$, then we have $GK(U(\mathfrak{a})) = 2r - 1 \leq GK(A_n) = 2n$, where GK( ) denotes Gel'fand-Kirillov dimension, and so $r \leq n$. However if $I \neq 0$, then since the localization of $U(\mathfrak{a})$ at the nonzero elements of $k[x_1^2]$ is a simple ring, we would have $x_1^2 - \alpha \in I$ for some scalar α. This would imply that x_1^2 is central in $\mathfrak{g}_0$, a contradiction.

Finally, we note that the proof of Theorem 7 works for certain other algebras.

Theorem 10. Let $\mathfrak{g}$ be a semisimple Lie algebra, and A a primitive factor algebra of $U(\mathfrak{g})$, then $A = k \oplus [A, A]$.

Proof. As before we have $[A, A] = [A, \mathfrak{g}]$. Also $A = \oplus V$, a direct sum of finite dimensional simple submodules under the adjoint representation. Since $[V, \mathfrak{g}]$ is a submodule of V for any such V, and the center of A equals k, we obtain $[A, A] = \oplus_{V \neq k} V$, and the result follows.

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