

# COMPLETE SETS OF REPRESENTATIONS OF CLASSICAL LIE SUPERALGEBRAS

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ABSTRACT. Descriptions of the complete sets of irreducible highest weight modules over complex classical Lie superalgebras are recorded. It is further shown that the finite dimensional irreducible modules over a (not necessarily classical) finite dimensional complex Lie superalgebra form a complete set if and only if the even part of the Lie superalgebra is reductive and the universal enveloping superalgebra is semiprime.

## 1. Introduction

In their study of  $sl(2,1)$ -supersymmetry, Arnal, Ben Amor, and Pinczon [1] obtain certain structural identities dependent upon their proof that the finite dimensional irreducible  $sl(2,1)$ -modules form a complete set. (A collection of ( $\mathbb{Z}_2$ -graded) modules over a finite dimensional complex Lie superalgebra is said to be *complete* if for every element  $u$  in the enveloping algebra there exists a module  $M$  such that  $u.M = 0$ .) That the finite dimensional irreducible modules over a finite dimensional complex reductive Lie algebra form a complete set follows from a theorem of Harish-Chandra [6]

A graded  $\mathfrak{g}$ -module is *irreducible* when there are no proper graded submodules, and a (two-sided) ideal of  $\mathbf{V}$  is *graded-primitive* when it is the annihilator of an irreducible graded  $\mathfrak{g}$ -module. It is straightforward to check that the finite dimensional irreducible graded modules form a complete set exactly when the intersection of the cofinite dimensional graded-primitive ideals of  $\mathbf{V}$  is equal to zero. However, it follows from [11, 3.1] that this last condition occurs exactly when the intersection of the cofinite dimensional primitive ideals of  $\mathbf{V}$  is equal to zero. Consequently, the (not necessarily graded) finite dimensional irreducible  $\mathfrak{g}$ -modules form a complete set if and only if the same holds true for the finite dimensional irreducible graded  $\mathfrak{g}$ -modules.

We will follow the conventions that *module* will refer to *left module*, unless otherwise stated, and that *graded* will always mean  $\mathbb{Z}_2$ -graded. However, a module will be assumed to be graded only when explicitly specified.

**2.2.** Suppose that  $\mathfrak{g}_0$  is reductive and that  $\mathbf{V}$  is a prime ring (i.e., the product of any two nonzero ideals is also nonzero). For example,  $\mathfrak{g}_0$  is reductive if  $\mathfrak{g}$  is classical simple (see, e.g., [15, p. 101]), and it is proved in [3] that the enveloping algebra of a classical simple Lie superalgebra of type other than  $b(n)$  is prime. That the finite dimensional irreducible  $\mathfrak{g}$ -modules now form a complete set is already implicit in [2], as can be explained as follows. To begin, it follows from [2, 4, Proposition 1] and its proof that each irreducible finite dimensional  $\mathfrak{g}_0$ -module  $L$  is a  $\mathfrak{g}_0$ -submodule of some irreducible finite dimensional  $\mathfrak{g}$ -module  $\tilde{L}$ . In particular, if  $I = \text{ann}_{\mathbf{U}} L = \{u \in \mathbf{U} \mid u.L = 0\}$  and  $J = \text{ann}_{\mathbf{V}} \tilde{L}$ , then  $J \subseteq \mathbf{U} \subseteq I$ . Consequently, letting  $N$  denote the intersection of all of the cofinite dimensional primitive ideals of  $\mathbf{V}$ , it must follow that  $N \subseteq \mathbf{U}$  is contained within the intersection of all of the cofinite dimensional primitive ideals of  $\mathbf{U}$ . Therefore, by our assumptions,  $N \subseteq \mathbf{U} = 0$ . However, it easily follows from [2, 3, Proposition 4] that  $N$  must now equal zero. Thus the irreducible finite dimensional  $\mathfrak{g}$ -modules form a complete set.

**2.3.** Recall that  $\mathbf{V}$  must be semiprime (i.e., there exist no nonzero nilpotent ideals) when the collection of irreducible  $\mathfrak{g}$ -modules forms a complete set.

**Theorem A.** *The intersection of the cofinite dimensional primitive ideals of  $\mathbf{V}$  is nilpotent (as an ideal of  $\mathbf{V}$ ) if and only if  $\mathfrak{g}_0$  is reductive. In particular, the finite dimensional irreducible  $\mathfrak{g}$ -modules form a complete set if and only if  $\mathfrak{g}_0$  is reductive and  $\mathbf{V}$  is semiprime.*

**2.4.** The proof of the preceding theorem follows immediately from the following ‘abstract’ result. First, recall from the Poincaré-Birkhoff-Witt (PBW) Theorem that  $\mathbf{U}$  and  $\mathbf{V}$  are noetherian  $\mathbb{C}$ -algebras (cf. [2], [4, 2.3.8]). Also,  $\mathbf{U}$  is integral [4, 2.3.9] and is therefore prime. Another consequence of the PBW Theorem is that  $\mathbf{V}$  is finitely generated and free as a right and left  $\mathbf{U}$ -module.

**Proposition B.** *Let  $k$  be a field, and let  $R$  be a prime noetherian  $k$ -subalgebra of a noetherian  $k$ -algebra  $S$  such that  $S$  is finitely generated as a left  $R$ -module.*

(i) *Suppose that  $S$  is free as a left  $R$ -module and that the irreducible finite dimensional (over  $k$ )  $R$ -modules form a complete set. Then the intersection of the cofinite dimensional primitive ideals of  $S$  is nilpotent (as an ideal of  $S$ ).*

(ii) *Suppose that the intersection of the cofinite dimensional primitive ideals of  $S$  is nilpotent. Then the finite dimensional irreducible  $R$ -modules form a complete set.*

*Proof.* (i) We prove this statement under the more general hypothesis that  $S$  is *torsionfree* [5, p. 103] rather than free as a left  $R$ -module. To begin, fix an arbitrary minimal prime ideal  $P$  of  $S$ . To prove (i) it suffices to show that  $P$  is equal to the intersection of those cofinite dimensional primitive ideals of  $S$  that contain  $P$  (see, e.g., [5, Chapter 2]). Next, it follows from [5, 7.7] that there exists an  $R$ - $S$ -bimodule subfactor  $B$  of  $S$  such that  $B$  is faithful and torsionfree as a left  $R$ -module, such that  $\text{ann } B_S = s \in S \mid B \cdot s = 0 = P$ , and such that  $B$  is torsionfree as a right  $S/P$ -module. (In the terminology of [16], the preceding situation is summarized by stating that  $B$  is an  $S$ -bond from  $R$  to  $S/P$ .) Now let  $\Delta$  denote the set of cofinite dimensional primitive ideals of  $R$ . Let  $\Delta^*$  denote the set of prime ideals  $P'$  of  $S$  containing  $P$  for which there exist  $R$ - $S$ -bimodule subfactors  $B'$  of  $B$  such that  $\text{ann } B'_S = P'$ , such that  $\text{ann}_R B' = Q'$  for some  $Q' \in \Delta$ , and such that  $B'$  is torsionfree as a left  $R/Q'$ -module and right  $S/P'$ -module. (In other words,  $B'$  is a  $B$ -bond from  $R/Q'$  to  $S/P'$ .) Observe in the above that  $B'$  is finite dimensional; hence from [5, 7.1] it follows that every member of  $\Delta^*$  is a cofinite dimensional primitive ideal. However, it now follows from [16, Theorem 3], and the assumption that the set of finite dimensional irreducible  $R$ -modules is complete, that the intersection of the primitive ideals in  $\Delta^*$  is equal to  $P$ . Thus (i) follows.

(ii) Since a finite product  $P_1 \cdots P_t$  of minimal prime ideals of  $S$  is equal to zero, it follows that  $(P_1 \cdots R) \cdots (P_t \cdots R) = 0$ , and so  $P_0 \cdots R = 0$  for some minimal prime ideal  $P_0$  of  $S$ . Therefore, in the terminology of (i), it follows from [5, 7.15] that there exists an  $S$ -bond  $B$  from  $R$  to  $S/P_0$ . Let  $\Delta$  denote the set of cofinite dimensional primitive ideals of  $S$  containing  $P_0$ , and let  $\Delta^*$  denote the set of prime ideals  $Q$  of  $R$  such that there exists a  $B$ -bond from  $R/Q$  to  $S/P$  for some  $P \in \Delta$ . As in the proof of (i), every prime ideal in  $\Delta^*$  is a cofinite dimensional primitive ideal. Finally, because the intersection of the cofinite dimensional primitive ideals of  $S$  is nilpotent, it follows that the intersection of the ideals in  $\Delta$  is equal to  $P_0$ . Therefore, again by [16, Theorem 3], the intersection of the ideals in  $\Delta^*$  is equal to zero, and (ii) follows.

**2.5.** (Not required in the sequel.) The following alternative to Proposition B is more natural from a ring-theoretic point of view, but uses a bit more machinery. Recall that  $\mathbf{U}$  and  $\mathbf{V}$  have finite Gelfand-Kirillov dimension (GK-dimension) over  $\mathbb{C}$ ; see [2] and [10] for definitions and background.

**Proposition B'.** *Let  $k$  be a field, and let  $R$  and  $S$  denote noetherian algebras of finite GK-dimension over a field  $k$ . Suppose that  $R$  is a  $k$ -subalgebra of  $S$  and that  $S$  is finitely generated as a left  $R$ -module. Set  $I$  equal to the intersection of all of the cofinite dimensional primitive ideals of  $R$ , and set  $J$  equal to the intersection of all of the cofinite dimensional primitive ideals of  $S$ . Then  $I$  is nilpotent if and only if  $J$  is nilpotent.*

*Proof.* Assume that  $I$  is nilpotent, let  $P$  be a minimal prime ideal of  $S$ , and recall the terminology of the preceding proof. By [5, 7.15], there exists an  $S$ -bond from  $R/Q$  to  $S/P$  for some prime ideal  $Q$  of  $R$ . By [10, 3.16, 5.3], it further follows that  $Q$  is a minimal prime ideal of  $R$ . In particular,  $Q$  is equal to the intersection of those cofinite dimensional primitive ideals containing  $Q$ . Therefore, the conclusion that  $J$  is nilpotent now follows in a similar fashion to the proof of Proposition B(i). The remainder of the proof follows in a fashion similar to the proof of Proposition B(ii).

### 3. Irreducible Highest Weight Modules

Retaining the notation of the previous section, we now assume for the remainder that  $\mathfrak{g}$  is a classical simple Lie superalgebra, that  $\mathbf{V} = U(\mathfrak{g})$ , and that  $\mathbf{U} = U(\mathfrak{g}_0)$ . Recall that  $\mathfrak{g}_0$  is a reductive Lie algebra.

**3.1.** We now construct Verma modules for  $\mathfrak{g}$ , following [14, 1], where the reader is referred for more details. To begin, fix a triangular decomposition  $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$ . In particular,  $\mathfrak{n}^-, \mathfrak{h}, \mathfrak{n}^+$  are graded subalgebras of  $\mathfrak{g}$  with  $\mathfrak{n}^\pm$  nilpotent,  $\mathfrak{h} \oplus \mathfrak{n}^+$  solvable, and  $\mathfrak{g}_0 = \mathfrak{n}_0^- \oplus \mathfrak{h}_0 \oplus \mathfrak{n}_0^+$  a triangular decomposition of  $\mathfrak{g}_0$  in the sense of, for example, [4, 1.10.14]. Set  $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}^+$  and  $\mathfrak{b}_0 = \mathfrak{h}_0 \oplus \mathfrak{n}_0^+$ . Next, fix  $\lambda \in \mathfrak{h}_0^*$ , and let  $\mathbb{C}_\lambda$  denote the one dimensional  $\mathfrak{b}_0$ -module for which  $\mathfrak{n}_0^+.c = 0$  and  $h.c = \lambda(h)c$  for all  $c \in \mathbb{C}$  and  $h \in \mathfrak{h}_0$ . There exists a unique irreducible graded  $\mathfrak{b}$ -module  $V_\lambda$  for which  $\mathfrak{n}^+.V_\lambda = 0$  and  $h.v = \lambda(h)v$  for all  $h \in \mathfrak{h}_0$  and  $v \in V_\lambda$  (cf. [7, 5.2], [14, 1.1]). Now set

$$M(\lambda) = \mathbf{U} \otimes_{U(\mathfrak{b}_0)} \mathbb{C}_\lambda,$$

$$\widetilde{M}(\lambda) = \mathbf{V} \otimes_{U(\mathfrak{b})} V_\lambda.$$

The unique irreducible  $\mathfrak{g}_0$ -module factor of  $M(\lambda)$  will be denoted by  $L(\lambda)$ . The  $\mathfrak{g}$ -module  $\widetilde{M}(\lambda)$  also has a unique irreducible graded factor, denoted  $\widetilde{L}(\lambda)$ . Set  $I(\lambda) = \text{ann}_{\mathbf{U}} L(\lambda)$  and  $J(\lambda) = \text{ann}_{\mathbf{V}} \widetilde{L}(\lambda)$ .

**3.2.** Let  $\mathbf{Z}$  denote the center of  $\mathbf{U}$ . If  $I$  is a nonzero ideal of  $\mathbf{U}$ , then  $I \cap \mathbf{Z} = 0$  (e.g., [4, 4.2.2]). Identifying  $\mathbf{Z}/I(\lambda) \cong \mathbf{Z}$  with  $\mathbb{C}$  (e.g., [4, 2.6.8]), let  $\chi_\lambda: \mathbf{Z} \rightarrow \mathbb{C}$  denote the map sending  $z \in \mathbf{Z}$  to  $z + I(\lambda)$ . In other words,  $\chi_\lambda$  is the central character corresponding to  $L(\lambda)$ .

We next record a direct consequence of the Harish-Chandra homomorphism.

**3.3 Lemma.** *Let  $\Lambda \subseteq \mathfrak{h}_0^*$ . Set*

$$I = \bigcap_{\lambda \in \Lambda} I(\lambda), \quad \text{and} \quad K = \bigcap_{\lambda \in \Lambda} \ker \chi_\lambda.$$

*Then  $I = 0$  if and only if  $\Lambda$  is Zariski dense in  $\mathfrak{h}_0^*$ , if and only if  $K = 0$ .*

*Proof.* From the decomposition  $U(\mathfrak{g}_0) = (\mathfrak{n}^- \mathbf{U} + \mathbf{U} \mathfrak{n}^+) \oplus U(\mathfrak{h}_0)$ , one obtains the projection  $\varphi: \mathbf{U} \rightarrow U(\mathfrak{h}_0)$ . We identify  $U(\mathfrak{h}_0)$  with  $S(\mathfrak{h}_0)$ , the algebra of polynomial functions on  $\mathfrak{h}_0^*$ . It then follows, for example, from [4, 7.4.4] that  $\chi_\lambda(z) = (\varphi(z))(\lambda)$  for all  $z \in \mathbf{Z}$ .

Now assume that  $\Lambda$  is dense in  $\mathfrak{h}_0^*$ , and fix  $z \in K$ . Consequently,  $(\varphi(z))(\lambda) = 0$  for all  $\lambda \in \Lambda$ . Hence,  $\varphi(z) = 0$ . But  $\varphi$  restricted to  $\mathbf{Z}$  is injective (e.g., [4, 7.4.5]), and so  $z = 0$ . Finally,  $I \cap \mathbf{Z} = K$ , and therefore  $I = K = 0$  by (3.2).

Now assume that  $\Lambda$  is not dense in  $\mathfrak{h}_0^*$ , and let  $W$  denote the Weyl group of  $\mathfrak{g}_0 = \mathfrak{n}_0^- \oplus \mathfrak{h}_0 \oplus \mathfrak{n}_0^+$ . Let  $W$  act on  $\mathfrak{h}_0^*$  by  $w.\lambda = w(\lambda + \rho) - \rho$  and on  $S(\mathfrak{h}_0)$  by  $w.f(\lambda) = f(w^{-1}.\lambda)$ , for all  $\lambda \in \mathfrak{h}_0^*$ ,  $f \in S(\mathfrak{h}_0)$ , and  $\rho$  equal to the half sum of the positive roots of  $\mathfrak{g}_0$  relative to the above triangular decomposition. By assumption, there exists a nonzero polynomial  $g \in S(\mathfrak{h}_0)$  such that  $g(\Lambda) = 0$ . Set  $f = \prod_{w \in W} w.g$ . From [4, 7.4.5] it then follows that there exists a nonzero  $z \in \mathbf{Z}$  such that  $f = \varphi(z)$ . By [4, 7.4.4],  $z \in \bigcap_{\lambda \in W.\Lambda} \ker \chi_\lambda$ , and so  $I$  and  $K$  are both nonzero.

**3.4 Remark.** Note that  $\lambda \in \mathfrak{h}_0^* \mid L(\lambda) \neq 0$  is Zariski dense in  $\mathfrak{h}_0^*$ . In particular, Harish-Chandra's theorem [6] that the finite dimensional irreducible modules over a semisimple Lie algebra form a complete set may be deduced from the proof of (3.2).

**3.5.** As mentioned earlier, it is proved in [3] that  $\mathbf{V}$  is prime if  $\mathfrak{g}$  is not of type  $b(n)$ . It is proved in [9], when  $\mathfrak{g}$  is of type  $b(n)$ , that  $\mathbf{V}$  is not semiprime and contains a unique minimal prime ideal. Consequently, in general, it follows that the prime radical of  $\mathbf{V}$  is equal to its unique (possibly zero) minimal prime ideal.

**Theorem C.** *Let  $\Lambda$  be a subset of  $\mathfrak{h}_0^*$ , let*

$$J = \bigcap_{\lambda \in \Lambda} J(\lambda) = \bigcap_{\lambda \in \Lambda} \text{ann}_{\mathbf{V}} \tilde{L}(\lambda),$$

*and let  $P$  denote the unique minimal prime ideal of  $\mathbf{V}$ . Then  $J = P$  if and only if  $\Lambda$  is Zariski dense in  $\mathfrak{h}_0^*$ .*

*Proof.* First, suppose that  $\Lambda$  is a dense subset of  $\mathfrak{h}_0^*$ . Note from (3.1), for each  $\lambda \in \Lambda$ , that there exists a vector  $v \in \tilde{L}(\lambda)$  such that  $\mathfrak{n}_0^+ \cdot v = 0$  and such that  $h \cdot v = \lambda(h) \cdot v$  for all  $h \in \mathfrak{h}_0$ . Therefore,  $L(\lambda)$  is a  $\mathfrak{g}_0$ -composition factor of  $\tilde{L}(\lambda)$ . Consequently,  $J(\lambda) \subseteq \bigcup_{\lambda \in \Lambda} I(\lambda)$ , and so

$$J \subseteq \bigcup_{\lambda \in \Lambda} I(\lambda) = 0,$$

by (3.3). However, it then follows from, for example, [12, 2.3] that  $J$  cannot strictly contain a minimal prime ideal of  $\mathbf{V}$ . Thus  $J = P$ , and so  $J = P$  since  $J$  is a semiprime ideal [11, 3.1].

Next suppose that  $\Lambda$  is not dense in  $\mathfrak{h}_0^*$ . Set  $t = 2^{\dim \mathfrak{g}_1}$ , and let  $\Pi$  denote the set of weights of the  $t$ -dimensional  $\mathfrak{g}_0$ -module  $\mathfrak{g}_1$ . Set

$$\Lambda' = \{ \lambda + \mu \mid \lambda \in \Lambda, \mu \in \Pi \},$$

and consider the set of  $\mathfrak{g}_0$ -modules

$$X = \bigoplus_{\nu \in \Lambda'} M(\nu).$$

As noted in [14, 1], for each  $\lambda \in \mathfrak{h}_0^*$ ,  $\tilde{M}(\lambda)$  is isomorphic as a  $\mathfrak{g}_0$ -module to a factor of  $\mathfrak{g}_1 \otimes \tilde{M}(\lambda)$ . Therefore, by [4, 7.6.14], there is a series

$$0 = M_0 \subset M_1 \subset \dots \subset M_s = \tilde{M}(\lambda),$$

where  $M_i/M_{i-1} \subseteq X$  for each  $1 \leq i \leq s$ , and where  $1 \leq s \leq t$ . Now let  $K = \bigcap_{\lambda \in \Lambda'} \ker \chi_\lambda$ . It then follows, for each  $\lambda \in \mathfrak{h}_0^*$ , that

$$K^t \tilde{M}(\lambda) = 0$$

(see, e.g., [4, 7.1]). Because  $\Lambda'$  is not dense in  $\mathfrak{h}_0^*$ , it follows from (3.3) that  $K$  is nonzero. However,  $K^t \tilde{L}(\lambda) = 0$ , for all  $\lambda \in \Lambda'$ , and so  $K^t \subseteq J$ . Because  $P$  is nilpotent and  $K$  is not, it therefore follows that  $J = P$ .

**3.6.** The following corollary now follows from [3].

**Corollary D.** *Suppose that  $\mathfrak{g}$  is a classical simple Lie superalgebra not of type  $b(n)$ . If  $\Lambda \subseteq \mathfrak{h}_0^*$ , then  $\{ \tilde{L}(\lambda) \mid \lambda \in \Lambda \}$  is a complete set of irreducible  $\mathfrak{g}$ -modules if and only if  $\Lambda$  is Zariski dense in  $\mathfrak{h}_0^*$ .*

**3.7 Remark.** It follows from [7, Theorem 8] and [8, Corollary to Theorem 2] that the set

$$\left\{ \lambda \in \mathfrak{h}_0^* \mid \tilde{L}(\lambda) \text{ is finite dimensional and } \lambda \text{ is typical} \right\}$$

is Zariski dense in  $\mathfrak{h}_0^*$  when  $\mathfrak{g}$  is a basic classical Lie superalgebra. In particular, Corollary D is a direct generalization of [1, Proposition III.1].

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