Finite Quantum Groups and Pointed Hopf Algebras

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Abstract

We show that under certain conditions a finite dimensional graded pointed Hopf algebra is an image of an algebra twist of a quantized enveloping algebra $U_q(\mathfrak{b})$ when q is a root of unity. In addition we obtain a classification of Hopf algebras H such that $G(H)$ has odd prime order $p > 7$ and grH is of Cartan type.

Throughout this paper K will denote an algebraically closed base field of characteristic zero. Recently there has been considerable interest in the structure of finite dimensional pointed Hopf algebras over K. For example if p is prime all Hopf algebras of dimension p are group algebras. Also any pointed Hopf algebra of dimension p^2 is either a group algebra or a Taft algebra, while those of dimension p^3 have been classified [AS1], [CD], [SvO], [Z]. In addition there are infinitely many isomorphism classes of pointed Hopf algebras of dimension p^4, [AS1], [BDG], see also [G].

If H is pointed the coradical filtration $\{H_n\}$ on H is a Hopf algebra filtration and the associated graded algebra $grH = \oplus_{n\geq 0} H_n/H_{n-1}$ is a Hopf algebra, see [M1], also [M2,Lemma 5.5.1]. In [AS2] pointed Hopf algebras H such that $H \cong grH$ are studied using methods from Lie theory and quantum groups. These Hopf algebras are crossed products $H = R * G$ where $G = G(H)$ is the group of grouplike elements in H, and R is an analog of the infinitesimal part of H in the classical theory of cocommutative Hopf

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algebras. Another key ingredient in understanding the structure of H is a matrix **b** known as the braiding matrix which is determined by the action of G on R.

Our first task is to understand finite dimensional graded pointed Hopf algebras in terms of something more familiar. In [AS2] this is done under certain conditions by twisting the coalgebra structure of a Frobenius-Lusztig kernel. We show that under similar conditions a finite dimensional graded pointed Hopf algebra is an image of an algebra twist of a quantized enveloping algebra $U_q(\mathfrak{b})$ when q is a root of unity. This approach seems to have certain technical advantages, for example we don't use quantum antisymmetrizers.

We also consider the problem of determining the finite dimensional pointed Hopf algebras H for which grH is known. This is known as the lifting problem. In particular we show that if $G(H)$ has prime order $p > 7$ and grH is of Cartan type, then $H \cong grH$. We also obtain some results for low primes; in particular we construct some apparently new Hopf algebras H with $G(H)$ of order 3 and grH of type $A_2 \times A_2$. Since the first version of this paper was written, Andruskiewitsch and Schneider have obtained further results on the structure of pointed Hopf algebras H such that $G(H)$ is an elementary abelian $p-$group with $p > 17$, [AS4].

1. POINTED HOPF ALGEBRAS AS IMAGES OF QUANTIZED ENVELOPING ALGEBRAS.

1.1 Let $H = \oplus_{n \geq o} H(n)$ be a graded pointed Hopf algebra with coradical KG. We say that H is *coradically graded* if the coradical filtration of H is given by $H_n = \oplus_{m \leq n} H(m)$, [CM2].

If H is coradically graded, the projection $\pi : H \longrightarrow H(0)$ is a Hopf algebra map. If Δ is the coproduct on H then $\rho = (1 \otimes \pi)\Delta$ makes H into a left $H(0)$ - Hopf module. Since ρ is an algebra maps it follows that the coinvariants $R = \{h \in H | \rho(h) = h \otimes 1\}$ form a subalgebra of H which is invariant under conjugation by elements of G. It follows from [M2, Theorem 1.9.4] that $H = R * G$ is a crossed product of R by G.

If $R(n) = H(n) \cap R$ then $R = \oplus_{n \geq 0} R(n)$ is a graded algebra with $R(0) = K$ by [AS1, Lemma 2.1].

If $x \in H$ and $\Delta x = g \otimes x + x \otimes h$ for grouplike elements $g, h \in G$ we say x is (h, g)-primitive. Since G is abelian, G acts on the space $P_g(H)$ of

$(1, g)$-primitives by conjugation. For $\chi \in G^* = \mathrm{Hom}(G, K^*)$ we set

$$P_g^\chi = P_g^\chi(H) = \{x \in P_g(H) | hxh^{-1} = \chi(h)x \text{ for all } h \in G\}.$$

Note that if G is finite then $\chi(h)$ is a root of unity for all $\chi \in G^*$ and $h \in G$. It is frequently useful to make the following assumptions.

(1) $R(1)$ has a basis $x_1, \ldots, x_n$ such that $x_i \in P_{g_i}^{\chi_i}$ for some $g_i \in G$, $\chi_i \in G^*$.

(2) H is generated as an algebra by G and $R(1)$.

(3) $G =< g_1, \ldots, g_n >$ is abelian.

Note that if $R(1)$ is finite dimensional and (3) holds then (1) also holds.

The algebra R is an Hopf algebra in the category ${}^{KG}_{KG}\mathcal{YD}$ of left Yetter-Drinfeld modules (also known as a braided bialgebra) over KG and H can be reconstructed by bosonization as a biproduct $H = R\#KG$, see [Mj],[R]. In [AS2] the braided bialgebra R is christened a Nichols algebra since algebras of this form were first studied by Nichols, [N]. By [AG, Prop. 3.2.12] R is determined by the subspace V regarded as an object in ${}^{KG}_{KG}\mathcal{YD}$, and as in [AS2] we write $R = \mathcal{B}(V)$ in this situation. The structure of V as a Yetter-Drinfeld module is given as follows: V is a left KG-comodule via the map $x_i \longrightarrow g_i \otimes x_i$, and a left KG-module via $g.x_j = \chi_j(g)x_j$. It is easily seen that the Yetter-Drinfeld compatability condition ([M2, 10.6.11]) holds. We set $b_{ij} = \chi_j(g_i)$ and call the matrix $\mathbf{b} = (b_{ij})$ the *braiding* of H and dim R(1) the *rank* of H.

As we shall see the relations in R depend largely on the matrix $\mathbf{b}$ and are independent of the elements χ_j and g_i. Thus we also call an $n \times n$ matrix $\mathbf{b} = (b_{ij})$ a *braiding of rank* n if all the entries b_{ij} are nonzero.

Lemma. If $\mathbf{b}$ is the braiding matrix of a finite dimensional Hopf algebra, then $b_{ii} \neq 1$ for all i.

Proof. See [AS1, Lemma 3.1] or [N, page 1538].

1.2 Following [AS2] we say that a braiding matrix $\mathbf{b} = (b_{ij})$ is of *Cartan type* if $b_{ii} \neq 1$ and there exists $a_{ij} \in \mathbb{Z}$ such that

$$b_{ij}b_{ji} = b_{ii}^{a_{ij}} \tag{1}$$

where $a_{ii} = 2$ for all i. If b_{ii} is a root of unity we assume that a_{ij} is the unique integer such that $-ord(b_{ii}) < a_{ij} \leq 0$. With this choice of the a_{ij} we say that $\mathbf{b}$ is of Cartan type (a_{ij}).

If $\mathbf{b}$ is a braiding of Cartan type (a_{ij}), we say that $\mathbf{b}$ is *admissible* if b_{ij} is a root of unity of odd order and a_{ij} is either zero or relatively prime to the order of b_{ij} for all i, j. Observe that if (a_{ij}) is the Cartan matrix of a simple Lie algebra $\mathfrak{g}$ then this condition simply means that the order of each b_{ij} is odd and, if $\mathfrak{g}$ is of type G_2 not divisible by 3.

Also we say that $\mathbf{b}$ is a braiding of FL-type (a_{ij}) if there exist positive integers $d_1, \ldots, d_n$ such that for all i, j

$$d_i a_{ij} = d_j a_{ji} \tag{2}$$

and

$$\text{there exists } q \in K^* \text{ such that } b_{ij} = q^{d_i a_{ij}} \tag{3}$$

Finally we say that a braiding matrix $\mathbf{b} = (b_{ij})$ has *exponent* e if e is the least positive integer such that $b_{ij}^e = 1$ for all i, j.

<u>1.3</u> Let L be a free abelian group with basis $e_1, \ldots, e_n$. If H is an L-graded Hopf algebra and $p : L \times L \to K^*$ an antisymmetric bicharacter, we obtain a new Hopf algebra by "twisting" H by p as follows. Let H' be an isomorphic copy of H as a coalgebra with canonical isomorphism $h \to h'$. The new algebra structure on H' is defined by

$$a'.b' = p(\alpha, \beta)(ab)'$$

for $a \in H_\alpha, b \in H_\beta$. By [HLT, Theorem 2.1] H' is a Hopf algebra.

Suppose in addition that $H = \oplus H(n)$ is a coradically graded Hopf algebra such that assumptions $(1)-(3)$ of 1.1 hold, and let $\mathbf{b} = (b_{ij})$ be the braiding of H. Assume also that H is an L-graded Hopf algebra with $\deg x_i = \deg g_i = e_i$. Suppose that $p : L \times L \to K^*$ is an antisymmetric bicharacter with $p(e_i, e_j) = p_{ij}$ for $i < j$. A short calculation shows that in the twisted algebra H' we have

$$g_i x_j g_i^{-1} = b_{ij} p_{ij}^2 x_j.$$

Thus the braiding $\mathbf{b}'$ in H' satisfies

$$b'_{ij} = b_{ij} p_{ij}^2.$$

Henceforth we denote the twisted Hopf algebra H' by $H^{(p)}$.

Motivated by the above calculation we say that two braidings $\mathbf{b}, \mathbf{b}'$ of rank n are *twist equivalent* if there exists $p_{ij} \in K^*$ for $1 \leq i, j \leq n$ such that

$$p_{ij}p_{ji} = 1 \text{ and } b'_{ij} = b_{ij}p_{ij}^2$$

for all i, j.

Obviously any braiding of FL-type is a braiding of Cartan type. Conversely we have the following result.

Lemma. If (a_{ij}) be a symmetrizable Cartan matrix and $\mathbf{b}$ an admissible braiding of Cartan type (a_{ij}), then $\mathbf{b}$ is twist equivalent to a braiding of FL-type.

Proof. There is a root of unity q of odd order and integers e_{ij} such that $b_{ij} = q^{e_{ij}}$ for all i, j. Thus q has a unique fourth root and we set $p_{ij} = q^{(e_{ji}-e_{ij})/4}$. Then $b'_{ij} = b_{ij}p_{ij}^2 = q^{(e_{ij}+e_{ji})/2} = b'_{ji}$ for all i, j. Therefore the result follows from [AS2, Lemma 4.3].

1.4. Let (a_{ij}) be a generalized Cartan matrix and suppose that there exist relatively prime positive integers d_i such that $d_i a_{ij} = d_j a_{ji}$ for all i, j. Define $[a] = (v^a - v^{-a})/(v - v^{-1})$, $[a]! = \Pi_{i=1}^{a}[i]$ and $\begin{bmatrix} a \\ i \end{bmatrix} = [a]!/[i]![a-i]!$. The result of substituting q for v in $\begin{bmatrix} a \\ i \end{bmatrix}$ is denoted $\begin{bmatrix} a \\ i \end{bmatrix}_q$. Let $v_i = v^{d_i}$. We define the quantized enveloping algebra $\mathbf{U} = U_v(\mathfrak{b})$ to be the $\mathbb{Q}(v)$-algebra with generators $K_1^{\pm 1}, \dots, K_n^{\pm 1}$ and $E_1, \dots, E_n$, subject to the relations

$$K_i K_i^{-1} = K_i^{-1} K_i = 1 \tag{1}$$

$$K_j E_i K_j^{-1} = v^{d_i a_{ij}} E_i \tag{2}$$

$$\sum_{r=0}^{1-a_{ij}} (-1)^r \begin{bmatrix} 1 - a_{ij} \\ r \end{bmatrix}_{v_i} E_i^{1-a_{ij}-r} E_j E_i^r = 0. \tag{3}$$

Then $\mathbf{U}$ is a Hopf algebra whose coproduct satisfies

$$\Delta K_i^{\pm 1} = K_i^{\pm 1} \otimes K_i^{\pm 1}$$

$$\Delta E_i = K_i \otimes E_i + E_i \otimes 1.$$

Note that $\mathbf{U}$ is a graded Hopf algebra if we set $\deg K_i = 0$ and $\deg E_i = 1$ for all i. It follows from [CM2, Theorem B] that $\mathbf{U}$ is coradically graded. Let $\Gamma =< K_1, \ldots, K_n >$ and define $\psi_j \in \Gamma^*$ by $\psi_j(K_i) = v^{d_i a_{ij}}$. If R is the subalgebra of $\mathbf{U}$ generated by $E_1, \ldots, E_n$ then $\mathbf{U} = R * \Gamma$ as in 1.1 and $E_i \in P_{K_i}^{\psi_i}$ for $1 \leq i \leq n$.

As in [J, 4.3] it is useful to consider also the Hopf algebra $\widetilde{\mathbf{U}}$ over $\mathbb{Q}(v)$ with generators $K_1^{\pm 1}, \ldots, K_n^{\pm 1}$, $E_1, \ldots, E_n$ satisfying only relations (1) and (2) above.

Let $\mathcal{A} = \mathbb{Q}[v, v^{-1}]$. We define a "nonrestricted $\mathcal{A}$-form" of $\mathbf{U}$ (c.f. [CP, 9.2]). This is the $\mathcal{A}$-subalgebra U of $\mathbf{U}$ generated by the elements $E_i, K_i^{\pm 1}$, and

$$[K_i : 0] = (K_i - K_i^{-1})/(v_i - v_i^{-1}).$$

The $\mathcal{A}$-subalgebra $\widetilde{U}$ of $\widetilde{\mathbf{U}}$ is defined similarly. The defining relations for U are the same as those for $\mathbf{U}$ except that we have the additional relations

$$K_i - K_i^{-1} = (v_i - v_i^{-1})[K_i : 0]$$

in U. Note that we can regard U and $\widetilde{U}$ as L-graded Hopf algebras ($L = \mathbb{Z}^n$) by setting $\deg K_i = \deg E_i = e_i$. If p is an antisymmetric bicharacter on L, then $U^{(p)}$ denotes the twisted algebra as in 1.3. Finally if $q \in K$, we set $U_q^{(p)} = U^{(p)} \otimes_{\mathbb{Q}[v^{\pm 1}]} K$ where K is a $\mathbb{Q}[v^{\pm 1}]$-algebra with v acting as q. The Hopf algebras $\widetilde{U}^{(p)}$ and $\widetilde{U}_q^{(p)}$ are defined analogously.

<u>1.5</u> Recall that if H is a Hopf algebra with antipode S, the adjoint action is defined by

$$(ada)(b) = \sum a_1 b S(a_2).$$

The next result is a key ingredient in our work. In the case of quantized enveloping algebras it says that the rather complex Serre relations of 1.4 (3) follow from the relatively simple relations in 1.4 (2) together with a mild assumption on the skew primitives. This might seem rather remarkable, although a similar situation obtains for Kac-Moody algebras, see [Kac,3.3].

<u>Lemma.</u> If $x_i \in P_{g_i}^{\chi_i}(H)(i = 1, 2)$ and r is a positive integer such that

$$\chi_2(g_1)\chi_1(g_2)\chi_1(g_1)^{r-1} = 1$$

then $(adx_1)^r(x_2)$ is $(1, g_1^r g_2)$-primitive.

Proof. A result similar to this is proved in [AS2, Lemma A.1.] for a braided adjoint action. The result in [AS2] translates into the Lemma after bosonization. Set $b_{ij} = \chi_j(g_i)$. Because of the importance of the result we indicate another proof in the case where $\mathbf{b}$ is an admissible rank 2 braiding of Cartan type (a_{ij}) (this is the only case that we shall need). By specialization it suffices to prove the result in the case where $H = \widetilde{\mathbf{U}}$ as in 1.4 and $x_i = E_i, g_i = K_i$ and $\chi_i = \psi_i$ for i = 1,2. If $\mathbf{b}$ is of FL-type this follows from [J, Lemma 4.10]. (Note that it is assumed in [J, Chapter 4] that (a_{ij}) has finite type, but this is not necessary for the proof of [J, Lemma 4.10]). In general since $\mathbf{b}$ is twist equivalent to a braiding of FL-type, the result follows from [CM1, Lemma 3.2].

1.6 Theorem. Let H be a finite dimensional coradically graded pointed Hopf algebra with braiding $\mathbf{b}$. Assume that $\mathbf{b}$ is admissible of Cartan type (a_{ij}) and that H satisfies (1)-(3) of Section 1.1. Let U be the $\mathcal{A}$-form of $U_v(\mathfrak{b})$ described in 1.4. Then there is a surjective map of Hopf algebras $U_q^{(p)} \longrightarrow H$ for some twist $U_q^{(p)}$ of U_q, and some $q \in K$.

Proof. By Lemma 1.3 and its proof there exist roots of unity p_{ij} such that the braiding $\mathbf{b}'$ given by $b'_{ij} = b_{ij}p_{ij}^{-2}$ is of FL-type. Hence there exists a root of unity q such that $p_{ij} \in (q)$ and $b'_{ij} = q^{d_i a_{ij}}$ for all i, j. Let L be a free abelian group with basis $e_1, \ldots, e_n$ and define $p : L \times L \to \mathbb{Q}[v^{\pm 1}]$ by $p(e_i, e_j) = v^{e_{ij}}$, where $p_{ij} = q^{e_{ij}}$ and $0 \le e_{ij} < ord(q)$.

By our assumptions, H is generated by $G =< g_1, \ldots, g_n >$ and $x_i \in P_{g_i}^{\chi_i}(H), 1 \le i \le n$. We claim there is a surjective algebra map $\theta : \widetilde{U}^{(p)} \longrightarrow H$ sending K_i to g_i and E_i to x_i. Clearly θ preserves relation (1) in 1.4. Since the braiding in $\widetilde{U}_q^{(p)}$ satisfies

$$\psi_j(K_i) = q^{d_i a_{ij} + 2e_{ij}}$$

and the braiding in H satisfies $b_{ij} = b'_{ij}p_{ij}^2 = q^{d_i a_{ij} + 2e_{ij}}$, it follows that the twisted version of relation (2) in 1.4 is preserved by θ and the claim follows. The relation (3) in 1.4 can be written in the form $(adE_i)^{1-a_{ij}}(E_j) = 0$ and by [CM1, Lemma 3.2] the same relation holds in $U_q^{(p)}$. Since $\chi_j(g_i)\chi_i(g_j)\chi_i(g_i)^{-a_{ij}}$ $= b_{ij}b_{ji}b_{ii}^{-a_{ij}} = 1$ we have $(adx_i)^{1-a_{ij}}(x_j) = 0$ in H since H is graded so θ

descends to an algebra map $U_q^{(p)} \longrightarrow H$.

To see that θ is a map of bialgebras note that the set

$$\{a \in U_q^{(p)} | \Delta\theta(a) = (\theta \otimes \theta)\Delta(a)\}$$

is a subalgebra of $U_p^{(p)}$ which contains the generators $K_i^{\pm 1}$ and E_i. Similarly θ is a map of Hopf algebras.

__1.7.__ We describe the kernel of the Hopf algebra map in the previous theorem. Let Φ^+ be the set of positive roots for the semisimple Lie algebra associated to a Cartan matrix (a_{ij}) of finite type. Using a reduced expression for the longest element in the Weyl group and Lusztig's braid group action we can construct root vectors $E_\alpha \in U_q$ for $\alpha \in \Phi^+$. Suitably ordered monomials in the E_α form a PBW basis for the subalgebra U_q^+ of U_q generated by $E_1, \ldots, E_n$, see [L] or [J, Theorem 8.24]. The same monomials form a basis for the twisted subalgebra $(U_q^+)^{(p)}$ of $U_q^{(p)}$.

Proposition. Suppose that H is a finite dimensional coradically graded pointed Hopf algebra of finite and indecomposable type $A = (a_{ij})$ such that $G(H) \cong (\mathbb{Z}/N)^s$ for some odd N with $N \neq 3$ if A has type G_2. Let $\theta : U_q^{(p)} \longrightarrow H$ be the surjective map of Hopf algebras described in Theorem 1.6 and $L = Ker\theta|_\Gamma$. Then $Ker\theta$ is the ideal generated by the elements $g-1, g \in L$ and $E_\alpha^N, \alpha \in \Phi^+$.
Proof. This follows from [AS4, Theorem 4.2].

__1.8.__ We illustrate Theorem 1.6 in case where $H = R * G$ is a graded Hopf algebra with $G = (g)$ a group of prime order p, $\dim R(1) = 2$ and $\mathbf{b}$ a braiding of finite Cartan type (a_{ij}). These possibilities are described in [AS 2, Section 5]. Let $d \in \{1, 2, 3\}$ and let p be an odd prime ($p > 3$ if $d = 3$). Suppose that q is a primitive p^{th} root of unity in K. Then q has a unique $2d^{th}$ root in K.
Define

$$A = \begin{pmatrix} 2 & -1 \\ -d & 2 \end{pmatrix}, \quad D = \begin{pmatrix} d & 0 \\ 0 & 1 \end{pmatrix}$$

Then DA is symmetric and any twist of the braiding $\mathbf{b}'$ for $U_v(\mathfrak{b})$ has the form

$$\begin{pmatrix} v^{2d} & v^{a-d} \\ v^{-a-d} & v^2 \end{pmatrix}.$$

Now specialize v to $q^{1/2\,d}$ and set $b = a/2d - 1/2$, $c = -a/2d - 1/2$. This means that $b + c + 1 \equiv 0 \pmod{p}$. Also the braiding matrix of the twisted specialization $U_q^{(p)}$ has the form

$$\begin{pmatrix} q & q^b \\ q^c & q^{1/d} \end{pmatrix}$$

Now consider a homomorphism from $U_q^{(p)}$ to H. We can assume that K_1 maps to g and K_2 to g^c. Thus such a map exists if and only if $dbc \equiv 1 \bmod p$ or equivalently $db(b+1) \equiv -1 \bmod p$. The last congruence imposes some conditions on p which can be found using quadratic reciprocity, see [AS2, Section 5].

By [AS2, Theorem 1.3] the number of isomorphism types of Nichols algebras with coradical of prime dimension p is equal to $(p-1)$ for type A_2 and $2(p-1)$ for type B_2 and G_2. The factor $p-1$ comes from the choice of a p^{th} root of unity q. With q fixed the two roots of the congruence $db(b+1)+1 \equiv 0 \bmod p$ give rise to 2 nonisomorphic Hopf algebras of types B_2 or G_2. There is a unique Hopf algebra of type A_2 because of the diagram automorphism in this case.

2. THE LIFTING PROBLEM.

2.1. Let H be a pointed Hopf algebra with coradical filtration $\{H_n\}$, and $H_0 = KG$. Then the graded Hopf algebra $grH = \oplus_{n\geq 0} H_n/H_{n-1}$ is coradically graded. Assuming the structure of grH is known we investigate the possibilities for H.

We assume that G is abelian and that $grH = R * G$ as before. By [M2, 5.4.1], we can find skew primitive elements $y_1, \ldots, y_n$ in H_1 such that the images x_i of these elements in grH form a basis for $R(1)$. By considering the action of G by conjugation we can assume further that $y_i \in P_{g_i}^{\chi_i}(H)$ for suitable g_i, χ_i. As before we set $b_{ij} = \chi_j(g_i)$.

Lemma. Suppose that $y_i \in P_{g_i}^{\chi_i}(H)$ for $i = 1, 2$, and that r is a positive integer such that

$$\chi_2(g_1)\chi_1(g_2)\chi_1(g_1)^{r-1} = 1. \tag{1}$$

If $(ady_1)^r(y_2) = a(g_1^r g_2 - 1)$ with $a \neq 0$ then $\chi_2 = \chi_1^{-r}$ and $\chi_1(g_1) = \chi_1(g_2)$

<u>**Proof.**</u> If $k \in G$ then k commutes with $(ady_1)^r(y_2)$ since G is abelian and $a \neq 0$. This forces $(\chi_1^r\chi_2)(k) = 1$. In particular $(\chi_1^{r-1}\chi_2)(g_1) = \chi_1^{-1}(g_1)$ and then (1) implies that $\chi_1(g_1) = \chi_1(g_2)$.

<u>**Corollary.**</u> With the hypothesis of the Lemma suppose that the braiding $\mathbf{b}$ in grH has rank 2 and finite Cartan type $A = (a_{ij})$ as in 2.2. Assume that H is finite dimensional and that the exponent of G is an odd prime p which is different from 3 if $d = 3$. Then either

1) $\mathbf{b}$ has exponent dividing $2d + 1$

or

2) $(ad\, y_1)^2(y_2) = (ad\, y_2)^{d+1}(y_1) = 0$.

<u>**Proof.**</u> By Lemma 1.5 $z_1 = (ady_1)^2(y_2)$ is $(1, g_1^2 g_2)$-primitive and $z_2 = (ady_2)^{d+1}(y_1)$ is $(1, g_1 g_2^{d+1})$-primitive. We show first that for $g = g_1 g_2^{d+1}$ all $(1, g)$-primitives are trivial. There are two cases to consider as follows. Note that $g_1 \neq 1 \neq g_2$.

1) If $g_1 g_2^{d+1} = g_1$ then $p|d + 1$. The only possibility is $d + 1 = p = 3$, but then the congruence $2b^2 + 2b + 1 \equiv 0 \mod 3$ from section 2.2 has no solution.

2) If $g_1 g_2^{d+1} = g_2$ then for $j = 1, 2$

$$b_{1j} = \chi_j(g_1) = \chi_j(g_2)^{-d} = b_{2j}^{-d}.$$

Thus

$$b_{22}^{-d} b_{21} = b_{12} b_{21} = b_{22}^{a_{21}} = b_{22}^{-d}.$$

Hence $b_{21} = 1$ and $b_{11} = b_{21}^{-d} = 1$. This is impossible by Lemma 1.1. Similarly all $(1, g_1^2 g_2)$-primitives are trivial.

Now suppose $z_1 = a(g_1^2 g_2 - 1)$ with $a \in K$, $a \neq 0$. By Lemma 2.1 $\chi_2 = \chi_1^{-2}$ and $\chi_1(g_1) = \chi_1(g_2)$. Set $q = b_{11} = b_{21}$. Then for $j = 1, 2$.

$$b_{j2} = \chi_2(g_j) = q^{-2},$$

and $b_{12}b_{21} = b_{22}^{a_{21}}$ gives $q^{-1} = q^{2d}$.

Similarly if $z_2 = b(g_1 g_2^{d+1} - 1)$ with $b \neq 0$ we have $\chi_1 = \chi_2^{-(d+1)}$ and $\chi_2(g_1) = \chi_2(g_2)$. Then if $q = b_{12} = b_{22}$ we have

$$b_{j1} = \chi_1(g_j) = b_{j2}^{-(d+1)} = q^{-(d+1)}$$

and $b_{12}b_{21} = b_{11}^{a_{12}}$ gives $q^{-d} = q^{d+1}$ and hence the result.

2.2. We apply our results to pointed Hopf algebras H such that $G = G(H)$ has odd prime order p and grH is a Nichols algebra of finite Cartan type. We first discuss the indecomposable case.

Theorem. Let H be a finite dimensional pointed Hopf algebra such that $G(H) = (g)$ has odd prime order p and grH is of finite indecomposable Cartan type. Assume that
1) If $p = 3$ or 7 then grH is not of type G_2
2) If $p = 5$ then grH is not of type B_2.
Then $H \cong grH$.

Proof. By [AS2,Section 5], grH has type A_1, A_2, B_2 or G_2. For type A_1 the only possibility is the Taft algebra which has no nongraded analog. We assume grH has rank 2 and that the Cartan matrix A is as described in Section 1.8. In particular, grH has generators g, x_1, x_2 satisfying

$$(adx_1)^2(x_2) = (adx_2)^{d+1}(x_1) = 0.$$

By [Mo, Theorem 5.4.1] we can choose $y_i \in P_{g_i}^{\chi_i}(i = 1, 2)$ such that the image of y_i in grH is x_i. If $d \neq 1$ or $p \neq 3$, Corollary 2.1 implies

$$(ady_1)^2(y_2) = (ady_2)^{d+1}(y_1) = 0. \tag{1}$$

If $d = 1$ and $p = 3$ we can assume

$$g_1 = g, \quad g_2 = g^b, \quad \chi_1 = \chi, \quad \chi_2 = \chi^c$$

where $\chi(g) = q$ is a primitive cube root of unity. From section 1.8 we have

$$b + c + 1 \equiv b(b + 1) \equiv 0 \bmod 3.$$

The only solution is $b = c = 1$. Then Lemma 1.5 implies that (1) holds in this case also.

To see this we modify the proof of [AS4, Lemma 6.9]. Let $U_q^{(p)}$ be the twisted quantized enveloping algebra used in the proof of Theorem 1.6. By the choice of the twist p and (1) there is a surjective Hopf algebra map $\phi : U_q^{(p)} \longrightarrow H$ such that $\phi(K_i) = g_i$ and $\phi(E_i) = y_i$. Set $y_\alpha = \phi(E_\alpha)$. We claim that $y_\alpha^p = 0$. To see this we modify the proof of [AS4, Lemma 6.9]. By [DCP, Section 19] the elements $E_\alpha^p, K_i^{\pm p}$ generate a Hopf subalgebra L of $U_q^{(p)}$. By [Mo, Cor. 5.3.5], $\phi(L)$ is a finite dimensional pointed Hopf algebra with trivial coradical. Thus $\phi(L) = 0$ and $y_\alpha^p = 0$.

2.3. We next extend Theorem 2.2 to the case where the Cartan matrix (a_{ij}) is decomposable. By [AS2, Section 5] the only new root systems that arise are subsystems of $A_2 \times A_2$.

We construct some examples of pointed Hopf algebras H such that grH has Cartan type $A_2 \times A_2$. Let q be a primitive cube root of unity and let $K < x_1, x_2 >$ be the free algebra on x_1, x_2. Consider the crossed product $\widetilde{B} = K < x_1, x_2 > *(g)$ where g has order 3 and $gx_ig^{-1} = qx_i$ for $i = 1, 2$.

Now let I be the ideal of $\widetilde{B}$ generated by the elements

$$x_i^3 \quad i = 1, 2$$

$$(x_1x_2 - qx_2x_1)^3$$

$$x_i^2x_j + x_ix_jx_i + x_jx_i^2 \quad i \neq j.$$

Similarly let $\widetilde{C} = K < y_1, y_2 > *(\chi)$ where χ has order 3 and $\chi y_i \chi^{-1} = q^{-1}y_i$ for $i = 1, 2$. Let J be the ideal of $\widetilde{C}$ generated by the elements

$$y_i^3 \quad i = 1, 2$$

$$(y_2y_1 - qy_1y_2)^3$$

$$y_i^2y_j + y_iy_jy_i + y_jy_i^2 \quad i \neq j.$$

Set $B = \widetilde{B}/I$ and $C = \widetilde{C}/J$. We denote the images of elements of $\widetilde{B}, \widetilde{C}$ in the factor algebras by the same symbol. We make $B, C, \widetilde{B}, \widetilde{C}$ into Hopf algebras via the coproducts

$$\Delta g = g \otimes g, \quad \Delta\chi = \chi \otimes \chi$$

$$\Delta x_i = g \otimes x_i + x_i \otimes 1$$

$$\Delta y_i = 1 \otimes y_i + y_i \otimes \chi^{-1}.$$

Thus B, C are coradically graded pointed Hopf algebras of type A_2 and $dimB = dimC = 3^4$. It is easy to check that there is a Hopf algebra isomorphism $\psi : \widetilde{B} \longrightarrow \widetilde{C}^{opp}$ defined by $\psi(g) = \chi$ and $\psi(x_i) = y_i\chi$. Note that $\psi(I) = J$.

Lemma. Given a 2×2 matrix $\Lambda = (\lambda_{ij})$ there are unique linear maps $\delta_i \in \widetilde{C}^*$ such that

(1) $\delta_i(ab) = \epsilon(a)\delta_i(b) + \delta_i(a)\gamma(b)$ for all $a, b \in \widetilde{C}$

(2) $\delta_i(y_j) = \lambda_{ij}$.

Furthermore $\delta_i(J) = 0$.

Proof. This is similar to [AS4, Lemma 5.19 (b)]. It suffices to show that there are algebra maps $T_i : C \longrightarrow M_2(K)$ satisfying

$$T_i(g) = \begin{pmatrix} 1 & 0 \\ 0 & q \end{pmatrix}, \quad T_i(x_j) = \begin{pmatrix} 0 & \lambda_{ij} \\ 0 & 0 \end{pmatrix}.$$

Then T_i will have the form

$$T_i(c) = \begin{pmatrix} \epsilon(c) & \delta_i(c) \\ 0 & \gamma(c) \end{pmatrix}.$$

We leave the details to the reader.

Now it is easy to see that there is an algebra map $\phi_\Lambda : \widetilde{B} \longrightarrow \widetilde{C}^*$ defined by

$$\phi_\Lambda(g) = \gamma, \;\; \phi_\Lambda(x_i) = \delta_i, \quad i = 1, 2$$

It follows that there is a pairing of Hopf algebras $(\;,\;)_\Lambda : \widetilde{C}^{opp} \times \widetilde{B} \rightarrow K$ defined by $(c, b)_\Lambda = \phi_\Lambda(b)(c)$ for all $b \in \widetilde{B}, c \in \widetilde{C}$. The pairing is determined by the rules

$$(y_j, x_i)_\Lambda = \lambda_{ij} \quad i, j = 1, 2$$

$$(y_i, g)_\Lambda = (\chi, x_i)_\Lambda = 0$$
$$(\chi, g)_\Lambda = q.$$

Let Λ^{tr} be the transpose of Λ. Then

$$(c, b)_\Lambda = (\psi(b), \psi^{-1}(c))_{\Lambda^{tr}}$$

for all $b \in \widetilde{B}, c \in \widetilde{C}$. Since $\delta_i(J) = 0$ it follows that

$$(J, \widetilde{B})_\Lambda = (\widetilde{C}, I)_\Lambda = 0.$$

Thus $(\ ,\ )_\Lambda$ induces a pairing of Hopf algebras $(\ ,\ ) : C^{opp} \times B \to K$.

As in [HLT, Section 2] we can form the Drinfeld double $D(B, C)$ of the pair (B, C). As a coalgebra $D(B, C) = C \otimes B$. The algebra structure is determined by the requirements that $1 \otimes B$ and $C \otimes 1$ are subalgebras of $D(B, C)$ and that

$$b \otimes c = (c_1, Sb_1)(c_3, b_3)c_2 \otimes b_2.$$

Here we use the abbreviated summation notation $\Delta b = b_1 \otimes b_2$. This easily gives

$$\chi x_i = q x_i \chi, \quad y_i g = q g y_i$$
$$x_i y_j - y_j x_i = \lambda_{ij}(g - \chi^{-1})$$

for $i, j = 1, 2$. It follows that $g\chi^{-1}$ is a central grouplike in $D(B, C)$. We denote the Hopf algebra obtained from $D(B, C)$ by factoring the ideal generated by $g\chi^{-1} - 1$ by $H(q, \Lambda)$. Finally let Λ_1 be the identity matrix and Λ_0 the matrix $\Lambda_0 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and set $H(q, \epsilon) = H(q, \Lambda_\epsilon)$.

Theorem. Let H be a finite dimensional pointed Hopf algebra such that $G(H)$ is cyclic of prime order p and grH has Cartan type $A_2 \times A_2$. Then either $H \cong grH$ or $H \cong H(q, \epsilon)$ for some primitive cube root q and $\epsilon = 0, 1$. Also

$$H(q, \epsilon) \cong H(q', \epsilon')$$

if and only if $q = q'$ and $\epsilon = \epsilon'$.

Proof. By [AS2, Theorem 1.3] graded pointed Hopf algebras of type $A_2 \times A_2$

and coradical of prime dimension p exist only if $p = 3$. Furthermore when $p = 3$ there are 4 isomorphism classes of such algebras. They are denoted $R(q, e)\#KG$ for $e = 1, 2$ and q a primitive cube root of 1 in [AS2, Section 6]. Suppose now that H is a pointed Hopf algebra such that

$$grH \cong R(q, e)\#KG.$$

By [AS2, (5.7)] we can assume there exist $x_i \in P_{g_i}^{\chi_i}(H)$, $(1 \leq i \leq 4)$ such that the images of $x_1, \ldots, x_4$ in grH from a basis for $R(1)$ and

$$g_1 = g_2 = g \quad , \quad g_3 = g_4 = g^e,$$

$$\chi_1 = \chi_2 = \chi \quad , \quad \chi_3 = \chi_4 = \chi^{-e}$$

where $\chi(g) = q$.

Set $y_{i-2} = x_i g_i^{-1}$ for $i = 3, 4$. The subalgebra B generated by g, x_1, x_2 is a Hopf algebra such that $G(B)$ has order 3 and grH is of type A_2. It follows from Theorem 2.2 that $H \cong grH$ and hence

$$(adx_1)^2(x_2) = (adx_2^2)(x_1) = x_1^3 = x_2^3 = (x_1x_2 - qx_2x_1)^3 = 0.$$

Similarly

$$(ady_1)^2(y_2) = (ady_2)^2(y_1) = y_1^3 = y_2^3 = (y_2y_1 - qy_1y_2)^3 = 0.$$

Also since $\chi_1(g_3)\chi_3(g_1) = 1$ it follows from Lemma 1.5 with $r = 1$ or by a direct calculation that $[x_j, y_k]$ is (g, g^{-1})-primitive for all j, k. If $[x_j, y_k] = 0$ for all j, k then $H \cong grH$. Otherwise $e = 1$ by Lemma 2.1 and we have

$$[x_j, y_k] = \lambda_{jk}(g - g^{-1})$$

for some 2×2 matrix Λ. Now suppose $P, Q \in GL_2(K)$ and define $x_i' = \sum_j p_{ij}x_j, y_\ell' = \sum_k q_{k\ell}y_k$. Then

$$[x_i', y_\ell'] = \lambda_{i\ell}'(g - g^{-1})$$

where $\Lambda' = P\Lambda Q$. Since $\Lambda \neq 0$ we can choose P, Q such that $\Lambda' = \Lambda_\epsilon$ for $\epsilon = 0$ or 1. Then replacing x_i' by x_i and y_i' by y_i we have the first claim in the Theorem.

If $\phi : H = H(q, \epsilon) \to H' = H(q', \epsilon')$ is an isomorphism then $q = q'$ by passing to the graded algebras and using [AS2, Lemma 6.5]. Denote the generators of $H(q', \epsilon')$ by $g, x'_1, x'_2, y'_1, y'_2$ Since $\phi(x_i)$ is a nontrivial $(1, \phi(g))$-primitive in H', we have $\phi(g) = g$ and $\phi(x_i) \in span\{g - 1, x'_1, x'_2, y'_1, y'_2\}$. Applying ϕ to the equation $gx_ig^{-1} = qx_i$ we see that ϕ maps $span\{x_1, x_2\}$ onto $span\{x_1, x_2\}$. Similarly ϕ maps $span\{y_1, y_2\}$ onto $span\{y'_1, y'_2\}$. The result follows easily.

2.4 Now we discuss decomposable case.

Theorem. Let H be a finite dimensional pointed Hopf algebra such that $G(H) = (g)$ has odd prime order p and grH is of finite decomposable Cartan type. Then either $H \cong grH$ or one of the following holds

(1) H is the Frobenius-Lusztig kernels $u_q(sl(2))$ where q is a p^{th} root of unity.

(2) $p = 3$ and H is one of the Hopf algebras described $H(q, \epsilon)$ in Theorem 2.2.

(3) $p = 3$ and H is the subalgebra of $H(q, 1)$ generated by x_1, x_2, y_1 and g.

Proof. By [AS2, Proposition 5,1] $gr(H)$ has type Cartan type $A_1 \times A_1, A_2 \times A_1$ or $A_2 \times A_2$ and in the last two cases $p = 3$. For type $A_1 \times A_1$ the only non-graded examples are the algebras $u_q(sl(2))$ by for example [AS1, Section 1]. For type $A_2 \times A_2$ the result follows from Theorem 2.2. The result for type $A_2 \times A_1$ is easily deduced from the proof of Theorem 2.2 and we leave the details to the reader.

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