

On the Center of the Enveloping Algebra of a Classical Simple Lie Superalgebra

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0.1. Let \mathfrak{g} be a semisimple Lie algebra with Cartan subalgebra \mathfrak{h} , Weyl group W and enveloping algebra $U(\mathfrak{g})$. Several results illustrate the important role played by the center $Z(\mathfrak{g})$ of $U(\mathfrak{g})$. First there is the Harish-Chandra homomorphism which shows that $\text{Spec}Z(\mathfrak{g}) \cong \mathfrak{h}^*/W$, and describes the action of $Z(\mathfrak{g})$ on highest weight modules. Next we recall the separation of variables Theorem of Kostant ([Ko], [D. 8.2.4]) which states that there is an $\text{ad}\mathfrak{g}$ -invariant subspace K of $U(\mathfrak{g})$ such that the multiplication map $K \otimes Z(\mathfrak{g}) \rightarrow U(\mathfrak{g})$ is an isomorphism of $\text{ad}\mathfrak{g}$ -modules. Kostant's theorem is a key ingredient in the proof of a result of Duflo, [D,8.4.3] stating that the annihilator of a Verma module is generated by its intersection with $Z(\mathfrak{g})$.

A version of the Harish-Chandra homomorphism have been given by Kac for basic classical simple Lie superalgebras. We recall the details in 1.1.

The proof of Kostant's theorem depends heavily on the fact that every finite dimensional \mathfrak{g} -module is completely reducible. By contrast if \mathfrak{g} is a simple Lie superalgebra, such that every finite dimensional \mathfrak{g} -module is completely reducible, then either \mathfrak{g} is a simple Lie algebra or \mathfrak{g} is an orthosymplectic Lie superalgebra $\mathfrak{g} = \text{osp}(1, 2r)$ for some $r \geq 1$ [Sch, Theorem 1, page 239].

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In this paper we show that for $\mathfrak{g} = osp(1, 2r)$ we again have $U(\mathfrak{g}) \cong K \otimes Z(\mathfrak{g})$ for K ad-invariant. In addition if E is any finite dimensional simple \mathfrak{g} -module, the multiplicity of E as a direct summand of K is equal to the dimension of the zero weight space of E . For the case where $\mathfrak{g} = osp(1, 2)$ these results are obtained in [Pi] where the subspace K is computed explicitly. However even for $\mathfrak{g} = osp(1, 2)$ the analog of Duflo's theorem is false. This follows from [Pi, Proposition 7.3] and can be seen as follows, [M2, Section 3]. For any $\lambda \in \mathfrak{h}^*$, the Verma module $\widetilde{M}(\lambda)$ for \mathfrak{g} is a direct sum $\widetilde{M}(\lambda) = M_0 \oplus M_1$ where M_0, M_1 are Verma modules for $\mathfrak{g}_0 = sl(2)$. The Casimir elements Q, C of $U(\mathfrak{g}_0)$ and $U(\mathfrak{g})$ are related by the following equation, see [Pi, Proposition 1.2] or [L, Proposition II.1]

$$(16C + 1) = (8Q - 8C + 1)^2$$

If λ is not regular then C acts on $\widetilde{M}(\lambda)$ as the scalar $-1/16$ and Q acts on M_0, M_1 by the same scalar $-3/16$. The image of $Q + \frac{3}{16}$ in $U(\mathfrak{g})/\ker \chi_\lambda U(\mathfrak{g})$ is nonzero and generates the ideal $ann \widetilde{M}(\lambda)/\ker \chi_\lambda U(\mathfrak{g})$.

Motivated by this example, we investigate algebraic relations between elements of $Z(\mathfrak{g}_0)$ and $Z(\mathfrak{g})$ and their connection with representation theory. We show that for any classical simple Lie superalgebra \mathfrak{g} , $\mathfrak{g} \neq P(n)$ every element of $Z(\mathfrak{g}_0)$ is a root of a polynomial with coefficients in $Z(\mathfrak{g})$ (Theorem 2.5). If $\mathfrak{g} = osp(1, 2r)$ this polynomial can be chosen to be monic, that is $Z(\mathfrak{g}_0)$ is integral over $Z(\mathfrak{g})$. Note however that neither of $Z(\mathfrak{g}_0)$ or $Z(\mathfrak{g})$ is contained in the other. As another application of our methods we show that when $\mathfrak{g} = osp(1, 2r)$ or $sl(r, 1)$, the algebra $U(\mathfrak{g})^{\mathfrak{g}_0}$ of $ad_{\mathfrak{g}_0}$ -invariants in $U(\mathfrak{g})$ is commutative.

Our work suggests the use of geometric methods to study the relationship between $Z(\mathfrak{g}_0)$ and $Z(\mathfrak{g})$. Let $A = (Z(\mathfrak{g}_0), Z(\mathfrak{g}))$ be the subalgebra of $U(\mathfrak{g})$ generated by $Z(\mathfrak{g}_0)$ and $Z(\mathfrak{g})$. The inclusion $B = Z(\mathfrak{g}) \subseteq A$ induces a map of spectra $\psi : Spec A \longrightarrow Spec B$, and we study the fibres of this map. When $\mathfrak{g} = osp(1, 2)$, this amounts to specializing C in the equation displayed above and looking at the resulting equation for Q .

We study a family of fibers when $\mathfrak{g} = sl(2, 1)$ in section 4. In what we shall call the typical case, the fiber is the union of a hyperbola in A^3 and two lines which meet the hyperbola at infinity. We show that the fiber in the atypical case can be regarded as the shadow at infinity cast by the fibers in

the typical case. This suggests a possible connection between our work and the noncommutative geometry of homogenized enveloping algebras studied for example in [Ba], [LS] and [LV].

In section 5 we study the case where $g = osp(1, 2r)$. In this case we show that $U(\mathfrak{g})^{\mathfrak{g}_0} = (Z(g_0), Z(g))$ and that $A = U(\mathfrak{g})^{\mathfrak{g}_0}$ is a free $B = Z(\mathfrak{g})$ -module of rank 2^r (Theorem 5.1 and Corollary 5.3). To state our main result in this case we need some more notation. Let Γ denote the set of sums of distinct odd positive roots. For $\lambda \in \mathfrak{h}^*$ and $\gamma \in \Gamma$, let $m_\lambda = ann_{Z(\mathfrak{g})} \tilde{M}(\lambda)$ and let $\chi_{\lambda-\gamma}^0$ be the central character afforded by the $U(\mathfrak{g}_0)$ -module $M(\lambda - \gamma)$.

Theorem. For $\lambda \in \mathfrak{h}^*$, the following are equivalent:

- (a) λ is regular.
- (b) The central characters $\chi_{\lambda-\gamma}^0, \gamma \in \Gamma$ are distinct.
- (c) The fiber of $\psi : \text{Spec } A \longrightarrow \text{Spec } B$ over m_λ is reduced.

The theorem follows by combining results in 3.9, 3.11, 5.2, and 5.6. As a by-product of the proof, we show that if $(\lambda + \rho, \alpha) = 0$, where α is an odd root, then $U(\mathfrak{g})m_\lambda$ is not semiprime. If in addition $(\lambda + \rho, \beta) \neq 0$ for all $\beta \in \overline{\Delta}_0^+$ (see 0.2 for notation), then $U(\mathfrak{g})m_\lambda$ is strictly contained in $ann_{U(\mathfrak{g})} \tilde{M}(\lambda)$. These results generalize [Pi, Proposition 7.3].

Some connections with representation theory are mentioned in Section 3. Here we are chiefly concerned with the structure of highest weight $U(\mathfrak{g})$ -modules when regarded as $U(\mathfrak{g}_0)$ -modules by restriction.

In the last section we collect some open problems arising from our work.

0.2. We fix the notation that we use throughout this paper. For any Lie superalgebra \mathfrak{k} we write \mathfrak{k}_0 (resp. \mathfrak{k}_1) for the even (resp. odd) part of \mathfrak{k} . Let $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ be a basic classical simple Lie superalgebra, and \mathfrak{h} a Cartan subalgebra of \mathfrak{g}_0 . The even (resp. odd) roots of \mathfrak{g} with respect to \mathfrak{h} are denoted Δ_0 (resp. Δ_1). Set $\Delta = \Delta_0 \cup \Delta_1$. By [Kac 1, 2.5.4] and [M1, 1.6, 1.8], there is a basis $\alpha_1, \dots, \alpha_n$ of simple roots of \mathfrak{g} . This means that $\alpha_1, \dots, \alpha_n$ are linearly independent and for every $\alpha \in \Delta$, we have either $\alpha \in Q^+$ or $-\alpha \in Q^+$ where $Q^+ = \sum_{i=1}^n \mathbb{N}\alpha_i$. For $\alpha, \beta \in \mathfrak{h}_0^*$, we write $\alpha \leq \beta$ if $\beta - \alpha \in Q^+$, and $\alpha < \beta$ if $\alpha \leq \beta$, and $\alpha \neq \beta$.

For $i = 0, 1$ we set $\Delta_i^+ = \Delta_i \cap Q^+$, $\Delta_i^- = -\Delta_i^+$, $\overline{\Delta}_0^+ = \{\alpha \in \Delta_0^+ | \alpha/2 \notin \Delta_1^+\}$ and $\overline{\Delta}_1^+ = \{\alpha \in \Delta_1^+ | 2\alpha \notin \Delta_0^+\}$. Let $\rho_0 = \frac{1}{2} \sum_{\beta \in \Delta_0^+} \beta$, $\rho_1 = \frac{1}{2} \sum_{\beta \in \Delta_1^+} \beta$ and $\rho = \rho_0 - \rho_1$. We fix a nondegenerate, \mathfrak{g} -invariant bilinear form $(\ , \)$ on \mathfrak{g} . If M is a $U(\mathfrak{g})$ -module, and $\alpha \in \mathfrak{h}_0^*$ we set $M^\alpha = \{m \in M | hm = \alpha(h)m \text{ for all } h \in \mathfrak{h}_0\}$. We say that $\alpha \in \mathfrak{h}_0^*$ is typical if $(\lambda + \rho, \alpha) \neq 0$ for all $\alpha \in \overline{\Delta}_1^+$, and that α is regular if $(\lambda + \rho, \alpha) \neq 0$ for all $\alpha \in \Delta_0^+$. If $\alpha \in \Delta_0$, we set $\alpha^\vee = 2\alpha/(\alpha, \alpha)$.

Let \mathfrak{h} be the centralizer of \mathfrak{h}_0 in \mathfrak{g} , $\mathfrak{n}^+ = \bigoplus_{\alpha \in Q^+ \setminus \{0\}} \mathfrak{g}^\alpha$ and $\mathfrak{n}^- = \bigoplus_{-\alpha \in Q^+ \setminus \{0\}} \mathfrak{g}^\alpha$. Then $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$ is a triangular decomposition of \mathfrak{g} in the sense of [M1, 1.1]. For $\lambda \in \mathfrak{h}^*$ there is a unique finite dimensional graded simple \mathfrak{b} -module V_λ such that $\mathfrak{n}^+ V_\lambda = 0$ and $hv = \lambda(h)v$ for all $h \in \mathfrak{h}_0$ and $v \in V_\lambda$. Also let $\mathbb{C}v_\lambda$ be the one dimensional \mathfrak{b}_0 module with $\mathfrak{n}_0^+ v_\lambda = 0$ and $hv_\lambda = \lambda(h)v_\lambda$ for $h \in \mathfrak{h}_0$. We define Verma modules for \mathfrak{g}_0 and \mathfrak{g} by

$$M(\lambda) = U(\mathfrak{g}_0) \otimes_{U(\mathfrak{b}_0)} \mathbb{C}v_\lambda$$

$$\widetilde{M}(\lambda) = U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} V_\lambda$$

The module $M(\lambda)$, (resp $\widetilde{M}(\lambda)$) has a unique simple (resp. graded simple) quotient which we denote by $L(\lambda)$ (resp. $\widetilde{L}(\lambda)$). We mention that if $\mathfrak{g} \neq Q(n)$ then $\mathfrak{h}_0 = \mathfrak{h}$, and $\dim_{\mathbb{C}} V_\lambda = 1$.

0.3. When $\mathfrak{g} = osp(1, 2r)$ it is convenient to use the following model for the root system of \mathfrak{g} [Kac 1, 2.5.4]. We identify \mathfrak{h}^* with \mathbb{C}^r with standard basis e_1, \dots, e_r and $(\ , \)$ with the usual inner product. Then a basis for the root system is given by $\alpha_i = e_i - e_{i+1}$ for $1 \leq i \leq r-1$ and $\alpha_r = e_r$, see [Kac 1, Section 2.5.4]. Then the positive even roots are

$$\Delta_0^+ = \{2e_i\} \cup \{e_i \pm e_j\}_{i < j}$$

and the positive odd roots are

$$\Delta_1^+ = \{e_i\}.$$

The Weyl group W acts as the group of all signed permutations of $\mathbf{r} = \{1, 2, \dots, r\}$. Let x_1, \dots, x_r be the basis of \mathfrak{h} dual to e_1, \dots, e_r . Then $S(\mathfrak{h})^W = \mathbb{C}[h_2, h_4, \dots, h_{2r}]$ where $h_d = \sum_{i=1}^r x_i^d$.

For $I \subseteq \mathbf{r}$, let $e_I = \sum_{i \in I} e_i$. If $\tilde{I} = \mathbf{r} \setminus I$ and $\gamma = e_I$, we write $\check{\gamma}$ for $e_{\tilde{I}}$. Note that $(\gamma, \check{\gamma}) = 0$.

1. The structure of $U(\mathfrak{g})$ as an $ad_{\mathfrak{g}}$ module

1.1. Our results depend on the extension of the Harish-Chandra homomorphism for Lie superalgebras given in [Kac 2], [Kac 4]. First we introduce some notation. Assume that \mathfrak{g} is classical simple and that $\mathfrak{g} \neq P(n)$.

Let $S(\mathfrak{g})$ be the symmetric algebra of \mathfrak{g} and $\pi : S(\mathfrak{g}) \rightarrow U(\mathfrak{g})$ the symmetrization map. This is an isomorphism of \mathfrak{g} -modules with respect to the adjoint action, [S, Lemma 1] so carries the subalgebra of \mathfrak{g} -invariants $S(\mathfrak{g})^{\mathfrak{g}}$ onto $Z(\mathfrak{g})$. Next, let $\zeta : Z(\mathfrak{g}) \rightarrow U(\mathfrak{h}) = S(\mathfrak{h})$ be the projection with respect to the decomposition

$$U(\mathfrak{g}) = U(\mathfrak{h}) \oplus (\mathfrak{n}^- U(\mathfrak{g}) + U(\mathfrak{g}) \mathfrak{n}^+)$$

and define $\alpha : S(\mathfrak{h}) \rightarrow S(\mathfrak{h})$ by $(\alpha F)(\lambda) = F(\lambda - \rho)$ for $F \in S(\mathfrak{h})$ and $\lambda \in \mathfrak{h}^*$. We write $\psi = \alpha\zeta$ and $\eta = \psi\pi$. The analogs of these maps for $U(\mathfrak{g}_0)$ are denoted $\pi_0, \zeta_0, \alpha_0, \psi_0$ and η_0 .

For $\lambda \in \mathfrak{h}^*$ we denote by χ_λ (resp. χ_λ^0) the central character afforded by the $U(\mathfrak{g})$ -module $\widetilde{M}(\lambda)$ (resp. by the $U(\mathfrak{g}_0)$ -module $M(\lambda)$). Thus for $z \in Z(\mathfrak{g})$ (resp. $z \in Z(\mathfrak{g}_0)$), z acts on $\widetilde{M}(\lambda)$ (resp. $M(\lambda)$) as the scalar $\chi_\lambda(z)$ (resp. $\chi_\lambda^0(z)$). Set $m_\lambda = \ker \chi_\lambda$ and $m_\lambda^0 = \ker \chi_\lambda^0$.

Theorem. (a) (Harish-Chandra) ψ_0 is an isomorphism from $Z(\mathfrak{g}_0)$ onto $S(\mathfrak{h})^W$, such that for all $z \in Z(\mathfrak{g}_0), \lambda \in \mathfrak{h}^*$

$$\chi_\lambda^0(z) = \psi_0(z)(\lambda + \rho_0)$$

(b) ψ is a monomorphism from $Z(\mathfrak{g})$ to $S(\mathfrak{h})^W$ such that for all $z \in Z(\mathfrak{g})$, and $\lambda \in \mathfrak{h}^*$

$$\chi_\lambda(z) = \psi(z)(\lambda + \rho)$$

(c) Let $S(\mathfrak{h})_0$ be the subalgebra of $S(\mathfrak{h})$ consisting of all functions ϕ on \mathfrak{h}^* such that $(\lambda, \alpha) = 0$ for an isotropic root α implies that $\phi(\lambda) = \phi(\lambda + t\alpha)$ for all $t \in \mathbb{C}$. Then the image of the map ψ in (b) is the fixed algebra $S(\mathfrak{h})_0^W$.

1.2. For the remainder of Section 1, assume that \mathfrak{g} is basic. The bilinear form $(\ , \)$ induces an isomorphism of \mathfrak{g} -modules $\mathfrak{g} \rightarrow \mathfrak{g}^*$. This map extends to an isomorphism of algebras and of \mathfrak{g} -modules $S(\mathfrak{g}) \rightarrow S(\mathfrak{g}^*)$ and hence of

fixed algebras $S(\mathfrak{g})^{\mathfrak{g}} \rightarrow S(\mathfrak{g}^*)^{\mathfrak{g}}$. Since $(\ , \)$ restricts to a nondegenerate form on \mathfrak{g}_0 and \mathfrak{h} we obtain in a similar way isomorphisms $S(\mathfrak{g}_0)^{\mathfrak{g}_0} \rightarrow S(\mathfrak{g}_0^*)^{\mathfrak{g}_0}$ and $S(\mathfrak{h}) \rightarrow S(\mathfrak{h}^*)$. All of these maps will be denoted by $x \rightarrow x^*$.

Let $\Lambda = \Lambda_{\mathfrak{g}_1}$ be the exterior algebra on \mathfrak{g}_1 with its natural grading $\Lambda = \bigoplus_{i=0}^m \Lambda^i$, where $\Lambda^1 = \mathfrak{g}_1$ and $m = 2^{\dim \mathfrak{g}_1}$. Then $S = S(\mathfrak{g}) = \bigoplus_{i=0}^m (\Lambda^i \otimes S(\mathfrak{g}_0))$, and we set $N = \bigoplus_{i=1}^m (\Lambda^i \otimes S(\mathfrak{g}_0))$ a nilpotent ideal of S . We write τ for the natural map $\tau : S \rightarrow S/N$ and identify S/N with $S(\mathfrak{g}_0)$.

There is a commutative diagram

$$\begin{array}{ccc} S(\mathfrak{g}) & \xrightarrow{\tau} & S(\mathfrak{g}_0) \\ * \downarrow & & \downarrow * \\ S(\mathfrak{g}^*) & \xrightarrow{\tau^*} & S(\mathfrak{g}_0^*) \end{array}$$

where τ^* is defined analogously to τ .

Now the inclusion $\mathfrak{h} \subseteq \mathfrak{g}_0$ induces a map $S(\mathfrak{g}_0^*) \rightarrow S(\mathfrak{h}^*)$, and this map in turn induces an isomorphism $\theta_0 : S(\mathfrak{g}_0^*)^{\mathfrak{g}_0} \rightarrow S(\mathfrak{h}^*)^W$ by [H, Theorem 23.1].

Set $\theta = \theta_0 \tau^*$. Then as in [H, page 131] we have the following diagram

$$\begin{array}{ccccc} S(\mathfrak{g})^{\mathfrak{g}} & \xrightarrow{\pi} & Z(\mathfrak{g}) & \xrightarrow{\psi} & S(\mathfrak{h})^W \\ * \downarrow & & & & \downarrow * \\ S(\mathfrak{g}^*)^{\mathfrak{g}} & & \xrightarrow{\theta} & & S(\mathfrak{h}^*)^W \end{array}$$

of filtered vector spaces such that for all $u \in S_n(\mathfrak{g})^{\mathfrak{g}}$

$$\eta(u)^* - \theta(u^*) \in S_{n-1}(\mathfrak{h}^*)^W.$$

Furthermore if $\mathfrak{g} = osp(1, 2r)$ then by [Theorem 1.1] the maps θ and ψ are algebra isomorphisms.

Of course, there is a similar diagram involving the maps π_0, ψ_0 and θ_0 .

Lemma. If $u \in S_n(\mathfrak{g})^{\mathfrak{g}}$ and $v = \eta_0^{-1}(\eta(u))$ then $u - v \in N + S_{n-1}(\mathfrak{g}_0)^{\mathfrak{g}_0}$.

Proof. Set $w = \tau(u) \in S_n(\mathfrak{g}_0)^{\mathfrak{g}_0}$. Then $\theta(u^*) = \theta_0 \tau^*(u^*) = \theta_0(w^*)$. However by the foregoing remarks $\eta(u)^* - \theta(u^*)$ and $\eta_0(w)^* - \theta_0(w^*)$ are contained in $S_{n-1}(\mathfrak{h}^*)^W$. Therefore $\eta_0(v) - \eta_0(w) = \eta(u) - \eta_0(w) \in S_{n-1}(\mathfrak{h})^W$, and hence $v - w \in S_{n-1}(\mathfrak{g}_0)^{\mathfrak{g}_0}$. Since $u - w \in N$ this proves the result.

Corollary.

- 1) $S(\mathfrak{g})^g \cap N = 0$
- 2) If $\mathfrak{g} = osp(1, 2r)$ then $S(\mathfrak{g})^g + N = S(\mathfrak{g}_0)^{g_0} + N$.

Proof.

- 1) If the intersection is nonzero, choose an element $u \in S_n(\mathfrak{g})^g \cap N, u \neq 0$ with n minimal. If $v = \eta_0^{-1}\eta(u)$ we can write $u - v = x + y$ with $x \in N$ and $y \in S_{n-1}(\mathfrak{g}_0)^{g_0}$. Then $u - x = v + y \in N \cap S(\mathfrak{g}_0) = 0$. This implies that $v = -y \in S_{n-1}(\mathfrak{g}_0)^{g_0}$ a contradiction.
- 2) Clearly $S(\mathfrak{g})^g + N \subseteq S(\mathfrak{g}_0)^{g_0} + N$. Conversely assume by induction that $S_{n-1}(\mathfrak{g}_0)^{g_0} \subseteq S(\mathfrak{g})^g + N$. If $v \in S_n(\mathfrak{g}_0)^{g_0}$, write $v = \eta_0^{-1}(x)$ with $x \in S_n(\mathfrak{h})^W$. Since η is surjective for $\mathfrak{g} = osp(1, 2r), x = \eta(u)$ for $u \in S_n(\mathfrak{g})^g$. By the lemma $u - v \in S_{n-1}(\mathfrak{g}_0)^{g_0} + N \subseteq S(\mathfrak{g})^g + N$, so $v \in S(\mathfrak{g})^g + N$.

1.3. Lemma. Suppose $S = \oplus_{i=0}^m S(i)$ is a graded ring, and R a graded subring of S such that $R \cap N = 0$, where $N = \oplus_{i=1}^m S(i)$. Assume that for each i , N^i/N^{i+1} is a free left R -module with basis the images of the elements $\{\omega_{i,\lambda} \in N^i | \lambda \in \Omega_i\}$. Then S is a free left R -module with basis $\{\omega_{i,\lambda} | (i, \lambda) \in \mathbb{N} \times \Omega_i\}$.

Proof. For $r \in R$, we write $r = r^0 + r^+$ with $r^0 \in R(0)$ and $r^+ \in N$. Fix i and set $N'_i = \sum R w_{j,\lambda}$ (sum over $j \geq i$ and $(j, \lambda) \in \mathbb{N} \times \Omega_j$). By reverse induction on i we show that $N'_i = N^i$. Observe that N'_i contains the elements $r w_{i,\lambda} = (r^0 + r^+) \omega_{i,\lambda}$, for $r \in R$, and by induction N'_i also contains $r^+ \omega_{i,\lambda}$. It follows that $N'_i + N^{i+1} = N^i$ and so $N'_i = N^i$.

Now suppose we have a relation $\sum r_{j,\lambda} \omega_{j,\lambda} = 0$ where not all the coefficients $r_{j,\lambda} \in R$ are zero, and among such relations assume that $i = \min\{j | r_{j,\lambda} \neq 0 \text{ for some } \lambda \in \Omega_j\}$ is chosen as small as possible. Then $\sum r_{i,\lambda} \omega_{i,\lambda} \in N^{i+1}$ gives $r_{i,\lambda}^0 = 0$ for all λ , but this is impossible since $R \cap N = 0$.

1.4. Now let $\mathfrak{g} = osp(1, 2r)$, $S = S(\mathfrak{g})$, $R = S(\mathfrak{g})^g$ and $R^+ = \oplus_{n \geq 1} R_n$. Since all finite dimensional \mathfrak{g} -modules are completely reducible we can choose an $ad\mathfrak{g}$ stable complement H to R^+S in S .

Theorem. The multiplication map induces an isomorphism of $ad\mathfrak{g}$ -modules $R \otimes H \rightarrow S$. In addition if P is any finite dimensional simple \mathfrak{g} -module, the multiplicity of P as a direct summand of H under the adjoint action is equal to the dimension of the weight space P^0 of P .

Proof. By [D, 8.2.2] There is a $ad\mathfrak{g}_0$ -stable subspace L of $S(\mathfrak{g}_0)$ such that $S(\mathfrak{g}_0)^{\mathfrak{g}_0} \otimes L \cong S(\mathfrak{g}_0)$ via the multiplication map. Write $S(i) = S(\mathfrak{g}_0) \otimes \Lambda^i$, $S = \bigoplus_{i=0}^m S(i)$ and $N = \bigoplus_{i=1}^m S(i)$ as in 1.2. Then as $ad\mathfrak{g}_0$ -modules $S(i) = S(\mathfrak{g}_0)^{\mathfrak{g}_0} \otimes L \otimes \Lambda^i$ and since $R \cong S(\mathfrak{g}_0)^{\mathfrak{g}_0}$ by Corollary 1.2, we see that N^i/N^{i+1} is a free R -module. Therefore S is a free R -module by Lemma 1.3. More precisely, if $H' = L \otimes \Lambda\mathfrak{g}_1$, the multiplication map $R \otimes H' \rightarrow S$ is an isomorphism of \mathfrak{g}_0 -modules and of graded vector spaces. Since $R = \mathbb{C} \oplus R^+$ this implies that $S = R^+S \oplus H'$.

If $V = \bigoplus_{n \geq 0} V_n$ is a graded vector space, set $P_V(t) = \sum_{n \geq 0} (\dim V_n) t^n$. Since $S = R \otimes H'$ we have $P_S(t) = P_R(t)P_{H'}(t)$. On the other hand since $S = R^+S \oplus H' = R^+S \oplus H'$ as graded vector spaces $P_H(t) = P_{H'}(t)$ so $P_S(t) = P_R(t)P_H(t)$. Since $S = RH$ by [Ko, Proposition 1, page 336], it follows that the multiplication map $R \otimes H \rightarrow S$ is bijective, and so is an isomorphism of $ad\mathfrak{g}$ -modules.

In addition $H \cong H'$ as \mathfrak{g}_0 -modules, and as a \mathfrak{g}_0 -module H' is isomorphic to the induced module $Ind_{\mathfrak{g}_0}^{\mathfrak{g}}(L) = U(\mathfrak{g}) \otimes_{U(\mathfrak{g}_0)} L$. Since H and $Ind_{\mathfrak{g}_0}^{\mathfrak{g}}(L)$ are locally finite dimensional \mathfrak{g} -modules with the same formal character, and all finite dimensional modules are completely reducible, $H \cong Ind_{\mathfrak{g}_0}^{\mathfrak{g}}(L)$ as \mathfrak{g} -modules. Now we use Frobenius reciprocity to prove the statement about multiplicities.

Write $L = \bigoplus_M (L : M)M$ where $(L : M)$ is the multiplicity of the simple \mathfrak{g}_0 -module M in L . Then $H \cong \bigoplus_M (L : M)Ind_{\mathfrak{g}_0}^{\mathfrak{g}}(M)$, and by [P, Exercise 2 (c), page 178], if P is any finite dimensional simple \mathfrak{g} -module then $(Ind_{\mathfrak{g}_0}^{\mathfrak{g}}(M) : P) = (P|_{\mathfrak{g}_0} : M)$. Thus

$$\begin{aligned} (H : P) &= \sum_M (L : M)(P|_{\mathfrak{g}_0} : M) \\ &= \sum_M \dim M^0(P|_{\mathfrak{g}_0} : M) = \dim P^0. \end{aligned}$$

Here we have also used the fact that $(L : M) = \dim M^0$, [D, 8.3.6].

1.5. Theorem. Let $\pi : S(\mathfrak{g}) \rightarrow U(\mathfrak{g})$ be the symmetrization map, and

$K = \pi(H)$. Then the multiplication map $K \otimes Z(\mathfrak{g}) \rightarrow U(\mathfrak{g})$ is an isomorphism of \mathfrak{g} -modules with respect to the adjoint action. Moreover if P is a finite dimensional simple \mathfrak{g} -module, the multiplicity of P in K equals $\dim P^0$.

Proof. This now follows as in the classical case [D,8.2.4].

2. Algebraic Relations between $Z(\mathfrak{g})$ and $Z(\mathfrak{g}_0)$

2.1. Lemma. If $g = \prod_{\beta \in \Delta_1^+} h_\beta$, then $gS(\mathfrak{h})^W \subseteq \text{Im} \psi$

Proof. This follows immediately from Theorem 1.1 (c).

2.2. Let β_1, \dots, β_r be the unique basis of simple roots of $[\mathfrak{g}_0, \mathfrak{g}_0]$ contained in Δ_0^+ . Note that the module $L(\lambda)$ will be finite dimensional if $\tilde{L}(\lambda)$ is finite dimensional, since in this case any element of V_λ will generate a finite dimensional \mathfrak{g}_0 -module. It is well known that $L(\lambda)$ is finite dimensional if and only if $(\lambda, \beta_i^\vee) \in \mathbb{N}$ for $i = 1, \dots, r$. We say that λ is dominant in this case. Necessary and sufficient conditions for $\tilde{L}(\lambda)$ to be finite dimensional are given in [Kac 1, Theorem 8]. From this result it follows that if λ is dominant and not too close to a reflecting hyperplane for the Weyl group, then $\tilde{L}(\lambda)$ is finite dimensional. Hence we conclude that

$$\Lambda' = \{\lambda \in \mathfrak{h}^* \mid \tilde{L}(\lambda) \text{ is finite dimensional and } \lambda \text{ is typical}\}$$

is Zariski dense in \mathfrak{h}^* .

We consider the decomposition of $\tilde{L}(\lambda)$ as a direct sum of simple \mathfrak{g}_0 -modules

$$\tilde{L}(\lambda) = \oplus m_\lambda(\mu) L(\mu)$$

Let Γ be the set of sums of distinct odd positive roots, and for $\gamma \in \Gamma$, let $K(\gamma)$ be the number of partitions of γ into distinct odd positive roots

Proposition. If λ is typical, then

$$m_\lambda(\mu) = (\dim V_\lambda) \sum_{w \in W} K(w(\lambda + \rho) - \mu - \rho)$$

Proof. If \mathfrak{g} is basic, this is deduced in [Kac 3, 2.11] from the character formula for typical modules [Kac 2].

Intuitively, since Γ is finite, we should expect that for λ “not too close to a wall”, the only term that could contribute to the above sum is the one with $w = 1$. More precisely we have

Corollary. (Compare [Pe, Corollary 2.1]) Set

$$a = \min\{0, 3(\gamma + \rho, \beta_i^\vee) | i = 1, \dots, r; \gamma \in \Gamma\},$$

$$b = \max\{0, (\gamma + \rho, \beta_i^\vee) | i = 1, \dots, r; \gamma \in \Gamma\}$$

and

$$\Lambda = \{\lambda \in \Lambda' | (\lambda + \rho, \beta_i^\vee) > b - a \text{ for } i = 1, \dots, r\}.$$

Then Λ is Zariski dense in \mathfrak{h}^* and for $\lambda \in \Lambda$ we have as a \mathfrak{g}_0 -module

$$\tilde{L}(\lambda) = \oplus_{\gamma \in \Gamma} K(\gamma) L(\lambda - \gamma)$$

Proof. For $\gamma \in \Gamma, \lambda \in \Lambda$ we have $(\lambda - \gamma, \beta_i^\vee) \geq -a \geq 0$. Also if $\Lambda \mathfrak{g}_1$ denotes the exterior algebra on \mathfrak{g}_1 , then γ is a weight of the finite dimensional \mathfrak{g}_0 -module $\Lambda \mathfrak{g}_1$, so $(\gamma, \beta_i^\vee) \in \mathbb{Z}$. Since $\lambda \in \Lambda'$ this implies that $\lambda - \gamma$ is dominant.

Suppose that μ is dominant, $\lambda \in \Lambda$ and $w \in W$ are such that $\gamma = w(\lambda + \rho) - \mu - \rho$ satisfies $K(\gamma) \neq 0$. By the proposition it suffices to show that $w = 1$.

Since μ is dominant we have

$$(\lambda + \rho, w^{-1}\beta_i^\vee) = (\gamma + \mu + \rho, \beta_i^\vee) \geq (\gamma + \rho, \beta_i^\vee) \geq a/3.$$

On the other hand if $w \neq 1$, then by [H, Lemma 10.3A] we have $w^{-1}\beta_i < 0$ for some i . Write $-w^{-1}\beta_i = \sum n_j \beta_j$ where $n_j \in \mathbb{N}$ and $\sum n_j \geq 1$. Since $(\beta_j, \beta_j)/(\beta_i, \beta_i) \geq 1/3$ for all j , we have

$$\begin{aligned} (\lambda + \rho, w^{-1}\beta_i^\vee) &\leq -(1/3)(\lambda + \rho, \sum n_j \beta_j^\vee) \\ &< (a/3) \sum n_j \leq a/3. \end{aligned}$$

This contradiction shows that $w = 1$.

2.3. Lemma. The set $\rho_1 - \Gamma = \{\rho_1 - \gamma \mid \gamma \in \Gamma\}$ is W -invariant.

Proof. This follows from the W -invariance of the function

$$\Pi_{\alpha \in \Delta_1^+}(e^{\alpha/2} + e^{-\alpha/2}) = \sum_{\gamma \in \Gamma} K(\gamma) e^{\rho_1 - \gamma}$$

The lemma allows us to define an action of W on Γ by

$$w * \gamma = \rho_1 - w(\rho_1 - \gamma)$$

for $w \in W$ and $\gamma \in \Gamma$.

2.4. We make frequent use of the following result.

Lemma. Suppose \mathfrak{g} is classical simple and not of type $P(n)$. If $\Lambda \subseteq \mathfrak{h}^*$ is Zariski dense, then $\cap_{\lambda \in \Lambda} \text{ann}_{U(\mathfrak{g})} \tilde{L}(\lambda) = 0$.

Proof. See [LM, Corollary D].

2.5. Theorem. If $x \in Z(\mathfrak{g}_0)$, there exist $z_i \in Z(\mathfrak{g}), i = 0, \dots, |\Gamma|$ with $\psi(z_{|\Gamma|}) = g$ such that $\sum_{i=0}^{|\Gamma|} x^i z_i = 0$.

Proof. For $\gamma \in \Gamma$, we define $q_\gamma \in S(\mathfrak{h})$ by

$$q_\gamma(\mu) = \psi_0(x)(\mu + \rho_1 - \gamma)$$

for $\mu \in \mathfrak{h}^*$. Note that x acts on $L(\lambda - \gamma)$ as the scalar $q_\gamma(\lambda + \rho)$. Thus if Λ is the Zariski dense subset of \mathfrak{h}^* given by Corollary 2.2 and $\lambda \in \Lambda$, we have $\tilde{L}(\lambda) = \oplus_{\gamma \in \Gamma} L(\lambda - \gamma)$, and so $\Pi(x - q_\gamma(\lambda + \rho)) \in \text{ann}_{U(\mathfrak{g})} \tilde{L}(\lambda)$. We observe that

$$\begin{aligned} q_{w*\gamma}(w(\lambda + \rho)) &= \psi_0(x)(w(\lambda + \rho_0 - \gamma)) \\ &= q_\gamma(\lambda + \rho) \end{aligned}$$

since $\psi_0(x)$ is W -invariant. Now let t be an indeterminate and consider the monic polynomial

$$\Pi_{\gamma \in \Gamma}(t - q_\gamma) = \sum_{i=0}^{|\Gamma|} t^i f_i \in S(\mathfrak{h})[t].$$

It follows that the coefficients f_i satisfy $f_i(w(\lambda + \rho)) = f_i(\lambda + \rho)$ for $w \in W$, that is $f_i \in S(\mathfrak{h})^W$. Therefore by Lemma 2.1 there exist $z_i \in Z(\mathfrak{g})$ such that $\psi(z_i) = gf_i$. Since z_i acts on $\tilde{L}(\lambda)$ as the scalar $(gf_i)(\lambda + \rho)$ we have that $\sum_{i=0}^{|\Gamma|} x^i z_i$ acts like

$$g(\lambda + \rho) \sum x^i f_i(\lambda + \rho) = g(\lambda + \rho) \Pi_\gamma(x - q_\gamma(\lambda + \rho)) = 0.$$

Since Λ is dense we have, by Lemma 2.4

$$\sum x^i z_i \in \cap_{\lambda \in \Lambda} \text{ann}_{U(\mathfrak{g})} \tilde{L}(\lambda) = 0.$$

Corollary. If $\mathfrak{g} = osp(1, 2r)$, then $Z(\mathfrak{g}_0)$ is integral over $Z(\mathfrak{g})$.

2.6. Let $\mathfrak{g} = osp(1, 2r)$. We require a more precise equation relating the degree two Casimir elements Q and C of $U(\mathfrak{g}_0)$ and $U(\mathfrak{g})$. We first normalize these so that C acts on $\tilde{L}(\lambda)$ as the scalar $(\lambda + 2\rho, \lambda)$ and Q acts on $L(\lambda)$ as the scalar $(\lambda + 2\rho_0, \lambda)$. Then $x = Q - C + 2(\rho, \rho_1)$ acts on the \mathfrak{g}_0 -submodule $L(\lambda - \gamma)$ as the scalar

$$\begin{aligned} & (\lambda + 2\rho_0 - \gamma, \lambda - \gamma) - (\lambda + 2\rho, \lambda) + 2(\rho, \rho_1) \\ &= 2(\lambda + \rho, \rho_1 - \gamma) - (\gamma, \tilde{\gamma}) \\ &= 2(\lambda + \rho, \rho_1 - \gamma). \end{aligned}$$

Set $\prod_{\gamma \in \Gamma} (t - 2(\lambda + \rho, \rho_1 - \gamma)) = \sum_{i=0}^{|\Gamma|} t^i f_i$, with $f_i \in S(\mathfrak{h})$. As in the proof of Theorem 2.5, there exist $z_i \in Z(\mathfrak{g})$ such that $\psi(z_i) = f_i$ and we have $\sum x^i z_i = 0$.

Lemma. Set $F(t) = \sum t^i z_i$. Then F is the minimum polynomial of x over $Z(\mathfrak{g})$.

Proof. By the above remarks, the minimum polynomial G of x over $Z(\mathfrak{g})$ divides F . Choose $\lambda \in \Lambda$ such that $(\lambda + \rho, \gamma - \gamma') \neq 0$ if $\gamma, \gamma' \in \Gamma$ and $\gamma \neq \gamma'$. Since x acts on $\tilde{L}(\lambda)$ with $|\Gamma|$ distinct eigenvalues, it follows that $F = G$.

2.7. Another application of our method is used to obtain the following result.

Theorem. Let \mathfrak{g} be classical simple and suppose that $K(\gamma) = 1$ for all $\gamma \in \Gamma$. Then $U = U(\mathfrak{g})^{\mathfrak{g}_0}$ is commutative.

Proof. Let Λ be the Zariski dense subset of \mathfrak{h}^* defined in Corollary 2.2 and set

$$\Lambda^+ = \{\lambda \in \Lambda \mid (\lambda + \rho, \gamma - \gamma') = 0, \text{ for all distinct } \gamma, \gamma' \in \Lambda.\}$$

Note that Λ^+ is Zariski dense in \mathfrak{h}^* , and if $\lambda \in \Lambda^+$, then $\tilde{L}(\lambda)$ is the direct sum of $|\Gamma|$ nonisomorphic irreducible \mathfrak{g}_0 -modules. Hence $D(\lambda) = \text{End}_{U(\mathfrak{g}_0)} \tilde{L}(\lambda) \cong \bigoplus \mathbb{C}^{|\Gamma|}$ is commutative. There is a homomorphism $U \longrightarrow D(\lambda)$ induced by left multiplication and it follows that $xy - yx \in \text{ann}_{U(\mathfrak{g})} \tilde{L}(\lambda)$ for all $x, y \in U$. Therefore the result follows from Lemma 2.4.

Remarks. 1) We observe that the hypothesis on $K(\gamma)$ holds in the theorem if $\mathfrak{g} = \mathfrak{sl}(m, 1)$ or $\mathfrak{g} = \mathfrak{osp}(1, 2r)$.

2) For any basic classical simple Lie superalgebra, a similar proof shows that $U(\mathfrak{g})^{\mathfrak{g}_0}$ is a P.I. ring.

2.8. We close this section with another application of Lemma 2.1. The result is probably well-known to experts in the field, but we have been unable to find it in the literature.

Proposition. $Z(\mathfrak{g})$ is Noetherian if and only if $\mathfrak{g} = \mathfrak{osp}(1, 2r)$ for some r .

Proof. If $\mathfrak{g} = \mathfrak{osp}(1, 2r)$, then $Z(\mathfrak{g}) \cong S(\mathfrak{h})^W$ by so $Z(\mathfrak{g})$ is Noetherian. For the converse, suppose $\mathfrak{g} \neq \mathfrak{osp}(1, 2r)$ and set $Z = \text{Im} \psi$. There is an odd root α of \mathfrak{g} such that $(\alpha, \alpha) = 0$. We can find an $x \in S(\mathfrak{h})^W$ such that $x(t\alpha) = f(t)$ is a nonconstant polynomial. If the ideal $\sum_{i \geq 0} gx^i Z$ of Z is finitely generated, then $gx^n \in \sum_{i=0}^{n-1} gx^i Z$ for some n . Therefore since Z is a domain we have an equation of the form $\sum_{i=0}^n x^i z_i = 0$ with $z_i \in Z$ and $z_n = 1$. However since $z_i(t\alpha) = z_i(0)$ for all $t \in \mathbb{C}$ this gives $\sum_{i=0}^n z_i(0) f^i = 0$, a contradiction.

3. Connections with Representation Theory

3.1. Let \mathfrak{g} be classical simple. We denote by \mathcal{O} , the category of \mathfrak{g}_0 -modules defined in [J, 4.3] and by $\tilde{\mathcal{O}}$ the category of graded \mathfrak{g} -modules which belong to the category \mathcal{O} when regarded as \mathfrak{g}_0 -modules by restriction, see [M1, 1.1]. For $\lambda \in \mathfrak{h}^*$ and M a \mathfrak{g}_0 -module, we set as in [J, 4.4 (3)],

$$pr_\lambda M = \{v \in M \mid \text{for all } z \in Z(\mathfrak{g}_0); (z - \chi_\lambda^0(z))^n v = 0, \text{ for } n \gg 0\}$$

Similarly if λ is typical and M a \mathfrak{g} -module we set

$$M_{(\lambda)} = \{v \in M \mid \text{for all } z \in Z(\mathfrak{g}), (z - \chi_\lambda(z))^n v = 0, \text{ for } n \gg 0\},$$

and let $\tilde{\mathcal{O}}_\lambda$ be the full subcategory of the category $\tilde{\mathcal{O}}$ consisting of modules M such that $M = M_{(\lambda)}$. Note the category $\tilde{\mathcal{O}}_\lambda$ is not quite analogous to the category \mathcal{O}_λ defined in [J, 4.4]. For generic irreducible \mathfrak{g} -modules, results similar to the next one are proved in [Pe 2, Section 2] using \mathcal{D} -module techniques.

Theorem. If λ is typical and $M \in \tilde{\mathcal{O}}_\lambda$ we have $M = \oplus_{\mu \in \lambda - \Gamma} pr_\mu M$.

Proof. By [M1, Cor. 1.1] any module in $\tilde{\mathcal{O}}$ has a finite filtration by graded modules whose factors are homomorphic images of Verma modules. Thus we may assume that $M = \tilde{M}(\lambda)$. As in [J, 4.4(4)] we have $M = \oplus_{\mu \in \mathfrak{h}^*} pr_\mu M$. If $\mu \notin \lambda - \Gamma$ we show that $pr_\mu M = 0$. By [Kap, Theorem 81], we can find $x \in m_\mu^0$ such that $x \notin m_{\lambda-\gamma}^0$ for all $\gamma \in \Gamma$. By Theorem 2.5, if $x \in Z(\mathfrak{g}_0)$, there exist elements $z_i \in Z(\mathfrak{g})$ with $\psi(z_{|\Gamma|}) = g$ such that $\sum_{i=0}^{|\Gamma|} x^i z_i = 0$. The action of x on $\tilde{M}(\lambda)$ satisfies

$$0 = \sum x^i \psi(z_i)(\lambda + \rho) = g(\lambda + \rho) \prod_{\gamma \in \Gamma} (x - \psi_0(x)(\lambda + \rho_0 - \gamma))$$

Since λ is typical, $g(\lambda + \rho) \neq 0$, so $\prod (x - \chi_{\lambda-\gamma}^0(x)) pr_\mu M = 0$. The result follows from this.

3.2 We now turn our attention to the Verma modules $\tilde{M}(\lambda)$. For the next result λ need not be typical.

Theorem. As a \mathfrak{g}_0 -module $\tilde{M}(\lambda)$ has a filtration $0 = M_0 \subset M_1 \subset \dots \subseteq$

$M_n = \widetilde{M}(\lambda)$ such that each factor M_{i+1}/M_i is isomorphic to a Verma module $M(\lambda - \gamma)$ with $\gamma \in \Gamma$. The module $M(\lambda - \gamma)$ occurs with multiplicity $(\dim V_\lambda)K(\gamma)$ in this filtration.

Proof. For ease of notation we shall assume that $\Delta_1^+ \cap \Delta_0^+ = \emptyset$ and that $\dim \mathfrak{g}^\alpha = 1$ for $\alpha \in \Delta$. This holds for “most” classical simple Lie superalgebras, see [Sch, Proposition 1, page 137], and we leave it to the reader to modify the argument which in the exceptional cases.

For each $\alpha \in \Delta^+ = \Delta_0^+ \cup \Delta_1^+$, let $e_{-\alpha}$ be a basis for $\mathfrak{g}^{-\alpha}$. Let \mathfrak{P} be the set of maps $\pi : \Delta^+ \rightarrow \mathbb{N}$ such that $\pi(\alpha) = 0$ or 1 for $\alpha \in \Delta_1^+$, and for $\pi \in \mathfrak{P}$, set $|\pi| = \sum_{\alpha \in \Delta^+} \pi(\alpha)\alpha$. Fix an ordering on Δ^+ , such that every element of Δ_0^+ precedes every element of Δ_1^+ . For $\pi \in \mathfrak{P}$, set $e_{-\pi} = \prod e_{-\alpha}^{\pi(\alpha)}$ where the product over $\alpha \in \Delta^+$ is taken with respect to the fixed order. Any $\pi \in \mathfrak{P}$ has a unique decomposition $\pi = \pi_0 + \pi_1$, where $\pi_0, \pi_1 \in \mathfrak{P}$, $\pi_0(\alpha) = 0$ for $\alpha \in \Delta_1^+$ and $\pi_1(\alpha) = 0$ for $\alpha \in \Delta_0^+$.

For any $\gamma \in Q^+$, set ${}^\gamma \widetilde{M}(\lambda) = \sum U(\mathfrak{n}_0^-) e_{-\pi} v_\lambda$ where the sum is over all $\pi \in \mathfrak{P}$, such that $\pi = \pi_1$, and $|\pi| \leq \gamma$. This sum is direct by the *PBW* Theorem. Also ${}^0 \widetilde{M}(\lambda) = 0$, and ${}^\gamma \widetilde{M}(\lambda) = \widetilde{M}(\lambda)$, if $\gamma > 2\rho_1$. If $\delta < \gamma$, then since $\widetilde{M}(\lambda)^{\lambda-\delta}$ has a basis consisting of all $e_{-\pi} v_\lambda$ with $|\pi| = \delta$, and $e_{-\pi} = e_{-\pi_0} e_{-\pi_1}$, we have $\widetilde{M}(\lambda)^{\lambda-\delta} \subseteq {}^\gamma \widetilde{M}(\lambda)$.

Now suppose $\pi = \pi_1 \in \mathfrak{P}$, with $|\pi| = \gamma$. If $\alpha \in \Delta_0^+$, we have $e_\alpha e_{-\pi} v_\lambda \in {}^\gamma \widetilde{M}(\lambda)$. It follows by induction on the partial order \leq , that ${}^\gamma \widetilde{M}(\lambda)$ is a $U(\mathfrak{g}_0)$ -module, and that the image of $e_{-\pi} v_\lambda$ in $\widetilde{M}(\lambda)/{}^\gamma \widetilde{M}(\lambda)$ is a highest weight vector for \mathfrak{g}_0 of weight $\lambda - \gamma$. Again by the *PBW* theorem the submodule generated by this vector is isomorphic to $M(\lambda - \gamma)$. The result now follows easily.

3.3 We can use the previous theorem to measure the size of $pr_\mu \widetilde{M}(\lambda)$. Let D be the division ring of fractions of $U(\mathfrak{n}_0)$, and for a $U(\mathfrak{n}_0)$ -module N , set

$$\text{rank}(N) = \dim_D(D \otimes_{U(\mathfrak{n}_0)} N).$$

Corollary. If $\lambda \in \mathfrak{h}^*$, and $\mu \in \mathfrak{h}^*/W$, then

$$\text{rank } pr_\mu(\widetilde{M}(\lambda)) = \dim V_\lambda \sum_{w \in W} K(\lambda + \rho_0 - w(\mu + \rho_0)).$$

Proof. This follows from Theorem 3.2 using the additivity of $\text{rank}(\cdot)$ on exact sequences, and the fact that $\chi_\mu^0 = \chi_{\lambda-\gamma}^0$ if and only if $\lambda - \gamma = w(\mu + \rho_0) - \rho_0$ for some $w \in W$.

3.4 Let \mathfrak{g} be classical simple. For $\lambda, \gamma \in \mathfrak{h}^*$ we set

$$Y(\lambda, \gamma) = \{y \in \widetilde{M}(\lambda)^{\lambda-\gamma} | \mathfrak{n}_0^+ y = 0\}.$$

Lemma. For all $m \in \mathbb{N}$ the set $\{\lambda \in \mathfrak{h}^* | \dim Y(\lambda, \gamma) \geq m\}$ is Zariski closed.

Proof. This is a routine adaptation of the proof of [D, 7.6.12].

Corollary. For all $\lambda \in \mathfrak{h}^*$, and $\gamma \in \Gamma$, $\dim Y(\lambda, \gamma) \geq K(\gamma)$.

Proof. Let Λ be the Zariski dense subset of \mathfrak{h}^* defined in 2.2, and

$$\Lambda_\gamma = \left\{ \lambda \in \Lambda \mid \begin{array}{l} \dim U(\mathfrak{n}^-)^{-\mu} = \dim \widetilde{L}(\lambda)^{\lambda-\mu} \\ \text{for all } \mu \leq \gamma \end{array} \right\}$$

It is easy to see that Λ_γ is Zariski dense in \mathfrak{h}^* . By Corollary 2.2, if $\lambda \in \Lambda_\gamma$ then $\widetilde{L}(\lambda)^{\lambda-\gamma}$ contains $K(\gamma)$ linearly independent maximal vectors. Since the map $\widetilde{M}(\lambda)^{\lambda-\mu} \rightarrow \widetilde{L}(\lambda)^{\lambda-\mu}$ is an isomorphism for $\mu \leq \gamma$, the result follows.

3.5. Proposition. For any $\lambda \in \mathfrak{h}^*$ and $\gamma \in \Gamma$, $\widetilde{M}(\lambda)$ contains a \mathfrak{g}_0 -submodule isomorphic to $M(\lambda - \gamma)$. Thus $\text{ann}_{U(\mathfrak{g}_0)} \widetilde{M}(\lambda) \subseteq m_{\lambda-\gamma}^0$.

Proof. By the *PBW* theorem $\widetilde{M}(\lambda)$ is free as a $U(\mathfrak{n}_0^-)$ -module. Hence for any $v \in Y(\lambda, \gamma)$, $U(\mathfrak{g}_0)v = U(\mathfrak{n}_0^-)v \cong M(\lambda - \gamma)$. Hence the result follows from Duflo's theorem.

3.6. In certain cases $\widetilde{M}(\lambda)$ is a direct sum of Verma modules for $U(\mathfrak{g}_0)$.

Theorem. Suppose $\lambda \in \mathfrak{h}^*$ is typical, $K(\gamma) = 1$ for all $\gamma \in \Gamma$, and the central characters $\chi_{\lambda-\gamma}^0$, $\gamma \in \Gamma$ are distinct. Then as a \mathfrak{g}_0 -module

$$\widetilde{M}(\lambda) = \oplus_{\gamma \in \Gamma} M(\lambda - \gamma).$$

Proof. The sum $\sum_{\gamma \in \Gamma} M(\lambda - \gamma)$ is contained in $\widetilde{M}(\lambda)$ by Proposition 3.5, and thus sum is direct by the assumption on central characters. The result follows since $\widetilde{M}(\lambda)$ and $\oplus_{\gamma \in \Gamma} M(\lambda - \gamma)$ have the same formal character.

Corollary. Let $A = (Z(\mathfrak{g}), Z(\mathfrak{g}_0))$ and $E(\lambda) = \text{End}_{U(\mathfrak{g}_0)} \widetilde{M}(\lambda)$. Under the hypotheses of the theorem, the map $A \longrightarrow E(\lambda)$ induced by left multiplication is surjective and $E(\lambda) \cong \mathbb{C}^{|\Gamma|}$.

Proof. This follows since for all $\gamma \in \Gamma$, we can choose $z_\gamma \in Z(\mathfrak{g}_0)$ such that $\chi_{\lambda-\gamma}^0(z_\gamma) \neq 0$ and $\chi_{\lambda-\beta}^0(z_\gamma) = 0$ for all $\beta \in \Gamma$, $\beta \neq \gamma$.

Remark. If $\mathfrak{g} = \mathfrak{sl}(2, 1)$, it is shown in [M3, 1.1] that $\widetilde{M}(\lambda)$ is a direct sum of Verma modules for \mathfrak{g}_0 if and only if $(\lambda + \rho, \alpha) \neq 0$, where α is the unique positive even root.

3.7. Let $\mathfrak{g} = \mathfrak{osp}(1, 2r)$. We can sharpen Theorem 3.6 in this case. Let L be any \mathbb{Z}_2 -graded factor module of $\widetilde{M}(\lambda)$. The grading on L can be expressed in terms of the root lattices for \mathfrak{g} and \mathfrak{g}_0 . In the notation of 0.3, set $Q = \sum_{i=1}^r \mathbb{Z}\alpha_i$ and $Q' = 2\mathbb{Z}\alpha_r + \sum_{i=1}^{r-1} \mathbb{Z}\alpha_i$. Then define $L(0) = \oplus_{\eta \in Q'} L^{\lambda-\eta}$ and $L(1) = \oplus_{\eta \in Q \setminus Q'} L^{\lambda-\eta}$.

Then $L = L(0) \oplus L(1)$ is a decomposition into \mathfrak{g}_0 -submodules such that $\mathfrak{g}_1 L(0) \subseteq L(1)$ and $\mathfrak{g}_1 L(1) \subseteq L(0)$.

When $L = \widetilde{M}(\lambda)$ (resp. $\widetilde{L}(\lambda)$), we denote the modules $\widetilde{M}(\lambda)(\epsilon)$ (resp. $\widetilde{L}(\lambda)(\epsilon)$) by $M(\lambda, \epsilon)$ (resp. $\widetilde{L}(\lambda, \epsilon)$). Then an argument similar to that used to prove Theorem 3.6 gives

Lemma. If all the central characters $\chi_{\lambda-\gamma}^0, \gamma \in \Gamma(\epsilon)$ are distinct, then as a \mathfrak{g}_0 -module.

$$\widetilde{M}(\lambda, \epsilon) = \oplus_{\gamma \in \Gamma(\epsilon)} M(\lambda - \gamma)$$

3.8. To investigate the condition that the central characters $\chi_{\lambda-\gamma}^0$ are distinct, we require a preliminary result. Recall the action of W on Γ defined in 2.3.

Lemma. If \mathfrak{g} is classical simple then W acts faithfully on Γ .

Proof. Suppose $w \in W$ and $w * \gamma = \gamma$ for all $\gamma \in \Gamma$. We have to show that $w = 1$. Since $0 \in \Gamma$ and $w(\rho_1 - \gamma) = \rho_1 - \gamma$ for all $\gamma \in \Gamma$, we have $w\gamma = \gamma$ for all $\gamma \in \Gamma$. As $W \subseteq GL(\mathfrak{h}^*)$ it is enough to show that Γ spans \mathfrak{h}^* . In fact Δ_1^+ spans \mathfrak{h}^* , since otherwise we could find a nonzero element $h \in \mathfrak{h}$ such that $[h, \mathfrak{g}_1] = 0$, and this would contradict [Sch, equation (2.12), page 93].

3.9 Proposition. If \mathfrak{g} is classical simple, and $\lambda \in \mathfrak{h}^*$ is such that the central characters $\chi_{\lambda-\gamma}^0$, for $\gamma \in \Gamma$, are distinct, then λ is regular.

Proof. Suppose that λ is not regular, so that $(\lambda + \rho, \alpha) = 0$ for some root α . By Lemma 3.8 $s_\alpha * \gamma = \gamma' \neq \gamma$ for some $\gamma \in \Gamma$. We claim that $\chi_{\lambda-\gamma}^0 = \chi_{\lambda-\gamma'}^0$. In fact for any $z \in Z(\mathfrak{g}_0)$ we have, since $\psi_0(z)$ is W -invariant,

$$\chi_{\lambda-\gamma}^0(z) = \psi_0(z)(\lambda + \rho_0 - \gamma) = \psi_0(z)(s_\alpha(\lambda + \rho_0 - \gamma)).$$

Since

$$\begin{aligned} s_\alpha(\lambda + \rho_0 - \gamma) &= \lambda + \rho + s_\alpha(\rho_1 - \gamma) \\ &= \lambda + \rho_0 - s_\alpha * \gamma = \lambda + \rho_0 - \gamma' \end{aligned}$$

the claim follows.

3.10. In certain cases there is a converse to Proposition 3.9. We abstract part of the argument needed to show this. Let Φ be a root system of type X_r ($X = C$ or D). We regard Φ as a subset of \mathbb{R}^r and use the notation of [H, Chapter 12], except that we denote the standard basis of \mathbb{R}^r by e_1, \dots, e_r . The Weyl group W of Φ acts as a subgroup of the group \widehat{W} of all signed permutations of $\mathbf{r} = \{1, \dots, r\}$ and hence there is a homomorphism $W \longrightarrow S_r$. The image of an element w of W in S_r is denoted \bar{w} . Let $\rho_1 = \frac{1}{2} \sum_{i=1}^r e_i$ and for a subset I of \mathbf{r} set $e_I = \sum_{i \in I} e_i$

Let $\Gamma = \{e_I | I \subseteq \mathbf{r}\}$, and for $\epsilon = 0$ or 1 ,

$$\Gamma(\epsilon) = \{e_I | I \subseteq \mathbf{r} \text{ and } |I| \equiv \epsilon \pmod{2}\}.$$

Lemma. Suppose that $X = C$ and $\Omega = \Gamma$ or $X = D$ and $\Omega = \Gamma(\epsilon)$. Then

if $\mu \in \mathbb{R}^r$ and $w(\mu - \gamma') = \mu - \gamma$ for some $\gamma, \gamma' \in \Omega, \gamma \neq \gamma'$ and $w \in W$, we have $(\mu - \rho_1, \alpha) = 0$ for some $\alpha \in \Phi$.

Proof. For $i = 1, \dots, r$ we have

$$(\mu - \gamma, we_i) = (\mu - \gamma', e_i).$$

Write $\gamma = \sum \epsilon_i e_i$ where $\epsilon_i = 0$ or 1 for all i . Assume for a contradiction that no α as in the statement of the lemma exists.

Consider the cycle decomposition of \bar{w} . Suppose if possible, that \bar{w} contains a cycle (l_1, \dots, l_t) of length $t > 1$. To avoid double subscripts later we set $l(i) = l_i$. Then with $l(t+1) = l(1)$ we have

$$(\mu - \gamma, e_{l(i+1)}) = \pm(\mu - \gamma', e_{l(i)})$$

for $i = 1, \dots, t$. We can take advantage of the fact that W acts transitively on $\rho_1 - \Gamma$ to simplify our notation. If $s \in W$ and $\lambda = s(\mu - \rho_1) + \rho_1$, then $sws^{-1}(\lambda - s_*\gamma') = (\lambda - s_*\gamma)$, and from $(\lambda - \rho_1, \alpha) = 0$, it follows that $(\mu - \rho_1, s^{-1}\alpha) = 0$.

Now suppose that $X = C$. By the above, we may assume $\gamma' = 0$. Then one of the following holds for $i = 1, \dots, t$

$$(\mu - \rho_1, e_{l(i+1)} - e_{l(i)}) = \epsilon_{l(i+1)} \quad (i.1)$$

$$(\mu - \rho_1, e_{l(i+1)} + e_{l(i)}) = \epsilon_{l(i+1)} - 1 \quad (i.2)$$

The combinations of (i.1) and (i+1.2) or (i.2) and (i+1.2) contradict the assumption on $\mu - \rho_1$. Hence for each i , (i.1) holds and $\epsilon_{l(i)} = 1$, but then summing the resulting equations again gives a contradiction.

If $\bar{w} = 1$, then $we_i = \pm e_i$ for all i . Since $\gamma \neq 0$, we may choose i so that $(\gamma, e_i) = 1$. In this case we have $(\mu - \gamma, e_i) = -(\mu, e_i)$, and so $(\mu - \rho_1, e_i) = 0$.

Next suppose that $X = D$, and $\epsilon = 0$. By transitivity of W we may assume $\gamma' = 0$. If $\bar{w} = 1$, then since $\gamma \neq 0$ we have $\epsilon_i = \epsilon_j = 1$ for some $i \neq j$. This implies $2(\mu, e_k) = 1$ for $k = i, j$ and hence $(\mu - \rho_1, e_i - e_j) = 0$.

Finally if $X = D$ and $\epsilon = 1$, choose $s \in \widehat{W} \setminus W$. Then $s_*\Gamma(1) = \Gamma(0)$ and $sws^{-1}(\lambda - s_*\gamma') = \lambda - s_*\gamma$ where $\lambda = s(\mu - \rho_1) + \rho_1$, so the result in this case follows from the case $\epsilon = 0$.

3.11 Corollary. Suppose $\mathfrak{g} = osp(1, 2r)$ and $\lambda \in \mathfrak{h}^*$.

- (a) The central characters $\chi_{\lambda-\gamma}^0, \gamma \in \Gamma$ are distinct if and only if λ is regular.
- (b) For $\epsilon = 0$ or 1 , the central characters $\chi_{\lambda-\gamma}^0, \gamma \in \Gamma(\epsilon)$ are distinct if and only if $(\lambda + \rho, \alpha) \neq 0$ for all $\alpha \in \overline{\Delta}_0^+$.

Proof. If λ is regular with respect to Δ_0 , then the central characters $\chi_{\lambda-\gamma}^0, \gamma \in \Gamma$ are distinct by Proposition 3.9. Conversely if γ, γ' are distinct elements of Γ such that $\chi_{\lambda-\gamma}^0 = \chi_{\lambda-\gamma'}^0$, then Lemma 3.10, with $\mu = \lambda + \rho_0$, and $X = C$, yields $(\lambda + \rho, \alpha) = 0$ for some $\alpha \in \overline{\Delta}_0^+$. This proves (a), and since the set of short roots in a root system of type C_r forms a root system of type D_r , (b) follows in a similar way.

Remark. If $\mathfrak{g} = \mathfrak{sl}(r, 1)$, statement (a) of the corollary is valid in this case also. The proof is similar to the proof of Lemma 3.10, but requires some minor changes since the invariant form (\cdot, \cdot) is not positive definite. We can identify \mathfrak{h}^* with \mathbb{C}^r in such a way that the positive odd roots of \mathfrak{g} correspond to the standard basis e_1, \dots, e_r of \mathbb{C}^r . Then (\cdot, \cdot) is given up to a scalar multiple by

$$\left(\sum_{i=1}^r x_i e_i, \sum_{i=1}^r y_i e_i \right) = \sum_{i \neq j} x_i y_j.$$

Let $\bar{e}_i = (-(r-2)e_i + \sum_{j \neq i} e_j)$. Then $(e_i, \bar{e}_j) = \delta_{ij}$. In addition $\bar{e}_i - \bar{e}_j = (e_j - e_i)$ and W permutes the \bar{e}_i in the same way that it permutes the e_i .

With e_I as before, set $\Gamma = \{e_I | I \subseteq \mathbf{r}\}$. If $w(\mu - \gamma) = \mu - \gamma'$ for some $\gamma, \gamma' \in \Gamma, \gamma \neq \gamma'$, then starting from

$$(\mu - \gamma, w\bar{e}_i) = (\mu - \gamma', \bar{e}_i)$$

for all $i = 1, \dots, r$, we obtain $(\mu - \rho_1, e_i - e_j)$ for some $i \neq j$.

4. A Geometric Approach to $Z(\mathfrak{g})$.

4.1. In view of Proposition 2.8 it seems at first that geometric methods will be of little use in the study of $Z(\mathfrak{g})$. However Theorem 2.5 and its proof suggest the following approach. Let $B = Z(\mathfrak{g})$ and $A = (Z(\mathfrak{g}_0), Z(\mathfrak{g}))$. The inclusion $B \subseteq A$ induces a map of spectra $\psi : \text{Spec} A \rightarrow \text{Spec} B$ and we consider the fibers of ψ . The behaviour of these fibers, at least in the examples we consider, is closely related to the representation theory of $U(\mathfrak{g})$.

Let P be a prime ideal of B . The fibre over the point of $\text{Spec} B$ corresponding to P is the closed subscheme $\text{Spec}(A/AP)$ of $\text{Spec} A$. Set $\tilde{A} = Z(\mathfrak{g}_0) \otimes_{\mathbb{C}} Z(\mathfrak{g})$ and let $\phi : \tilde{A} \longrightarrow A$ be the multiplication map $a \otimes b \longrightarrow ab$. Then ϕ is surjective and if $I = Z(\mathfrak{g}_0) \otimes P$, then $\phi(I) = AP$. Now $\tilde{A}/I \cong Z(\mathfrak{g}_0) \otimes (B/P)$ and ϕ induces a surjective map from \tilde{A}/I onto A/AP . Thus A/AP is a finitely generated algebra provided that B/P is finitely generated.

Now suppose P is a maximal ideal of B . Then since B has countable dimension over \mathbb{C} , $B/P \cong \mathbb{C}$. Let $\bar{\cdot} : B \longrightarrow \mathbb{C}$ be the natural map.

Lemma. In this situation $A/AP \cong Z(\mathfrak{g}_0)/J$ where

$$J = \{ \sum a_i \bar{b}_i \mid \sum a_i \otimes b_i \in \ker \phi \}$$

Proof. The commutative algebra A is generated by the subalgebras $Z(\mathfrak{g})$ and $Z(\mathfrak{g}_0)$ subject to the relations given by $\ker \phi$. Since $B/P \cong \mathbb{C}$ via $\bar{\cdot}$, we may eliminate the generators for $Z(\mathfrak{g})$ at the expense of introducing relations on $Z(\mathfrak{g}_0)$ corresponding to the ideal J .

4.2. We illustrate the foregoing remarks by computing a family of fibers when $\mathfrak{g} = \mathfrak{sl}(2, 1)$. First we need some notation, see [M3]. The Cartan subalgebra \mathfrak{h} of \mathfrak{g} has a basis consisting of the diagonal matrices h, z with entries $1, -1, 0$ and $1, 1, 2$ respectively. For $\lambda \in \mathfrak{h}^*$ we write $\lambda = (a, b)$ where $\lambda(h) = a, \lambda(z) = b$. There are Casimir elements $Q \in U(\mathfrak{g}_0)$ and $K \in U(\mathfrak{g})$ such that Q acts on $L(\lambda)$ as the scalar $a(a+2)$ and K acts on $\tilde{L}(\lambda)$ as the scalar $(a-b)(a+b+2)$. Let $L_1 = Q - z(z+2)$ and $L_2 = Q - z(z-2)$. Then we have $Z(\mathfrak{g}_0) = \mathbb{C}[z, Q] = \mathbb{C}[L_1, L_2]$. By Theorem 1.1 there is a central element C in the localization of $U(\mathfrak{g})$ with respect to the powers of K such that K and C are algebraically independent and $B = Z(\mathfrak{g})$ is the subring $\mathbb{C} + K\mathbb{C}[K, C]$ of $\mathbb{C}[K, C]$. This is also shown by a direct calculation in [ABP, Proposition IV.4.1.]. Also $\lambda = (a, b)$ is typical if and only if $(a-b)(a+b+2) \neq 0$, and in this case $\tilde{L}(\lambda)$ becomes a $\mathbb{C}[K, C]$ -module with C acting as the scalar b . For $\mu \in \mathbb{C}$, set $P_\mu = (K - \mu)\mathbb{C}[K, C] \cap Z(\mathfrak{g})$. Our aim is to compute the fibers of the map $\psi : \text{Spec} A \longrightarrow \text{Spec} B$ over the points P_μ .

First we need to describe the algebra A . Since K is central in $U(\mathfrak{g})$, and $U(\mathfrak{g})$ is prime [Be], K is not a zero divisor. Hence A and \tilde{A} embed in their respective localizations A_K and \tilde{A}_K and we can extend ϕ to a surjective map

of algebras $\phi_K : \tilde{A}_K \longrightarrow A_K$. Clearly $\ker \phi = \tilde{A} \cap \ker \phi_K$. We omit the tensor signs in writing elements of \tilde{A}_K . Let I_1, I_2, I_3 be the ideals of \tilde{A}_K given by $I_1 = (L_1 - K, z - C)$, $I_2 = (x, z - C - 1)$ and $I_3 = (L_2 - K, z - C - 2)$ where $x = (L_1 - K)(L_2 - K) - 4K$.

Proposition. $\ker \phi_K = I_1 \cap I_2 \cap I_3$.

Proof. We have $\Gamma = \{0, \beta, \alpha + \beta, \alpha + 2\beta\}$ where $\alpha = (2, 0)$ and $\beta = (-1, -1)$. Let Λ be the dense subset of \mathfrak{h}^* given in Corollary 2.2. Then for $\lambda \in \Lambda$ we have as \mathfrak{g}_0 -modules

$$\tilde{L}(\lambda) = \oplus_{i=1}^3 L^{(i)}(\lambda)$$

where $L^{(1)}(\lambda) = L(\lambda)$, $L^{(2)}(\lambda) = L(\lambda - \beta) \oplus L(\lambda - \alpha - \beta)$ and $L^{(3)}(\lambda) = L(\lambda - \alpha - 2\beta)$. Since all $\lambda \in \Lambda$ are typical, the map ϕ_K allows us to regard $\tilde{L}(\lambda)$ as an \tilde{A}_K -module. Now by Lemma 2.4 $\ker \phi_K = \cap_{\lambda \in \Lambda} \text{ann}_{\tilde{A}_K} \tilde{L}(\lambda)$. Therefore it suffices to show that $I_i = \cap_{\lambda \in \Lambda} \text{ann}_{\tilde{A}_K} L^{(i)}(\lambda)$ for $i = 1, 2, 3$. We show this equality holds for $i = 2$. The other cases are easier and left to the reader. Note first that $L_1 - K$ acts on $L(\lambda - \beta)$ as the scalar $2(a - b)$ and $L_2 - K$ as $2(a + b + 2)$, so $xL(\lambda - \beta) = 0$. Similar calculations show that $I_2 L^{(2)}(\lambda) = 0$ for all $\lambda \in \Lambda$.

Note that $\tilde{A}_K/I_2 \cong \mathbb{C}[L_1, L_2, K^{\pm 1}]/(x)$. Also x is a monic polynomial of degree 2 in K with coefficients in $\mathbb{C}[L_1, L_2]$. Thus to obtain the desired equality we must show that if

$$y = \sum a_{ij} L_1^i L_2^j + K \sum b_{ij} L_1^i L_2^j$$

annihilates $L^{(2)}(\lambda)$ for all $\lambda \in \Lambda$, then $y = 0$. In the above expression for y all but finitely many coefficients a_{ij}, b_{ij} are zero.

Now let $F_1 = (a - b)$, $F_2 = (a + b + 4)$, $G_1 = (a + b + 2)$, $G_2 = (a - b + 2)$, $F = F_1 F_2$ and $G = G_1 G_2$. Then if $\lambda = (a, b)$ and $yL(\lambda - \beta) = 0$ we obtain

$$0 = \sum a_{ij} F^i G^j + F_1 G_1 \sum b_{ij} F^i G^j. \quad (1)$$

This is a polynomial equation in $\lambda = (a, b)$ which is valid for all $\lambda \in \mathfrak{h}^*$ since it is valid on the dense subset Λ . Now choose $\lambda = (a, a)$ such that $G(\lambda) \neq 0$. Then $F_1(\lambda) = F(\lambda) = 0$ and we obtain

$$0 = \sum a_{0j} G^j(\lambda).$$

Thus $a_{0j} = 0$ for all j . Similarly choosing $\lambda = (a, -a - 2)$ shows that $a_{i0} = 0$ for all i . Next we can divide each term in equation (1) by $F_1 G_1$ to obtain

$$0 = F_2 G_2 \sum a_{ij} F^{i-1} G^{j-1} + \sum b_{ij} F^i G^j.$$

Arguing as before we obtain $b_{0j} = b_{i0}$ and each term in this equation is divisible by $F_2 G_2$. Continuing in this way shows $y = 0$ as required.

Remarks. 1) It follows from the proposition that $(L_1 - K)(L_2 - K)x = 0$. An equivalent equation is obtained in [ABP, Proposition III.5.1.], and further relations on central elements are given in [ABP, Proposition VI.6.1]. The proofs in [ABP] use a version of Lemma 2.4 for $\mathfrak{g} = \mathfrak{sl}(2, 1)$.

2) Rather surprisingly perhaps, the subalgebra $A = (Z(\mathfrak{g}_0), Z(\mathfrak{g}))$ of $U(\mathfrak{g})$ is finitely generated. To see this note that $Z(\mathfrak{g})$ is generated over \mathbb{C} by the elements $D_i = KC^i$ for $i \geq 0$. In A_K we have $(z - C)(z - C - 1)(z - C - 2) = 0$, which yields for all $i \geq 0$ that $D_{i+3} = 3D_{i+2}(z - 1) - D_{i+1}(3z^2 - 6z + 2) + D_i(z^3 - 3z^2 + 2z)$ in A . Thus A is generated over \mathbb{C} by z, Q, D_0, D_1 and D_2 .

4.3. We now consider the inclusion $B = Z(\mathfrak{g}) \subseteq A = (Z(\mathfrak{g}_0), Z(\mathfrak{g}))$ and the corresponding map on spectra $\psi : \text{Spec} A \longrightarrow \text{Spec} B$. For $\mu \in \mathbb{C}$ we compute the fiber $\text{Spec}(A/AP_\mu)$ of over P_μ .

Let $t = z - C$, and define idempotents $e_1, e_2, e_3 \in A_K$ by

$$e_1 = (t - 1)(t - 2)/2, \quad e_2 = t(2 - t)$$

and

$$e_3 = t(t - 1)/2.$$

Then $A_K = A_1 \oplus A_2 \oplus A_3$ where $A_i = A_K e_i \cong \tilde{A}_K / I_i$.

First consider the typical case where $\mu \neq 0$. We have $A/AP_\mu \cong A_K/A_K P_\mu$ and $\text{Spec}(A/AP_\mu)$ is the disjoint union of 3 irreducible components $X_i = \text{Spec}(\tilde{A}_K / (I_i, K - \mu))$. It is easy to see that X_2 is a hyperbola in A^3 and X_1 and X_3 are lines meeting X_2 at infinity.

In the atypical case $\mu = 0$, and P_0 is the maximal ideal $K\mathbb{C}[K, C]$ of $Z(\mathfrak{g})$ we claim that $A/AP_0 \cong \mathbb{C}[L_1, L_2]/(L_1 L_2)$. Note that $K e_i \in AP_0$ for each i . Also $(L_1 - K)e_1 \in \cap_{i=1}^3 \phi(I_i) = 0$. Thus $L_1 L_2 e_1 = L_2 K e_1 \in AP_0$. Similarly $L_1 L_2 e_3 = L_1 K e_3 \in AP_0$ and $L_1 L_2 e_2 = (L_1 + L_2 + 4 - K)K e_2 \in AP_0$. Thus $L_1 L_2 \in AP_0$. Thus it suffices to show that if $f \in AP_0 \cap Z(\mathfrak{g}_0)$ then f is divisible by $L_1 L_2$. Consider the finite dimensional atypical modules $\tilde{L}(\lambda)$. These

are annihilated by P_0 and contain $L(\lambda)$ as a \mathfrak{g}_0 -module direct summand. If $\lambda = (a, a)$ where $a \in \mathbb{N}$, then L_2 acts on $L(\lambda)$ as the scalar $-4(a+2)$, while if $\lambda = (a, -a-2)$, $a \in \mathbb{N}$, L_1 acts on $L(\lambda)$ as the scalar $4a$. The result follows from these remarks by varying a .

Remarks. 1) If $\mu \neq 0$, the projective closure of the component X_2 of $\text{Spec}(A/AP_\mu)$ is defined by the equations $(L_1 - \mu T)(L_2 - \mu T) = 4\mu T^2$, and $z - C = T$, where $z = (L_2 - L_1)/4$. Thus the shadow at infinity cast by X_2 is defined by $L_1 L_2 = z - C = 0$. Similarly the shadows of X_1 (resp. X_3) are by $z - C = 0$ and $L_1 = 0$ (resp. $L_2 = 0$). Then the fiber $\text{spec}(A/AP_0)$ may be regarded as the shadow at infinity cast by the typical fibers $\text{spec}(A/AP_\mu)$, $\mu \neq 0$.

2) Our work has implications for the structure of $U(\mathfrak{g})$ -modules when regarded as $U(\mathfrak{g}_0)$ -modules by restriction. For example we have

Corollary. If M is a $U(\mathfrak{g})$ -module such that $P_0 M = 0$, then $L_1 L_2 M = 0$.

5. The case of $\mathfrak{g} = \mathfrak{osp}(1, 2r)$.

5.1. Theorem. If $\mathfrak{g} = \mathfrak{osp}(1, 2r)$, there is a subspace M of $U(\mathfrak{g})^{\mathfrak{g}_0}$ such that $1 \in M$, $\dim M = 2^r$ and

$$U(\mathfrak{g})^{\mathfrak{g}_0} = Z(\mathfrak{g}) \otimes_{\mathbb{C}} M.$$

In particular $U(\mathfrak{g})^{\mathfrak{g}_0}$ is a free $Z(\mathfrak{g})$ -module of rank 2^r .

Proof. Let L be an $\text{ad}\mathfrak{g}_0$ -stable subspace of $U(\mathfrak{g}_0)$ such that $U(\mathfrak{g}_0) = Z(\mathfrak{g}_0) \otimes L$ as $\text{ad}\mathfrak{g}_0$ -modules. Let $\Lambda = \Lambda_{\mathfrak{g}_1} \subseteq U(\mathfrak{g})$. By the proof of Theorem 1.5, there is an isomorphism of $\text{ad}\mathfrak{g}_0$ -modules

$$U(\mathfrak{g}) = Z(\mathfrak{g}) \otimes (L \otimes \Lambda).$$

Therefore

$$U(\mathfrak{g})^{\mathfrak{g}_0} = Z(\mathfrak{g}) \otimes (L \otimes \Lambda)^{\mathfrak{g}_0}$$

We now use some representation theory to compute $\dim_{\mathbb{C}}(L \otimes \Lambda)^{\mathfrak{g}_0}$. Write $\Lambda = \bigoplus_{i=0}^{2r} \Lambda^i$, where $\Lambda^i = \Lambda^i \mathfrak{g}_1$, and let $\lambda_1, \dots, \lambda_r$ be the fundamental weights for \mathfrak{g}_0 . It is well known that as \mathfrak{g}_0 modules $\Lambda^i = L(\lambda_i) \oplus \Lambda^{i-2}$, for $2 \leq i \leq r$ [Sch, p.253, (A.16)] and that $\Lambda^{2r-i} \cong (\Lambda^i)^* \cong \Lambda^i$.

From the description of the root system given in 0.3, it follows easily that the zero weight space $(\Lambda^{2i})^0$ of Λ^{2i} has dimension $\binom{r}{i}$ for $1 \leq i \leq [r/2]$, and that $(\Lambda^{2i-1})^0 = 0$ for all i . Hence $\dim L(\lambda_{2i})^0 = \binom{r}{i} - \binom{r}{i-1}$ for $1 \leq i \leq [r/2]$.

Now if λ, μ are dominant weights, we have an isomorphism of \mathfrak{g}_0 -modules, [Sch, page 43, (3.31)]

$$L(\lambda) \otimes L(\mu) \cong \text{Hom}(L(\mu)^*, L(\lambda))$$

so taking invariants

$$(L(\lambda) \otimes L(\mu) : \mathbb{C}) = \begin{cases} 1 & \text{if } L(\lambda)^* \cong L(\mu) \\ 0 & \text{otherwise} \end{cases}$$

Thus $(L \otimes L(\lambda_i) : \mathbb{C}) = 0$ for i odd and

$$(L \otimes L(\lambda_{2i}) : \mathbb{C}) = \binom{r}{i} - \binom{r}{i-1}$$

if $1 \leq i \leq [r/2]$. Since $L(\lambda_{2i})$ occurs in Λ with multiplicity $(r - 2i + 1)$, and the multiplicity of \mathbb{C} is $r + 1$, it follows that

$$\begin{aligned} \dim(L \otimes \Lambda)^{\mathfrak{g}_0} &= (L \otimes \Lambda : L(0)) \\ &= (r + 1) + \sum_{i=1}^{[r/2]} \left\{ \binom{r}{i} - \binom{r}{i-1} \right\} (r - 2i + 1) = 2^r. \end{aligned}$$

Corollary. The algebras $U(\mathfrak{g})^{\mathfrak{g}_0}$ and $(Z(\mathfrak{g}), Z(\mathfrak{g}_0))$ have the same fields of fractions.

Proof. Let $x = Q - C + 2(\rho, \rho_1)$ be as in Lemma 2.6. Then the result follows since $Z(\mathfrak{g})[x]$ and $U(\mathfrak{g})^{\mathfrak{g}_0}$ are both free $Z(\mathfrak{g})$ -modules of rank 2^r .

5.2. Recall the Theorem about $\mathfrak{g} = osp(1, 2r)$ stated in 0.1. The equivalence of the hypotheses (a) and (b) follows from Corollary 3.11. We next show the

implication (b) \implies (c) is an easy consequence of our work so far.

Lemma. Suppose $\lambda \in \mathfrak{h}^*$ and all central characters $\chi_{\lambda-\gamma}^0$, with $\gamma \in \Gamma$ are distinct. Then A/Am_λ is reduced.

Proof. The map $A = U(\mathfrak{g})^{\mathfrak{g}_0} \longrightarrow \text{End}_{U(\mathfrak{g}_0)} \widetilde{M}(\lambda) \cong \oplus \mathbb{C}^{2^r}$ is surjective, since by Corollary 3.6, it is already surjective when we restrict the domain to $(Z(\mathfrak{g}_0), Z(\mathfrak{g})) \subseteq A$. Since Am_λ is contained in the kernel, Theorem 5.1 implies that the induced map $A/Am_\lambda \longrightarrow \mathbb{C}^{2^r}$ is an isomorphism.

5.3. Let $A = (Z(\mathfrak{g}_0), Z(\mathfrak{g}))$. We show how to extend the Harish-Chandra isomorphism $\psi : Z(\mathfrak{g}) \longrightarrow S(\mathfrak{h})^W$ to A . Let V be the open subset of \mathfrak{h}^* given by

$$V = \{\lambda + \rho \mid \lambda \text{ is regular}\}.$$

For $\lambda \in V$, we have by Theorem 3.6 and Corollary 3.11

$$\widetilde{M}(\lambda) = \oplus_{\gamma \in \Gamma} M(\lambda - \gamma)$$

and elements of A act by scalar multiplication on each component of this decomposition. Hence for all $\gamma \in \Gamma$, there is a homomorphism $\chi_{\lambda, \gamma} : A \longrightarrow \mathbb{C}$ such that

$$am = \chi_{\lambda, \gamma}(a)m$$

for all $a \in A$ and all m in the \mathfrak{g}_0 -submodule $M(\lambda - \gamma)$ of $\widetilde{M}(\lambda)$. Next we define a homomorphism $\Psi_\gamma : A \longrightarrow \mathcal{O}(V)$ by

$$\Psi_\gamma(a)(\lambda + \rho) = \chi_{\lambda, \gamma}(a)$$

for $a \in A$ and $\lambda \in V$.

To compare the Ψ_γ we require some automorphisms σ_γ of $S(\mathfrak{h})$. If $\gamma \in \Gamma$, we can write in the notation of 0.3, $\rho_1 - \gamma = (1/2) \sum_{i=1}^r \epsilon_i e_i$, where $\epsilon_i = \pm 1$. We define $\sigma_\gamma \in \text{Aut} S(\mathfrak{h})$ by

$$(\sigma_\gamma f)(\mu_1, \dots, \mu_r) = f(\epsilon_1 \mu_1, \dots, \epsilon_r \mu_r).$$

Theorem. The map $\sigma_\gamma \Psi_\gamma : A \longrightarrow S(\mathfrak{h})^{S_r}$ is an isomorphism which is independent of γ . Moreover if $z \in Z(\mathfrak{g})$ we have $\psi(z) = (\sigma_\gamma \Psi_\gamma)(z)$.

Proof. As in 0.3 we write $S(\mathfrak{h}) = \mathbb{C}[x_1, \dots, x_r]$ and set $h_d = \sum_{i=1}^r x_i^d$. Since the maps ψ and ψ_0 of 1.1 are isomorphisms, there exist $z_d \in Z(\mathfrak{g})$ and $z'_d \in Z(\mathfrak{g}_0)$ such that $\psi(z_d) = \psi_0(z'_d) = h_{2d}$ for all d .

Fix $\lambda \in V, \gamma \in \Gamma$, and write $\lambda + \rho = \mu = (1/2) \sum_{i=1}^r \mu_i e_i$ and $\rho_1 - \gamma = (1/2) \sum_{i=1}^r \epsilon_i e_i$. Then $(\sigma_\gamma \Psi_\gamma)(z_d) = h_{2d}$ and

$$\begin{aligned} \Psi_\gamma(z'_d)(\mu) &= \psi_0(z'_d)(\mu + \rho_1 - \gamma) \\ &= \left(\frac{1}{2}\right)^d \left(\sum_{i=0}^{2d} \binom{2d}{i} h_{2d-i}(\epsilon_1 \mu_1, \dots, \epsilon_r \mu_r), \right) \end{aligned}$$

whence $(\sigma_\sigma \Psi_\gamma)(z'_d) = \sum_{i=0}^{2d} \binom{2d}{i} h_{2d-i}$.

Since A is generated by z_1, \dots, z_r and z'_1, \dots, z'_r it follows that the map $\sigma_\gamma \Psi_\gamma$ is independent of γ and has image $\mathbb{C}[h_1, \dots, h_r] = S(\mathfrak{h})^{S_r}$. On the other hand, since A is a domain whose Krull dimension is equal to that of $S(\mathfrak{h})^{S_r}$, it follows that $\sigma_\gamma \Psi_\gamma$ is injective.

Corollary. If $g = osp(1, 2r)$, then $(Z(\mathfrak{g}_0), Z(\mathfrak{g})) = U(\mathfrak{g})^{\mathfrak{g}_0}$.

Proof. By Corollary 5.1, $A = (Z(\mathfrak{g}_0), Z(\mathfrak{g}))$ and $B = U(\mathfrak{g})^{\mathfrak{g}_0}$ have the same fields of fractions. The result follows since B is a finitely generated A -module by Theorem 5.1, and A is integrally closed by Theorem 5.3.

5.4. Lemma. There exists an element $T \in (Z(\mathfrak{g}), Z(\mathfrak{g}_0))$ such that for all $\lambda \in \Lambda, T$ acts on $\tilde{L}(\lambda, \epsilon)$ as the scalar $(-1)^\epsilon \prod_{i=1}^r (\lambda + \rho, e_i)$.

Proof. We have $\tilde{L}(\lambda) = \oplus_{\gamma \in \Gamma} L(\lambda - \gamma)$. Fix $\gamma \in \Gamma$ and write $\rho_1 - \gamma = (\frac{1}{2}) \sum_{i=1}^r \epsilon_i e_i$ where $\epsilon_i = \pm 1$. Then $L(\lambda - \gamma) \subseteq \tilde{L}(\lambda, \epsilon)$ if and only if $|\{i | \epsilon_i = -1\}| \equiv \epsilon \pmod{2}$. Choose $T \in (Z(g), Z(g_0))$ such that

$$(\sigma_\gamma \Psi_\gamma)(T) = x_1 x_2 \dots x_r \in S(\mathfrak{h})^{S_r}.$$

It follows that T acts on $L(\lambda - \gamma)$ as the scalar $(-1)^\epsilon \prod_{i=1}^r (\lambda + \rho, e_i)$, using Theorem 5.3.

5.5. For $d = 1, \dots, r$ let a_d be the d^{th} elementary symmetric function of

$x_1^2, x_2^2, \dots, x_r^2$. Then $S(\mathfrak{h})^W = \mathbb{C}[a_1, \dots, a_r]$. Therefore by Theorem 1.1, there is a unique element $z \in Z(\mathfrak{g})$ such that $\psi(z) = a_r$.

Lemma. With T as in Lemma 5.4,

- (a) $T^2 = z$
- (b) $xT + Tx = 0$ for all $x \in \mathfrak{g}_1$,
- (c) $U(\mathfrak{g})T = TU(\mathfrak{g})$

Proof. Suppose Λ is the dense subset of \mathfrak{h}^* given in Corollary 2.2, and $\lambda \in \Lambda$. By Theorem 1.1, z acts on $\tilde{L}(\lambda)$ by the scalar $\prod_{i=1}^r (\lambda + \rho, e_i)^2$, so (a) follows from Lemma 5.4 and Lemma 2.4. Also $\mathfrak{g}_1 \tilde{L}(\lambda, \epsilon) \subseteq \tilde{L}(\lambda, 1 - \epsilon)$ for $\epsilon = 0, 1$ and therefore $xT + Tx \in \text{ann}_{U(\mathfrak{g})} \tilde{L}(\lambda)$ for $x \in \mathfrak{g}_1$ by Lemma 5.4. Thus (b) follows from Lemma 2.4 also, and (c) follows from (b) and the fact that $T \in U(\mathfrak{g})^{\mathfrak{g}_0}$.

5.6. Theorem. Suppose $(\lambda + \rho, \alpha) = 0$ for some $\alpha \in \Delta_1^+$. Then

- (a) A/Am_λ is not reduced
- (b) $U(\mathfrak{g})m_\lambda$ is not semiprime.
- (c) If in addition $(\lambda + \rho, \beta) \neq 0$ for all $\beta \in \overline{\Delta}_0^+$, then $U(\mathfrak{g})m_\lambda$ is strictly contained in $\text{ann}_{U(\mathfrak{g})} \tilde{M}(\lambda)$.

Proof. We have $\alpha = e_i$ for some i . Thus under our assumption $T^2 = z \in m_\lambda$. If $T \in Am_\lambda$, then since A is a free $Z(\mathfrak{g})$ -module with 1 as part of a free basis by Theorem 5.1, we would have $T^2 = z \in Am_\lambda^2 \cap Z(\mathfrak{g}) = m_\lambda^2$. However since ψ is an isomorphism, this would imply that a_r belongs to the square of a maximal ideal of $\mathbb{C}[a_1, \dots, a_r]$. This contradiction proves (a). Since $U(\mathfrak{g})T$ is a two-sided ideal whose square is contained in $U(\mathfrak{g})m_\lambda$, (b) follows in a similar way using Theorem 1.5 in place of Theorem 5.1.

Finally if the condition in (c) holds, then by Corollary 3.11 (b) and Lemma 3.7, $\tilde{M}(\lambda) = \bigoplus_{\gamma \in \Gamma} M(\lambda - \gamma)$. Hence the proof of Lemma 5.4 goes through with $\tilde{M}(\lambda, \epsilon)$ in place of $\tilde{L}(\lambda, \epsilon)$, and we obtain $T\tilde{M}(\lambda) = 0$. Since $T \notin U(\mathfrak{g})m_\lambda$

this proves (c).

5.7. It remains to show that A/Am_λ is not reduced if $(\lambda + \rho, \alpha) = 0$ for some $\alpha \in \overline{\Delta}_0^+$. Here we exploit the fact that W acts transitively on $\overline{\Delta}_0^+ \cup -\overline{\Delta}_0^+$.

In the notation of 0.3, set $\sigma = e_r$ and $\tau = e_{r-1} - e_r$.

Lemma. If $(\mu + \rho, \tau) = 0$, then the image of $A = (Z(\mathfrak{g}), Z(\mathfrak{g}_0))$ in $E(\mu) = \text{End}_{U(\mathfrak{g}_0)} \widetilde{M}(\mu)$ is not reduced.

Proof. It suffices to show that the image of A in $\text{End}_{U(\mathfrak{g}_0)} pr_{\mu-\sigma} \widetilde{M}(\mu)$ is not reduced. For each $\eta \in \Delta^+$, fix bases $e_\eta, e_{-\eta}$ for \mathfrak{g}^η and $\mathfrak{g}^{-\eta}$ such that $(e_\eta, e_{-\eta}) = 1$.

Set $v = e_{-\sigma} v_\mu$ and $w = e_{-\sigma-\tau} v_\mu$. Let η be a positive even root. Note that $e_\eta v = 0$ and that $e_\eta w = 0$ unless $\eta = \tau$, in which case $e_\eta w$ is a nonzero multiple of v . Set $N = U(\mathfrak{g}_0)v$ and $M = U(\mathfrak{g}_0)v + U(\mathfrak{g}_0)w$. It follows easily that $N \cong M(\mu - \sigma)$ and that M/N is a homomorphic image of $M(\mu - \sigma - \tau)$. A consideration of formal characters shows that $M/N \cong M(\mu - \sigma - \tau)$. Since $s_\tau * \sigma = \sigma + \tau$, it follows from the proof of Proposition 3.9, that the modules M/N and N afford the same central character. Thus $M \subseteq pr_{\mu-\sigma} \widetilde{M}(\lambda)$.

Now let Q be the degree two Casimir element of $U(\mathfrak{g}_0)$ and $c = \chi_{\mu-\sigma}(Q)$. Up to a scalar multiple, Q takes the form

$$Q = x + \sum_{\eta > 0} e_{-\eta} e_\eta$$

where $x \in U(\mathfrak{h})$. By the above remarks that $(Q - c)N = (Q - c)(M/N) = 0$. Since $\mathbb{C}w$ is invariant under $U(\mathfrak{h})$, it follows that $(Q - c)w$ is a nonzero multiple of $e_{-\tau}v$, and that $(Q - c)^2 w = 0$. This proves the result.

Corollary. If $\lambda \in \mathfrak{h}^*$ and $(\lambda + \rho, \alpha) = 0$ for some $\alpha \in \overline{\Delta}_0^+$, then A/Am_λ is not reduced.

Proof. There exists $w \in W$ such that $w\alpha = \tau$ is as in the lemma. If $\mu = w(\lambda + \rho) - \rho$ then $(\mu + \rho, \tau) = 0$, so since $m_\lambda = m_\mu$, A/Am_λ is not reduced.

6. Some Open Problems.

We state the problems in maximum generality although solutions in special cases would often be interesting. Thus let \mathfrak{g} be a classical simple Lie superalgebra, and $\lambda \in \mathfrak{h}_0^*$.

- 1) When do we have equality $U(\mathfrak{g})^{\mathfrak{g}_0} = (Z(\mathfrak{g}), Z(\mathfrak{g}_0))$?
- 2) Are the algebras in 1) finitely generated? Is $U(\mathfrak{g})^{\mathfrak{g}_0}$ always commutative?
- 3) Is every minimal primitive ideal of $U(\mathfrak{g})^{\mathfrak{g}_0}$ generated as a two sided ideal by its intersection with $U(\mathfrak{g})^{\mathfrak{g}_0}$? If $\mathfrak{g} = \mathfrak{sl}(2, 1)$ this is shown in [Ben], while if $\mathfrak{g} = \mathfrak{osp}(1, 2)$, it follows easily from [Pi].
- 4) Find necessary and sufficient conditions for $\text{ann}_{U(\mathfrak{g})} \widetilde{M}(\lambda)$ to be generated by its intersection with $Z(\mathfrak{g})$.
- 5) Let $\mathfrak{g} = \mathfrak{osp}(1, 2r)$. Does the converse to Lemma 3.7 hold? If the image of $U(\mathfrak{g})^{\mathfrak{g}_0}$ (or of $(Z(\mathfrak{g}), Z(\mathfrak{g}_0))$) in $\text{End}_{U(\mathfrak{g}_0)} \widetilde{M}(\lambda)$ is reduced do we have $(\lambda + \rho, \alpha) \neq 0$ for all $\alpha \in \overline{\Delta}_0^+$?

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