Noncommutative Deformations of Type A Kleinian Singularities and Hilbert Schemes.

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Abstract

Let $H_{\mathbf{k}}$ be a symplectic reflection algebra corresponding to a cyclic subgroup $\Gamma \subseteq SL_2\mathbb{C}$ of order n and $U_{\mathbf{k}} = eH_{\mathbf{k}}e$ the spherical subalgebra of $H_{\mathbf{k}}$. We show that for suitable $\mathbf{k}$ there is a filtered $\mathbb{Z}^{n-1}$-algebra R such that

(1) there is an equivalence of categories $R - \mathrm{qgr} \simeq U_{\mathbf{k}}$-mod ,

(2) there is an equivalence of categories $grR - \mathrm{qgr} \simeq \mathsf{Coh}(Hilb_\Gamma\mathbb{C}^2)$.

Here $\mathsf{Coh}(Hilb_\Gamma\mathbb{C}^2)$ is the category of coherent sheaves on the Γ-Hilbert scheme. and for a graded algebra $\mathcal{R}$, we write $\mathcal{R}-\mathrm{qgr}$ for the quotient category of finitely generated graded $\mathcal{R}$-modules modulo torsion.

1 Introduction

This paper is motivated by a construction of Iain Gordon and Toby Stafford concerning the representation theory of a symplectic reflection algebra H_c and the spherical subalgebra U_c of H_c. This construction addresses issues raised in Conjecture 1.6 in [GK]. We refer to [EG] for the definition of symplectic reflection algebra and to [GS1] for background and the analogy for with the Beilinson-Bernstein theorem.

Suppose that G is a finite subgroup of $SL_2\mathbb{C}$, and let $\Gamma = G \wr S_m$, the wreath product of G by the symmetric group S_m. The group Γ acts on $V = (\mathbb{C}^2)^m$ preserving the natural sympletic structure. Let $Y_{\Gamma,m}$ be the set of Γ-invariant ideals I in the Hilbert scheme of $m|G|$ points in $\mathbb{C}^2$ such that the quotient $\mathbb{C}[x,y]/I$ is isomorphic to a direct sum of m copies of the regular representation of G. By [W, Theorem 4.2] there is a crepant resolution

$$Y_{\Gamma,m} \longrightarrow V/\Gamma.$$

This gives the bottom arrow in the diagram below. The algebra U_c has a filtration such that the associated graded algebra is isomorphic to $\mathcal{O}(V/\Gamma)-$mod. The passage

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to associated graded modules is indicated by the vertical arrow on the left. Gordon and Stafford suggest that there is a suitable category that will complete the diagram.

$$\begin{array}{ccc} U_c\text{–}mod & \xrightarrow{\simeq} & ??? \\ {\scriptstyle gr}\downarrow & & \downarrow{\scriptstyle gr} \\ \mathcal{O}(V/\Gamma)\text{–mod} & \longrightarrow & \mathcal{O}_{Y_{\Gamma,m}}\text{–mod} \end{array}$$

The main theorem in [GS1] shows that this is possible in the crucial special case where $G = 1$, and so $\Gamma = S_m$. Applications of this result are given in [GS2]. Another special case arises when $m = 1$, and so $G = \Gamma$. Then V/Γ is a Kleinian singularity and the algebras H_c were actually introduced earlier by Hodges [H] if Γ is cyclic, and by Crawley-Boevey and Holland [CBH] in general. The purpose of this paper is to solve the problem of Gordon and Stafford when Γ is cyclic of order n. Note that when Γ is the symmetric group or a finite subgroup of $SL_2\mathbb{C}$ then $Y_{\Gamma,m}$ is the same as the Γ-Hilbert scheme $Hilb_\Gamma\mathbb{C}^2$.

To state the main theorem requires some notation. There is an action of Γ on the first Weyl algebra $\mathbb{C}[\partial, y]$ and on the localization $\mathbb{C}[\partial, y^{\pm 1}]$. For $\mathbf{k} \in \mathbb{C}^{n-1}$, we construct $U_\mathbf{k}$ and $H_\mathbf{k}$ as subalgebras of the crossed product $\mathbb{C}[\partial, y^{\pm 1}] * \Gamma$. Our parameterization of these algebras is different from what is usually used, hence the change in notation. Then for suitable $\mathbf{k}', \mathbf{k} \in \mathbb{C}^{n-1}$, we find $U_{\mathbf{k}'} - U_\mathbf{k}$-bimodules $B(\mathbf{k}', \mathbf{k})$ and give a sufficient condition for these bimodules to induce a Morita equivalence. Next following [GS1], we assemble the bimodules $B(\mathbf{k}', \mathbf{k})$ to form a Morita $\mathbb{Z}^{n-1}$-algebra R. This is an algebra, without identity which is graded by $\mathbb{Z}^{n-1} \times \mathbb{Z}^{n-1}$. Since the algebras $U_\mathbf{k}$ and the bimodules $B(\mathbf{k}', \mathbf{k})$ are contained in $\mathbb{C}[\partial, y^{\pm 1}] * \Gamma$, they have a filtration given by the order of the differential operators. The algebra R inherits this filtration and we write grR for the associated graded algebra. The associated graded algebra of $U_\mathbf{k}$ is isomorphic to $\mathcal{O}(V/\Gamma)$. We write $\mathsf{Coh}(Hilb_\Gamma\mathbb{C}^2)$ for the category of coherent sheaves on $Hilb_\Gamma\mathbb{C}^2$. For a graded algebra $\mathcal{R}$, we write $\mathcal{R} - \mathrm{qgr}$ for the quotient category of finitely generated graded $\mathcal{R}$-modules modulo torsion. This notation is explained more fully at the start of Section 3. We can now state our main result. The definition of dominance is immediately given before Theorem 5.3.

Main Theorem. If $\mathbf{k}$ is dominant then
(1) there is an equivalence of categories $R - \mathrm{qgr} \simeq U_\mathbf{k}$-mod ,
(2) there is an equivalence of categories $grR - \mathrm{qgr} \simeq \mathsf{Coh}(Hilb_\Gamma\mathbb{C}^2)$.

A brief outline of the proof is as follows. In the next section we recall the construction of $\mathsf{Coh}(Hilb_\Gamma\mathbb{C}^2)$ as a toric variety from [IN]. We also show that there is a $\mathbb{N}^{n-1}$ -graded ring S such that the category $S-$qgr is equivalent to $\mathsf{Coh}(Hilb_\Gamma\mathbb{C}^2)$. We construct a $\mathbb{Z}^{n-1}$-algebra $\widehat{S}$ such that the categories $\widehat{S}-$qgr and $S-$qgr are equivalent. The algebra grR is also $\mathbb{N}^{n-1}$-graded and it is not hard to show that $\widehat{S} \subseteq grR$. The proof of (2) is completed using a Poincaré series argument as in [GS1, Section 6] to show that $\widehat{S} = grR$. The proof of (1) uses a generalization of the $\mathbb{Z}$-algebra machinery developed in [GS1, Section 5].

I would like to thank Iain Gordon for suggesting the problem considered in this paper, and for outlining the approach which led to a solution. The material in Section 3 and the proof of Lemma 4.2 are due to him. I also thank Mitya Boyarchenko of the University of Chicago for some helpful correspondence. Mitya has informed me that he is close to a solution of this problem for general Kleinian singularities.

2 Toric Varieties

Let Γ be a finite subgroup of $SL_2\mathbb{C}$. The Γ-Hilbert scheme $Hilb_\Gamma\mathbb{C}^2$ is the scheme which parameterizes Γ-invariant ideals I of $C[u,v]$ such that $\mathbb{C}[u,v]/I$ is isomorphic to the regular representation of Γ. The Hilbert-Chow morphism

$$Hilb_\Gamma\mathbb{C}^2 \longrightarrow \mathbb{C}^2/\Gamma$$

gives a minimal (equivalently crepant) resolution of the quotient singularity, [W]. For Γ cyclic of order n we construct this morphism using toric varieties.

Let $N = \mathbb{Z}^2, M = Hom(N, \mathbb{Z})$, and write $< \ , >: M \times N \longrightarrow \mathbb{Z}$. for the natural bilinear pairing. Set $v_i = (1, i)$ for $0 \leq i \leq n$. Then let Δ be the fan in $N_\mathbb{R} = N \otimes_\mathbb{Z} \mathbb{R}$ with one dimensional cones $\mathbb{R}^+ v_i$ and 2 dimensional cones $\sigma_i = \mathbb{R}^+ v_{i-1} + \mathbb{R}^+ v_i$ for $1 \leq i \leq n$. The cone $\sigma_i^\vee$ in $M_\mathbb{R}$ dual to σ_i is $\sigma_i^\vee = \mathbb{R}^+(i, -1) + \mathbb{R}^+(1-i, 1)$ and we set

$$A_i = \mathbb{C}[M \cup \sigma_i^\vee], \quad X_i = Spec A_i.$$

Let $X = X(\Delta) = \cup_{i=1}^n X_i$ be the toric variety determined by Δ and let $T = T_N$ be the dense torus acting on X. Since $det \begin{pmatrix} i & -1 \\ 1-i & 1 \end{pmatrix} = 1$, it follows from [F, Proposition, page 29] that X is nonsingular. We write elements of A_i multiplicatively by setting

$$x = (1, 0), \quad z = (0, 1).$$

Then $A_i = \mathbb{C}[x^i z^{-1}, x^{1-i} z]$ and the maximal ideal $\mathbf{m}_i$ of A_i corresponding to the T fixed point $p_i \in X_i$ is

$$\mathbf{m}_i = (x^i z^{-1}, x^{1-i} z). \tag{2.1}$$

In order to relate X to the singularity it is convenient to introduce new indeterminates u, v satisfying

$$v^n = z, \quad uv = x,$$

so that $u^n = x^n z^{-1}$. Then

$$A_i = \mathbb{C}[u^i v^{i-n}, v^{n+1-i} u^{1-i}].$$

Note that the fan Δ is obtained by subdividing the fan Δ' with the single 2-dimensional cone $\sigma = \mathbb{R}^+(1, 0) + \mathbb{R}^+(1, n)$. Also $\sigma^\vee = \mathbb{R}^+(0, 1) + \mathbb{R}^+(n, -1)$, so the toric variety X' corresponding to Δ' is $Spec\ C^\Gamma$ where $C = \mathbb{C}[u, v]$. It follows easily that the map $X \longrightarrow X'$ is the minimal resolution of the singularity.

For $1 \leq i \leq n$, let U_i be the set of Γ-invariant ideals of $\mathbb{C}[u,v]$ such that the elements

$$1, u, \dots, u^{i-1}, v, v^2, \dots, v^{n-i}$$

form a basis for the factor algebra $\mathbb{C}[u,v]/I$. If I is a Γ-invariant ideal of $\mathbb{C}[u,v]$ such that the factor algebra is isomorphic to $\mathbb{C}\Gamma$, then $u^n, uv, v^n \in I$, and it follows that $I \in U_i$ for some i. Furthermore if $I \in U_i$ then for a unique $(a,b) \in \mathbb{C}^2$ we have

$$u^i \equiv av^{n-i},\ v^{n-i+1} \equiv bu^{i-1} \mod I.$$

This identifies U_i with $\mathbb{C}^2$ and $\mathcal{O}(U_i)$ with $\mathbb{C}[u^i v^{i-n}, v^{n+1-i}u^{1-i}] = A_i$. The open sets U_i are glued together in the same way as the X_i and it follows easily that the resolution $X \longrightarrow X'$ constructed above is the Hilbert-Chow morphism, [IN]. This map is equivariant for the action of a dense torus $T = \mathbb{C}^\times \times \mathbb{C}^\times$. Let $\mathbb{X}(T) = \mathbb{Z}\chi_1 \oplus \mathbb{Z}\chi_2$ be the character group of T. If T acts rationally on a vector space V we define the weight space decomposition of V to be $V = \bigoplus_{\chi \in \mathbb{X}(T)} V(\chi)$, where $V(\chi) = \{v \in V | \tau \cdot v = \chi(\tau)v \text{ for all } \tau \in T\}$. If $dim V(\chi) < \infty$ for all $\chi \in \mathbb{X}(T)$ we define the Poincaré series $\mathcal{H}_V(q,t)$ of V by

$$\mathcal{H}_V(q,t) = \sum_{r,s} \dim V(r\chi_1 + s\chi_2)\, q^r t^s.$$

We assume that T acts on C so that $u \in C(\chi_1)$ and $v \in C(\chi_2)$. We have

$$x \in C(\chi_1 + \chi_2); \quad z \in C(n\chi_2). \tag{2.2}$$

Let D_i be the divisor on X corresponding to v_i. By [F, 3.4] there is an exact sequence.

$$0 \longrightarrow M \xrightarrow{\alpha} \oplus_{i=0}^n \mathbb{Z}D_i \xrightarrow{\beta} Pic(X) \longrightarrow 0.$$

where $\alpha(m) = \sum_{i=0}^n < m, v_i > D_i$ and β sends a divisor D to its class $[D]$ in $Pic(X)$. It follows that $Pic(X)$ is generated by the D_i which are subject to the relations

$$\sum_{i=0}^n D_i \ = \ \sum_{i=0}^n iD_i \ = \ 0. \tag{2.3}$$

This implies that $Pic(X) = \oplus_{i=1}^{n-1} \mathbb{Z}D(i)$, where for $1 \leq i \leq n-1$, we define

$$D(i) = \sum_{j=0}^{i-1} (i-j) D_{n-j}. \tag{2.4}$$

For $\mathbf{b} \in \mathbb{Z}^{n-1}$ set

$$D(\mathbf{b}) = \sum_{i=1}^{n-1} b_i D(i). \tag{2.5}$$

The polynomial ring $\mathbb{C}[x_0, \dots, x_n]$ is graded by $Pic(X)$ where we define

$$\deg(x_i) = [D_i] \in Pic(X).$$

We recall Proposition 1.1 in [Cox]. Let

$$S_{\mathbf{b}} = span\{s \in \mathbb{C}[x_0, \ldots, x_n] | deg(s) = [D(\mathbf{b})]\}.$$

Then there is an isomorphism $S_{\mathbf{b}} \cong H^0(X, \mathcal{O}(D(\mathbf{b})))$

$$\phi_{\mathbf{b}} : S_{\mathbf{b}} \longrightarrow H^0(X, \mathcal{O}(D(\mathbf{b})))$$

and a commutative diagram

$$\begin{tikzcd}
S_{\mathbf{b}} \otimes S_{\mathbf{c}} \arrow[r] \arrow[d] & S_{\mathbf{b}+\mathbf{c}} \arrow[d] \\
H^0(X, \mathcal{O}(D(\mathbf{b}))) \otimes H^0(X, \mathcal{O}(D(\mathbf{c}))) \arrow[r] & H^0(X, \mathcal{O}(D(\mathbf{b}+\mathbf{c})))
\end{tikzcd}$$

where the horizontal arrows are multiplication and the vertical arrows are $\phi_{\mathbf{b}} \otimes \phi_{\mathbf{c}}$ and $\phi_{\mathbf{b}+\mathbf{c}}$.

Lemma 2.1. *If* $\mathbf{b} \in \mathbb{N}^{n-1}$, *and* $D = D(\mathbf{b})$, *then*
(1) $\mathcal{O}(D(\mathbf{b}))$ *is generated by its global sections*
(2) $H^i(X, \mathcal{O}(D)) = 0$ *for* $i > 0$.

Proof. (1) Let ψ_D be the piecewise linear function associated to $D = D(\mathbf{b})$ in [F, Section 3.4]. It is easy to see that if $\mathbf{b} \in \mathbb{N}^{n-1}$, then ψ_D is convex. The result follows from [F, Proposition, page 68].
(2) This follows from (1) and [F, Corollary, page 74].

Corollary 2.2. *For* $\mathbf{b}, \mathbf{c} \in \mathbb{N}^{n-1}$ *we have*

$$H^0(X, \mathcal{O}(D(\mathbf{b})))H^0(X, \mathcal{O}(D(\mathbf{c}))) = H^0(X, \mathcal{O}(D(\mathbf{b}+\mathbf{c}))).$$

Proof. This follows by part (1) of the Lemma and the first exercise on page 69 of [F].

Our goal is to compute a graded Poincaré series for $H^0(X, \mathcal{O}(D))$. To do this we need the Atiyah-Bott-Lefschetz theorem, which we state next.

Theorem 2.3. *Let* $T = \mathbb{C}^\times \times \mathbb{C}^\times$ *be a 2 dimensional torus acting on a smooth surface* X *such that the fixed point set* X^T *is finite. Let* $\mathcal{A}$ *be a* T*-equivariant locally free sheaf on* X. *Let* m_x *be the maximal ideal of the local ring at* x, $\mathcal{A}_x$ *the stalk at* x, *and* $\mathcal{A}(x) = \mathcal{A}_x / m_x \mathcal{A}_x$. *If* x *is a* T*-fixed point then* T *acts on* $\mathcal{A}(x)$. *Suppose that* τ *acts on with eigenvalues* $v_x(q,t)$ *and* $w_x(q,t)$ *on the cotangent space* m_x/m_x^2 *at* x. *Then we have*

$$\sum_{i \geq 0} (-1)^i \mathcal{H}_{H^i(X,\mathcal{A})}(q,t) = \sum_{x \in X^T} \mathcal{H}_{\mathcal{A}(x)}(q,t)/(1 - v_x(q,t))(1 - w_x(q,t)). \tag{2.6}$$

Proof. See [AB] for the compact case. This form is a special case of [Ha, Theorem 3.1].

The denominator in (2.6) is equal to $det_{m_x/m_x^2}(1-\tau)$. We apply Theorem 2.3 when $\mathcal{A}$ is the invertible sheaf corresponding to the divisor $D = D(\mathbf{b})$ on X as in (2.5).

Lemma 2.4. *If $\mathbf{b} \in \mathbb{N}^{n-1}$ and $D = D(\mathbf{b})$, then $\mathcal{O}(D)(X_{i+1})$ is a free $\mathcal{O}(X_{i+1})$-module generated by $x^{m_1}z^{m_2}$ where*

$$m_1 = \sum_{j=n-i}^{n-1} (n-j)b_j, \qquad m_2 = -\sum_{j=n-i}^{n-1} b_j. \tag{2.7}$$

Proof. We have $D = \sum_{i=0}^{n} a_i D_i$, where

$$a_k = \sum_{j=n+1-k}^{n-1} (j+k-n)b_j. \tag{2.8}$$

As in [F, page 66] we set

$$P_D(\sigma_{i+1}) = \{u \in M_{\mathbb{R}} | < u, v_j > \ \geq -a_j \ \text{ for } \ j = i, i+1\}.$$

Then

$$H^0(X_{i+1}, \mathcal{O}(D)) = \oplus_{u \in P_D(\sigma_{i+1}) \cap M} \mathbb{C} x^{u_1} z^{u_2}.$$

Note that if $u = (u_1, u_2) \in M_{\mathbb{R}}$, then $u \in P_D(\sigma_{i+1})$ if and only if

$$u_1 + iu_2 \geq -a_i$$

and

$$u_1 + (i+1)u_2 \geq -a_{i+1}.$$

It follows that $\mathcal{O}(D)(X_{i+1})$ is freely generated over $\mathcal{O}(X_{i+1})$ by $x^{m_1}z^{m_2}$ where $m_1 + im_2 = -a_i$ and $m_1 + (i+1)m_2 = -a_{i+1}$, that is

$$m_1 = ia_{i+1} - (i+1)a_i \quad \text{ and } \quad m_2 = a_i - a_{i+1}. \tag{2.9}$$

Now the result follows easily from (2.8).

Theorem 2.5. *If $\mathbf{b} \in \mathbb{N}^{n-1}$ and $N(\mathbf{b}) = H^0(X, \mathcal{O}(D(\mathbf{b})))$, then*

$$\mathcal{H}_{N(\mathbf{b})}(q,t) = \sum_{i=0}^{n-1} \frac{q^{\sum_{j=n-i}^{n-1}(n-j)b_j}\, t^{-\sum_{j=n-i}^{n-1} jb_j}}{(1-t^{i+1-n}q^{i+1})(1-t^{n-i}q^{-i})}. \tag{2.10}$$

Proof. By Lemma 2.1, the higher cohomology of $\mathcal{O}(D)$ vanishes, so the left side of (2.6) equals $\mathcal{H}_{N(\mathbf{b})}(q,t)$. On the other hand from (2.2) and Lemma 2.4 we have

$$\mathcal{H}_{\mathcal{A}(p_{i+1})} = q^{m_1} t^{m_1+nm_2} = q^{\sum_{j=n-i}^{n-1}(n-j)b_j}\, t^{-\sum_{j=n-i}^{n-1} jb_j}, \tag{2.11}$$

and by (2.1) that the determinant of $(1-\tau)$ on the cotangent space at p_{i+1} is

$$(1-t^{i+1-n}q^{i+1})(1-t^{n-i}q^{-i}).$$

Therefore the result follows from Theorem 2.3.

3 Multi-homogeneous coordinate rings

Let $\mathcal{R} = \bigoplus_{\mathbf{i}\in\mathbb{N}^m} \mathcal{R}(\mathbf{i})$ be a $\mathbb{N}^m$-graded algebra. We introduce several abelian categories that we will need. First we write $\mathcal{R}$-Grmod for the category of $\mathbb{Z}^m$-graded $\mathcal{R}$–modules with degree zero homomorphisms, and $\mathcal{R}$-grmod for the full subcategory of $\mathcal{R}$-Grmod consisting finitely generated graded $\mathcal{R}$–modules.

If $\mathbf{i}, \mathbf{j} \in \mathbb{Z}^m$ we write $\mathbf{i} \geq \mathbf{j}$ if $\mathbf{i} - \mathbf{j} \in \mathbb{N}^m$. We say that a property (P) holds for *large enough* $\mathbf{b} \in \mathbb{Z}^m$ if there exists $\mathbf{j} \in \mathbb{Z}^m$ such that (P) holds for all $\mathbf{b} \geq \mathbf{j}$. A finitely generated graded $\mathcal{R}$–module $M = \bigoplus_{\mathbf{i}\in\mathbb{Z}^m} M(\mathbf{i})$ is *bounded* if $M(\mathbf{i}) = 0$ for large enough $\mathbf{i}$, and a graded $\mathcal{R}$–module is *torsion* if it is the direct limit of a system of bounded modules. We denote the Serre subcategories of $\mathcal{R}$-grmod and $\mathcal{R}$-Grmod consisting of bounded and torsion modules by $\mathcal{R}$-tors and $\mathcal{R}$-Tors respectively. Finally, we define the quotient categories

$$\mathcal{R}\text{–Qgr} = \mathcal{R}\text{-}\mathsf{Grmod}/\mathcal{R}\text{-}\mathsf{Tors} \qquad \mathcal{R}\text{–qgr} = \mathcal{R}\text{-}\mathsf{grmod}/\mathcal{R}\text{-}\mathsf{tors}.$$

Let $S_{\mathbf{b}} = H^0(X, \mathcal{O}(D(\mathbf{b})))$ as in Section 2 and set $S = \bigoplus_{\mathbf{b}\in\mathbb{N}^{n-1}} S_{\mathbf{b}}$. We show that there is an equivalence of categories between S–qgr and $\mathsf{Coh}(X)$, the category of coherent sheaves on X. To do this adapt the proof of an equivalence of categories for twisted homogeneous coordinate rings, and for twisted multi-homogeneous coordinate rings obtained in [AV, Theorem 1.3] and [Chan, Theorem 3.4] respectively. The proof works in greater generality.

Let X be a variety, projective over $\operatorname{Spec} S$ with S a a noetherian k-algebra . Assume we have a system of invertible sheaves $(\mathcal{L}_1, \ldots, \mathcal{L}_m)$ on X with the following properties:

1. There exists $N \gg 0$ such that for all $n \geq N$ we have $H^p(X, \mathcal{L}_i^n) = 0$ for each i and for all positive integers p. (Note that by Grothendieck's vanishing theorem we need only consider $p \leq \dim X$, so for each $1 \leq i \leq m$ and for each $1 \leq p \leq \dim X$ we can look for a positive integer $N(i,p)$ with the cohomology vanishing property above, and then set N to be the maximum value amongst the $N(i,p)$.)

2. For a coherent sheaf $\mathcal{F}$ on X there exists a sequence of integers $(c_1, \ldots, c_m)$ such that for all $(b_1, \ldots, b_m) \geq (c_1, \ldots, c_m)$, we have $H^p(X, \mathcal{F} \otimes \mathcal{L}_1^{b_1} \cdots \mathcal{L}_m^{b_m}) = 0$ for all positive integers p.

We call such a system an *ample system*. To ease notation we write

$$\mathcal{L}(\mathbf{b}) = \mathcal{L}_1^{b_1} \cdots \mathcal{L}_m^{b_m} \tag{3.1}$$

for $\mathbf{b} = (b_1, \ldots, b_m) \in \mathbb{Z}^m$. Define the following graded algebras:

$$\mathcal{L} = \bigoplus_{\mathbf{b} \in \mathbb{Z}^m} \mathcal{L}(\mathbf{b}); \quad A = \bigoplus_{\mathbf{b} \in \mathbb{N}^m} H^0(X, \mathcal{L}(\mathbf{b})).$$

Then $\mathcal{L}$ is a strongly $\mathbb{Z}^m$–graded $\mathcal{O}_X$–algebra with $\mathcal{L}(0) = \mathcal{O}_X$. The algebra A is an $\mathbb{N}^m$–graded $\mathcal{O}_X(X)$–algebra. A *graded* $\mathcal{L}$–module $\mathcal{M}$ has the form

$$\mathcal{M} = \bigoplus_{\mathbf{i} \in \mathbb{Z}^m} \mathcal{M}(\mathbf{i})$$

with each $\mathcal{M}(\mathbf{i})$ a quasi-coherent $\mathcal{O}_X$–sheaf, such that $\mathcal{L}(\mathbf{i}) \cdot \mathcal{M}(\mathbf{j}) \subseteq \mathcal{M}(\mathbf{i}+\mathbf{j})$. We have then the categories $\mathcal{L}$-Grmod of graded $\mathcal{L}$–modules with degree zero homomorphisms, and its subcategory $\mathcal{L}$-grmod consisting of finitely generated $\mathcal{L}$–modules.

Theorem 3.1. *There is an equivalence of categories between* $\mathsf{Qcoh}\, X$ *and* $A-\mathrm{Qgr}$ *which restricts to an equivalence between* $\mathsf{Coh}\, X$ *and* $A-\mathrm{qgr}$*.*

The proof goes in two stages. First we have

Lemma 3.2. *The correspondences*

$$\begin{array}{ccc} \mathcal{M} & \longrightarrow & \mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{M} \\ \mathcal{N}(0) & \longleftarrow & \mathcal{N} \end{array}$$

induce inverse equivalences of categories between $\mathsf{Qcoh}\, X$ *and* $\mathcal{L}$-Grmod*.*

Proof. The proof is easily adapted from the corresponding result about strongly graded rings, [D].

Lemma 3.3. *Let $\mathcal{F}$ be a coherent sheaf on X. Then for large enough* $\mathbf{i}$*, $\mathcal{F} \otimes \mathcal{L}(\mathbf{i})$ is generated by global sections.*

Proof. (Adapted from [Hart, Proposition III.5.3].) Let $P \in X$ be a closed point and $\mathcal{I}_P$ its ideal sheaf. We have a short exact sequence

$$0 \longrightarrow \mathcal{I}_P \mathcal{F} \longrightarrow \mathcal{F} \longrightarrow \mathcal{F} \otimes k(P) \longrightarrow 0.$$

Tensoring this by $\mathcal{L}(\mathbf{i})$ gives

$$0 \longrightarrow \mathcal{I}_P \mathcal{F} \otimes \mathcal{L}(\mathbf{i}) \longrightarrow \mathcal{F} \otimes \mathcal{L}(\mathbf{i}) \longrightarrow \mathcal{F} \otimes \mathcal{L}(\mathbf{i}) \otimes k(P) \longrightarrow 0.$$

By hypothesis we can find $\mathbf{j}$ such that $H^1(X, \mathcal{I}_P \mathcal{F} \otimes \mathcal{L}(\mathbf{i})) = 0$ for all $\mathbf{i} \geq \mathbf{j}$. Therefore the mapping

$$H^0(X, \mathcal{F} \otimes \mathcal{L}(\mathbf{i})) \longrightarrow H^0(X, \mathcal{F} \otimes \mathcal{L}(\mathbf{i}) \otimes k(P))$$

is surjective for all $\mathbf{i} \geq \mathbf{j}$. So, by Nakayama's lemma, the stalk at P of $\mathcal{F} \otimes \mathcal{L}(\mathbf{i})$ is generated by global sections. Hence, there exists an open neighbourhood U of P such that $\mathcal{F} \otimes \mathcal{L}(\mathbf{i})$ is generated by global sections on all of U. However this open set depends on the choice of $\mathbf{i}$. So pick the U corresponding to $\mathbf{j}$.

If we refine the argument in the above paragraph to work with $\mathcal{F} = \mathcal{O}_X$ we can see (using property (1) of our ample system) that we can find neighbourhoods $V_1, \ldots, V_m$ of P and integers $\ell_1, \ldots, \ell_m$ such that for each i, $\mathcal{L}_i^{n_i}$ is generated by global sections on V_i for all $n_i \geq \ell_i$.

Therefore on $U \cap V_1 \cap \ldots \cap V_m$ the sheaf $\mathcal{F} \otimes \mathcal{L}(\mathbf{j}) \otimes \mathcal{L}(\mathbf{n})$ is generated by its global sections for any $\mathbf{n} \geq \mathbf{l}$. Covering X with a finite number of open sets of the above form proves the lemma.

To complete the proof of Theorem 3.1 we consider the functors

$$F : A\text{–mod} \longrightarrow \mathcal{L}\text{–Grmod} : \quad M \longrightarrow \mathcal{L} \otimes_A M$$
$$G : \mathcal{L}\text{–Grmod} \longrightarrow A\text{–mod} : \quad \mathcal{M} \longrightarrow H^0(X, \mathcal{M})_{\geq 0} = \bigoplus_{\mathbf{i} \in \mathbb{N}^m} H^0(X, \mathcal{M}(\mathbf{i})).$$

Lemma 3.4. *The functors F and G induce an equivalence of categories between $\mathcal{L} -$ Grmod and A–Qgr.*

Proof. The proof in [AV] goes through verbatim using Lemma 3.3 in place of [AV, 3.2 (ii)].

Proof of Theorem 3.1. This now follows by combining Lemmas 3.2 and 3.4. In particular the equivalences are given as follows:

$$\mathsf{Qcoh}\, X \longrightarrow A\text{–Qgr} \quad : \quad \mathcal{F} \mapsto \bigoplus_{\mathbf{i} \in \mathbb{N}^n} H^0(X, \mathcal{F} \otimes \mathcal{L}(\mathbf{i})),$$

and

$$A\text{–Qgr} \longrightarrow \mathsf{Qcoh}\, X \quad : \quad M \longrightarrow (\mathcal{L} \otimes_A M)(0).$$

Now suppose that X is the minimal resolution the cyclic singularity of type A_{n-1} as in Section 2. The map $X \longrightarrow \mathbb{C}^2/\Gamma$ is a projective morphism, since it is obtained by successive blow-ups. For $1 \leq i \leq n-1$ set $\mathcal{L}_i = \mathcal{O}(D(i))$. Then if we define $\mathcal{L}(\mathbf{b})$ for $\mathbf{b} \in \mathbb{Z}^{n-1}$ as in equation (3.1), we have $\mathcal{L}(\mathbf{b}) = \mathcal{O}(D(\mathbf{b}))$.

Lemma 3.5. *The invertible sheaves $\mathcal{O}(D(i))$ for $1 \leq i \leq n-1$ form an ample system on X.*

Proof. Property (1) in the definition of an ample system holds by Lemma 2.1. To check property (2) suppose that $\mathcal{F}$ be a coherent sheaf on X. We show that for large enough $\mathbf{i}$ we have $H^p(X, \mathcal{F} \otimes \mathcal{L}(\mathbf{i})) = 0$ for all positive p. By [Hart, Corollary II. 5.18] the coherent sheaf $\mathcal{F}$ can be written as a quotient, $\mathcal{E}/\mathcal{G}$, of a finite direct sum of twisted structure sheaves. Now by [F, Proposition 3.4], the group of T–Cartier divisors maps onto the Picard group of X, and because X is smooth every T–Cartier divisor is T–Weil. Hence every invertible sheaf (and in particular any twisted structure sheaf) has the form $\mathcal{L}(\mathbf{k})$ for some $\mathbf{k} \in \mathbb{Z}^{n-1}$.

We have a short exact sequence

$$0 \longrightarrow \mathcal{G} \otimes \mathcal{L}(\mathbf{i}) \longrightarrow \mathcal{E} \otimes \mathcal{L}(\mathbf{i}) \longrightarrow \mathcal{F} \otimes \mathcal{L}(\mathbf{i}) \longrightarrow 0$$

which leads to a long exact sequence in cohomology. For large enough $\mathbf{i}$ we can force the vanishing of $H^p(X, \mathcal{E} \otimes \mathcal{L}(\mathbf{i}))$ for positive p so we can deduce that

$$H^p(X, \mathcal{F} \otimes \mathcal{L}(\mathbf{i})) \cong H^{p+1}(X, \mathcal{G} \otimes \mathcal{L}(\mathbf{i})).$$

Since $\mathcal{G}$ is coherent we have by induction that the cohomology group on the right hand side vanishes, as required.

4 $\mathbb{Z}^m$-algebras.

We need a routine generalization of the notion of a $\mathbb{Z}$-algebra introduced in [BP]. Let A be a free abelian group of rank m. Thus $A \cong \mathbb{Z}^m$, and we write A^+ for the submonoid of A corresponding to $\mathbb{N}^m$. We define a partial order on A by writing $\mathbf{i} \geq \mathbf{j}$ if $\mathbf{i} - \mathbf{j} \in A^+$. A *(triangular) $\mathbb{Z}^m$-algebra* is an algebra $B = \bigoplus B(\mathbf{i}, \mathbf{j})$ where each $B(\mathbf{i}, \mathbf{j})$ is an additive subgroup of B, and the sum is over all $\mathbf{i}, \mathbf{j} \in A$ such that $\mathbf{i} \geq \mathbf{j}$. The multiplication in B resembles matrix multiplication, that is we have $B(\mathbf{i}, \mathbf{j})B(\mathbf{p}, \mathbf{q}) = 0$ if $\mathbf{j} \neq \mathbf{p}$, and

$$B(\mathbf{i}, \mathbf{j})B(\mathbf{j}, \mathbf{l}) \subseteq B(\mathbf{i}, \mathbf{l}). \tag{4.1}$$

whenever $\mathbf{i} \geq \mathbf{j} \geq \mathbf{l}$.

There are two kinds of $\mathbb{Z}^m$-algebras that will be of interest to us. Suppose first that $\mathcal{R} = \bigoplus_{\mathbf{b} \in \mathbb{N}^m} \mathcal{R}_{\mathbf{b}}$ is an $\mathbb{N}^m$-graded algebra and set $\widehat{\mathcal{R}} = \bigoplus_{\mathbf{i} \geq \mathbf{j} \geq 0} \widehat{\mathcal{R}}(\mathbf{i}, \mathbf{j})$ where $\widehat{\mathcal{R}}(\mathbf{i}, \mathbf{j}) = \mathcal{R}_{\mathbf{i}-\mathbf{j}}$. The multiplication in $\widehat{\mathcal{R}}$ is induced from that in $\mathcal{R}$. We call $\widehat{\mathcal{R}}$ the *$\mathbb{Z}^m$-algebra arising from $\mathcal{R}$.*

Let B be a $\mathbb{Z}^m$-algebra. We consider the category $B-$Grmod of $\mathbb{N}^m$-graded left B-modules $M = \bigoplus_{\mathbf{j} \geq 0} M(\mathbf{j})$ such that $B(\mathbf{i}, \mathbf{j})M(\mathbf{j}) \subseteq M(\mathbf{j})$ for all $\mathbf{i} \geq \mathbf{j}$ and $B(\mathbf{i}, \mathbf{j})M(\mathbf{k}) = 0$ if $\mathbf{k} \neq \mathbf{j}$. Morphisms in $B-$Grmod are graded homomorphisms of degree zero. It is now clear how to define the categories $B-$grmod, $B-$Qgr and $B-$qgr by analogy with the definitions we made for $\mathbb{N}^m$-graded algebras.

Returning to the algebras $\mathcal{R}$ and $\widehat{\mathcal{R}}$, let $\mathcal{R}-\mathrm{Grmod}_{\geq 0}$ be the full subcategory of $\mathcal{R}-$Grmod consisting of all $\mathbb{N}^m$-graded modules. We define the categories $\widehat{\mathcal{R}}-$Qgr and $\widehat{\mathcal{R}}-$qgr in the obvious way. For M an object in $\mathcal{R}-$Grmod we set $M_{\geq 0} = \bigoplus_{\mathbf{j} \geq 0} M(\mathbf{j})$ and write $\pi(M)$ for the image of M in $\mathcal{R} - Qgr$.

Lemma 4.1. *(1) The identity map yields equivalences of categories*

$$\mathcal{R}-\mathrm{Grmod}_{\geq 0} \longrightarrow \widehat{\mathcal{R}}-\mathrm{Grmod},\ \mathcal{R} - \mathrm{grmod}_{\geq 0} \longrightarrow \widehat{\mathcal{R}} - \mathrm{grmod}.$$

(2) The equivalences in (1) induce equivalences

$$\mathcal{R}-\mathrm{Qgr} \longrightarrow \widehat{\mathcal{R}}-\mathrm{Qgr} \quad \mathcal{R}\mathrm{qgr} \longrightarrow \widehat{\mathcal{R}}-\mathrm{qgr}$$

Proof. (1) is immediate. For (2) the only point to note is that $\pi(M) = \pi(M_{\geq 0})$, since $M/M_{\geq 0}$ is torsion.

For the second class of $\mathbb{Z}^m$-algebras, let $\mathbf{w_1}, \dots, \mathbf{w_m}$ be linearly independent elements of $\mathbb{C}^m$, and set $A = \oplus_{i=1}^m \mathbb{Z}\mathbf{w_i}$, $A^+ = \oplus_{i=1}^m \mathbb{N}\mathbf{w_i}$. Suppose that Q is a $\mathbb{C}$-algebra and that for $\mathbf{i}, \mathbf{j} \in \mathbb{C}^m$ with $\mathbf{i} \geq \mathbf{j}$ we are given subrings $R_\mathbf{i}, R_\mathbf{j}$ and $R_\mathbf{i} - R_\mathbf{j}$ sub-bimodules $B(\mathbf{i}, \mathbf{j})$ of Q such that $B(\mathbf{i}, \mathbf{i}) = R_\mathbf{i}$ and (4.1) holds.

Fix $\mathbf{k} \in \mathbb{C}^m$ and set

$$R(\mathbf{k}) = \oplus_{\mathbf{i} \geq \mathbf{j} \geq \mathbf{k}} B(\mathbf{i}, \mathbf{j}).$$

If the bimodules $B(\mathbf{i}, \mathbf{j})$ in this sum induce a Morita equivalence between $R_\mathbf{i}$ and $R_\mathbf{j}$, we say that $R(\mathbf{k})$ is *the Morita $\mathbb{Z}^m$-algebra associated to the data* $(R_\mathbf{i}, B(\mathbf{i}, \mathbf{j}))$. For this to happen it is necessary that equality holds in (4.1).

Lemma 4.2. *Suppose that $R(\mathbf{k})$ is the Morita $\mathbb{Z}^m$-algebra associated to the data $R_\mathbf{i}, B(\mathbf{i}, \mathbf{j})$, with $R = R_\mathbf{k}$ Noetherian, then*
(1) each finitely generated left $R(\mathbf{k})$-module is Noetherian,
(2) the association $\phi : M \longrightarrow \oplus_{\mathbf{i} \geq \mathbf{k}} B(\mathbf{i}, \mathbf{k}) \otimes_R M$ induces an equivalence of categories between $R-\mathrm{mod}$ and $R(\mathbf{k})-\mathrm{qgr}$.

Proof. Part (2) of the lemma follows the proof of [GS1, Lemma 5.5 (2)] verbatim, so we have only to check part (1). As in the proof of the [GS1, Lemma 5.5] it is enough to show that

$$M = \bigoplus_{\mathbf{i} \geq \mathbf{b}} B(\mathbf{i}, \mathbf{b})$$

is noetherian. So let $L = \bigoplus_{\mathbf{i} \geq \mathbf{b}} L(\mathbf{i}) \subseteq M$ be a graded submodule. For $\mathbf{i} \geq \mathbf{b}$ set

$$X(\mathbf{i}) = B(\mathbf{i}, \mathbf{b})^* \otimes L(\mathbf{i}) \subseteq R_\mathbf{b}.$$

As $R_\mathbf{b}$ is noetherian we have $\sum_{\mathbf{i} \geq \mathbf{b}} X(\mathbf{i}) = \sum_{\mathbf{j} \geq \mathbf{i} \geq \mathbf{b}} X(\mathbf{i})$ for some $\mathbf{j}$. Observe that there are only finitely many values of $\mathbf{i}$ between $\mathbf{b}$ and $\mathbf{j}$. Therefore, if $\mathbf{k} \geq \mathbf{j}$ we have

$$L(\mathbf{k}) = B(\mathbf{k}, \mathbf{b})X(\mathbf{k}) \subseteq \sum_{\mathbf{j} \geq \mathbf{i} \geq \mathbf{b}} B(\mathbf{k}, \mathbf{b})X(\mathbf{i}) = \sum_{\mathbf{j} \geq \mathbf{i} \geq \mathbf{b}} B(\mathbf{k}, \mathbf{i})B(\mathbf{i}, \mathbf{b})X(\mathbf{i}) = \sum_{\mathbf{j} \geq \mathbf{i} \geq \mathbf{b}} B(\mathbf{k}, \mathbf{i})L(\mathbf{i}).$$

Now fix $1 \leq t \leq m$. For each a such that $b_t \leq a < j_t$ we have a $\mathbb{Z}^{m-1}$–algebra $R^{(t,a)}$ defined as

$$R^{(t,a)} = \bigoplus_{\mathbf{i} \geq \mathbf{j}, i_t = j_t = a} B(\mathbf{i}, \mathbf{j}).$$

The $R^{(t,a)}$–module

$$\bigoplus_{\mathbf{i} \geq \mathbf{b}, i_t = a} B(\mathbf{i}, \mathbf{b})$$

is finitely generated noetherian by induction on m. Therefore we can find a finite set $I(t, a) \subseteq \{\mathbf{i} : i_t = a\}$ such that for all $\mathbf{k}$ with $k_t = a$ we have

$$L(\mathbf{k}) \subseteq \sum_{\mathbf{i} \in I(t,a)} B(\mathbf{k}, \mathbf{i})L(\mathbf{i}).$$

Thus L is generated by the $L(\mathbf{i})$ where $\mathbf{i}$ belongs to the finite set $\{\mathbf{i} : \mathbf{b} \leq \mathbf{i} \leq \mathbf{j}\} \cup \bigcup_{1 \leq t \leq m, b_t \leq a < j_t} I(t, a)$. Since each $L(\mathbf{i})$ is finitely generated as a $R_{\mathbf{i}}$-module the proof is finished.

The $\mathbb{Z}^m$-algebras we require can be constructed from certain bimodules that we call *basic.* These are the bimodules $B(\mathbf{j} + \mathbf{w_p}, \mathbf{j})$ for $\mathbf{j} \in \mathbb{C}^m$ and $1 \leq p \leq m$. The basic bimodules are required to satisfy a suitable compatibility condition. Namely for all $\mathbf{j} \in \mathbb{C}^m$ and $1 \leq p, q \leq m$ we require

$$B(\mathbf{j} + \mathbf{w_p} + \mathbf{w_q}, \mathbf{j} + \mathbf{w_p})B(\mathbf{j} + \mathbf{w_p}, \mathbf{j}) = B(\mathbf{j} + \mathbf{w_p} + \mathbf{w_q}, \mathbf{j} + \mathbf{w_q})B(\mathbf{j} + \mathbf{w_q}, \mathbf{j}). \quad (4.2)$$

Then we define $B(\mathbf{j}, \mathbf{k})$ for $\mathbf{j} \geq \mathbf{k}$ as follows. Choose $\mathbf{r_0}, \ldots, \mathbf{r_s} \in A$ such that $\mathbf{r_0} = \mathbf{k}$, $\mathbf{r_s} = \mathbf{j}$ and $\mathbf{r_i} = \mathbf{r_{i-1}} + \mathbf{w_{t(i)}}$ where $t(i) \in \{1, \ldots m\}$ for $1 \leq i \leq s$. Then set

$$B(\mathbf{j}, \mathbf{k}) = B(\mathbf{r_s}, \mathbf{r_{s-1}}) \ldots B(\mathbf{r_i}, \mathbf{r_{i-1}}) \ldots B(\mathbf{r_1}, \mathbf{r_0}). \quad (4.3)$$

By equation (4.2) this definition is independent of the choice of the $\mathbf{r_i}$, and it is clear that (4.1) holds.

5 Morita Theory for Spherical Subalgebras.

Suppose that $\Gamma = (\gamma)$ is cyclic of order n and that Γ acts on the first Weyl algebra $\mathbb{C}[\partial, y]$ so that in the crossed product we have

$$y\gamma = \omega\gamma y, \quad \gamma\partial = \omega\partial\gamma$$

where $\omega = e^{2\pi i/n}$. We do some computations in $Q = \mathbb{C}[\partial, y^{\pm 1}] * \Gamma$. For $0 \leq i \leq n-1$ set

$$e_i = (1/n)\sum_{j=0}^{n-1}(\omega^i\gamma)^j$$

and note that $ye_i = e_{i+1}y$ and $e_i\partial = \partial e_{i+1}$, where the indices are read mod n. Fix $\mathbf{k} = (k_1, \ldots, k_{n-1}) \in \mathbb{C}^{n-1}$, set $k_0 = 0$, and then define k_j for $j \in \mathbb{Z}$ by $k_j = k_i$ where $j \equiv i \mod n$ and $0 \leq i \leq n-1$. Then set $e = e_0$ and

$$d_{\mathbf{k}} = \partial - y^{-1}\sum_{i=1}^{n-1} k_i e_i.$$

Let $H_{\mathbf{k}}$ be the subalgebra of the crossed product generated by $d_{\mathbf{k}}, y$ and Γ, and set $U_{\mathbf{k}} = eH_{\mathbf{k}}e$. We define

$$\theta = y\partial = yd_{\mathbf{k}} + \sum_{i=1}^{n-1} k_i e_i \in H_{\mathbf{k}}.$$

Note that $Q = H_{\mathbf{k}}[y^{-1}]$ for all $\mathbf{k}$. By induction we have

$$d_{\mathbf{k}}^p y^p = \prod_{i=1}^{p}(\theta + i - \sum_{j=0}^{n-1} k_{i+j}e_j), \quad y^p d_{\mathbf{k}}^p = \prod_{i=0}^{p-1}(\theta - i - \sum_{j=0}^{n-1} k_{j-i}e_j). \quad (5.1)$$

Since the e_i are orthogonal idempotents which commute with θ, it follows that

$$e_j d_{\mathbf{k}}^p y^p = e_j \prod_{i=1}^{p} (\theta + i - k_{i+j}), \quad e_j y^p d_{\mathbf{k}}^p = e_j \prod_{i=0}^{p-1} (\theta - i - k_{j-i}). \tag{5.2}$$

Let $\mathbf{v_1}, \ldots, \mathbf{v_{n-1}}$ be the standard basis for $\mathbb{C}^{n-1}$, set $\mathbf{w_p} = n \sum_{i=1}^{p} \mathbf{v_i}, A = \oplus_{i=1}^{n-1} \mathbb{Z}\mathbf{w_i}$, and $A^+ = \oplus_{i=1}^{n-1} \mathbb{N}\mathbf{w_i}$. If $\mathbf{j} \geq \mathbf{k}$, then $\mathbf{j} - \mathbf{k} \in \sum_{j=1}^{n-1} \mathbb{N}\mathbf{w_j}$. It is convenient to define $F(\mathbf{b}) = \sum_{j=1}^{n-1} b_j \mathbf{w_j} = n \sum_{i=1}^{n-1} (\sum_{j=i}^{n-1} b_j) \mathbf{v_j}$. Then we have isomorphisms

$$\begin{aligned} Pic(X) \longleftarrow \mathbb{Z}^{n-1} &\longrightarrow \oplus_{i=1}^{n-1} \mathbb{Z}\mathbf{w_i} \\ D(\mathbf{b}) \longleftarrow \mathbf{b} &\longrightarrow F(\mathbf{b}). \end{aligned} \tag{5.3}$$

Lemma 5.1. *Fix $p \in \mathbb{N}$ with $1 \leq p \leq n-1$, and set $\mathbf{k}' = \mathbf{k} + \mathbf{w_p}$. Then*

$$y^p e H_{\mathbf{k}} e = e_p H_{\mathbf{k}'} e_p y^p. \tag{5.4}$$

Proof. To simplify notation write $d = d_{\mathbf{k}}$ and $d_1 = d_{\mathbf{k}'}$. We have

$$e_p \prod_{i=1}^{p} (\theta + i - p - k_i) = e_p \prod_{j=n-p+1}^{n} (\theta + j - k'_{p+j})$$

and

$$e_p \prod_{i=p+1}^{n} (\theta + i - p - k_i) = e_p \prod_{j=1}^{n-p} (\theta + j - k'_{p+j}).$$

To see this set $j = i + n - p$, (resp. $j = i - p$) in the left side of the first (resp. second) equation above to obtain the right side. Since $y\theta y^{-1} = \theta - 1$ it follows from (5.2) that

$$y^p e d^n y^{-p} = e_p \prod_{i=1}^{n} (\theta + i - p - k_i) y^{-n} = e_p \prod_{i=1}^{n} (\theta + i - k'_{p+i}) y^{-n} = e_p d_1^n. \tag{5.5}$$

The left side of (5.4) equals $y^p e \mathbb{C}[y^n, yd, d^n]$ and the right side equals $e_p \mathbb{C}[y^n, yd_1, d_1^n] y^p$. By (5.5) we have

$$y^p e d^n = e_p d_1^n y^p \tag{5.6}$$

and it easy to see that

$$y^p e y d = e_p (y d_1 - \kappa) y^p,$$

for some $\kappa \in \mathbb{C}$. Therefore

$$y^p e y^{an} (yd)^b d^{cn} = e_p y^{an} (y d_1 - \kappa)^b d_1^{cn} y^p.$$

The result follows from this.

We use Lemma 5.1 to define the basic bimodules for our $\mathbb{Z}^{n-1}-$algebras. Fix p with $1 \leq p \leq n-1$, and $\mathbf{k} \in \mathbb{C}^{n-1}$, and set $\mathbf{k'} = \mathbf{k} + \mathbf{w_p}$ and

$$B(\mathbf{k'}, \mathbf{k}) = B_p(\mathbf{k}) = eH_{\mathbf{k'}}e_p y^p,$$

By Lemma 5.1,

$$B_p(\mathbf{k}) \cdot U_{\mathbf{k}} = eH_{\mathbf{k'}}e_p \cdot y^p eH_{\mathbf{k}}e = eH_{\mathbf{k'}}e_p H_{\mathbf{k'}}e_p y^p \subseteq B_p(\mathbf{k})$$

It follows that $B_p(\mathbf{k})$ is a $U_{\mathbf{k'}} - U_{\mathbf{k}}$-bimodule. Similarly $C_p(\mathbf{k}) = y^{-p}e_p H_{\mathbf{k'}}e$ is a $U_{\mathbf{k}} - U_{\mathbf{k'}}$ bimodule and we have a Morita context

$$\begin{bmatrix} U_{\mathbf{k'}} & B_p(\mathbf{k}) \\ C_p(\mathbf{k}) & U_{\mathbf{k}} \end{bmatrix} \tag{5.7}$$

Theorem 5.2. *(1) $B_p(\mathbf{k})C_p(\mathbf{k}) = U_{\mathbf{k'}}$ provided $\{i - k_i\}_{i=1}^p \cap \{j - k_j\}_{j=p+1}^n = \emptyset$.*
(2) $C_p(\mathbf{k})B_p(\mathbf{k}) = U_{\mathbf{k}}$ provided $\{i - k_i\}_{i=1}^p \cap \{j + n - k_j\}_{j=p+1}^n = \emptyset$.

Proof. By (5.2) $B_p(\mathbf{k})C_p(\mathbf{k})$ contains the elements

$$ed^p_{\mathbf{k'}}e_p y^p \cdot y^{-p}e_p y^p e = ed^p_{\mathbf{k'}}y^p = eg(\theta).$$

and

$$ey^{n-p}d^{n-p}_{\mathbf{k'}} = eh(\theta).$$

where $g(\theta) = \Pi_{i=1}^p(\theta + i - n - k_i)$ and $h(\theta) = \Pi_{j=p+1}^n(\theta + j - n - k_j)$. Since g and h are relatively prime if and only if $\{i - k_i\}_{i=1}^p \cap \{j - k_j\}_{j=p+1}^n = \emptyset$, this proves (1) and the proof of (2) is similar.

We can express the conditions in Theorem 5.2 in terms of a root systems. We embed $\mathbb{C}^{n-1}$ in $\mathbb{C}^n$ as $\mathbb{C}^{n-1} \times \{0\}$, and let $\mathbf{v}_1, \ldots, \mathbf{v}_n$ be the standard basis for $\mathbb{C}^n$. Define a symmetric bilinear form (,) on $\mathbb{C}^n$ by $(\mathbf{v}_i, \mathbf{v}_j) = \delta_{i,j}$. The set

$$\Phi = \{\mathbf{v}_i - \mathbf{v}_j | i \neq j\}$$

forms a root system of type A_{n-1}. As a base for the root system we choose $B = \{\alpha_1, \ldots, \alpha_{n-1}\}$, where $\alpha_i = \mathbf{v}_i - \mathbf{v}_{i+1}$ for $1 \leq i \leq n-1$. Let Φ^+ denote the corresponding set of positive roots. Given $\mathbf{a} \in \mathbb{C}^{n-1}$, we set $\Phi_{\mathbf{a}} = \{\alpha \in \Phi | (\mathbf{a}, \alpha) \in \mathbb{Z}\}$.

Fix $\mathbf{k} = (k_1, \ldots, k_{n-1})$, and let $\rho = (n-1, \ldots, 2, 1) \in \mathbb{C}^{n-1}$. Then $(\mathbf{k} + \rho, \mathbf{v}_i - \mathbf{v}_j) = (k_i - i) - (k_j - j)$. Set $a_i = (n - i + k_i)/n$ for $1 \leq i \leq n-1, a_n = 0$ and $\mathbf{a} = (a_1, \ldots, a_{n-1})$. We have

$$\mathbf{k} + \rho = n\mathbf{a}. \tag{5.8}$$

Note also that $(\mathbf{w_p}, \alpha_i) = n\delta_{i,p}$ for $1 \leq i, p \leq n-1$. Let $\mathbf{1} = \sum_{i=1}^n \mathbf{v}_i$. It follows that $(\mathbf{w_p} - p\mathbf{1})/n$ is the p^{th} fundamental weight corresponding to the basis B. We say that $\mathbf{k}$ is *dominant* if $(\mathbf{k} + \rho, \alpha) > 0$ for all $\alpha \in \Phi_{\mathbf{a}} \cap \Phi^+$. For similar definitions in the enveloping algebra context see [J, Section 2.5].

Theorem 5.3. *If* $\mathbf{k}$ *is dominant and* $\mathbf{k}' = \mathbf{k} + \mathbf{w_p}$, *then* $\mathbf{k}'$ *is dominant and* $U_{\mathbf{k}}$ *and* $U_{\mathbf{k}'}$ *are Morita equivalent.*

Proof. The Morita equivalence follows from Theorem 5.2, and it is easy to check that $\mathbf{k}'$ is dominant.

We check that the basic bimodules $B(\mathbf{k}', \mathbf{k})$ satisfy (4.2).

Lemma 5.4. *If* $\mathbf{k}' = \mathbf{k} + \mathbf{w_p}$, $\mathbf{k}^{''} = \mathbf{k} + \mathbf{w_q}$, *and* $\mathbf{k}^{'''} = \mathbf{k} + \mathbf{w_p} + \mathbf{w_q}$, *then*

$$B(\mathbf{k}^{'''}, \mathbf{k}^{'})B(\mathbf{k}^{'}, \mathbf{k}) = B(\mathbf{k}^{'''}, \mathbf{k}^{''})B(\mathbf{k}^{''}, \mathbf{k}).$$

Proof. We first show that

$$B(\mathbf{k}^{'}, \mathbf{k}) = U_{\mathbf{k}'} e d^p_{\mathbf{k}'} y^p + U_{\mathbf{k}'} e y^n. \tag{5.9}$$

Since $e_0 H_{\mathbf{k}'} e_p = e\mathbb{C}[d_{\mathbf{k}'}, y] e_p$, and

$$e d^a_{\mathbf{k}'} y^b e_p = e e_{p+b-a} d^a_{\mathbf{k}'} y^b,$$

it follows that $eH_{\mathbf{k}'} e_p$ is spanned by all elements of the form $e d^a_{\mathbf{k}'} y^b e_p$ with $a \equiv p + b \; mod \; n$. Equation (5.9) follows easily from this. Thus

$$B(\mathbf{k}^{'''}, \mathbf{k}^{'})B(\mathbf{k}^{'}, \mathbf{k}) = B(\mathbf{k}^{'''}, \mathbf{k}^{'}) d^p_{\mathbf{k}'} y^p e + B(\mathbf{k}^{'''}, \mathbf{k}^{'}) y^n e$$

Using the analog of (5.9) for $B(\mathbf{k}^{'''}, \mathbf{k}^{'})$, we see that $B(\mathbf{k}^{'''}, \mathbf{k}^{'})B(\mathbf{k}^{'}, \mathbf{k})$ is generated as a left $U_{\mathbf{k}'''}$- module by the elements

$$e d^q_{\mathbf{k}'''} y^q \cdot d^p_{\mathbf{k}'} y^p, \quad e y^n \cdot d^p_{\mathbf{k}'} y^p,$$

$$e d^q_{\mathbf{k}'''} y^q \cdot y^n, \quad e y^n \cdot y^n.$$

Similarly $B(\mathbf{k}^{'''}, \mathbf{k}^{''})B(\mathbf{k}^{''}, \mathbf{k})$ is generated as a left $U_{\mathbf{k}'''}$- module by the elements

$$e d^p_{\mathbf{k}'''} y^p \cdot d^q_{\mathbf{k}''} y^q, \quad e y^n \cdot d^q_{\mathbf{k}''} y^q,$$

$$e d^p_{\mathbf{k}'''} y^p \cdot y^n, \quad e y^n \cdot y^n.$$

Assume that $p \leq q$. We have the following identities,

$$e y^n d^p_{\mathbf{k}'} y^p \quad = \quad e d^p_{\mathbf{k}'''} y^{p+n} \tag{5.10}$$

and

$$e d^q_{\mathbf{k}'''} y^q d^p_{\mathbf{k}'} y^p = e d^p_{\mathbf{k}'''} y^p d^q_{\mathbf{k}''} y^q. \tag{5.11}$$

These identities follows easily from (5.2). For example both sides of (5.11) are equal to

$$e\Pi^p_{i=1}(\theta + i - 2n - k_i)\Pi^q_{i=1}(\theta + i - n - k_i).$$

It remains to show that

$$ed^q_{\mathbf{k}'''}y^{q+n} \in B(\mathbf{k}''', \mathbf{k}'')B(\mathbf{k}'', \mathbf{k}) \tag{5.12}$$

and

$$ey^n d^q_{\mathbf{k}''}y^q \in B(\mathbf{k}''', \mathbf{k}')B(\mathbf{k}', \mathbf{k}). \tag{5.13}$$

By (5.2) $ey^n d^q_{\mathbf{k}''}y^q$ is a $\mathbb{C}[\theta]$-multiple of $ey^n d^p_{\mathbf{k}'}y^p$. This gives (5.13) and the proof of (5.12) is similar.

By Lemma 5.4 it now makes sense to define the bimodules $B(\mathbf{j}, \mathbf{k})$ for $\mathbf{j} \geq \mathbf{k}$ using equation (4.3).

Theorem 5.5. *If* $\mathbf{k}$ *is dominant then* $R(\mathbf{k}) = \oplus_{\mathbf{i}\geq\mathbf{j}\geq\mathbf{k}} B(\mathbf{i}, \mathbf{j})$ *is a Morita* $\mathbb{Z}^{n-1}$*-algebra* $R_{\mathbf{k}} = U_{\mathbf{k}}$.

6 Proof of the Main Theorem.

Suppose that $R = R(\mathbf{k})$ is as in Theorem 5.5. The algebra $D = \mathbb{C}[y^{\pm 1}, \partial] * \Gamma$ is equal to $D = \mathbb{C}[y^{\pm 1}, d_{\mathbf{k}}] * \Gamma$ for all $\mathbf{k} \in \mathbb{C}^{n-1}$. We consider the differential operator filtration on D defined by $D_N = \bigoplus_{i=0}^N \mathbb{C}[y^{\pm 1}] * \Gamma\partial^i$. We have $D_N = \bigoplus_{i=0}^N \mathbb{C}[y^{\pm 1}] * \Gamma d^i_{\mathbf{k}}$. If $a = \sum_{j=0}^N f_j(y)\partial^j \in \mathbb{C}[y^{\pm 1}, \partial]$ with $f_i(y) \in \mathbb{C}[y^{\pm 1}], f_N \neq 0$ we set $gr(a) = f_N(u)v^N$. We extend gr to a linear map from $D = \mathbb{C}[y^{\pm 1}, \partial] * \Gamma$ to $\mathbb{C}[u^{\pm 1}, v] * \Gamma$ such that $gr(\gamma a) = \gamma gr(a)$. Note in particular that $u = gr\ y, v = gr\ d_{\mathbf{k}}$ and $gr\ d_{\mathbf{k}} = v^n = z$. Also

$$gr(bd^n_{\mathbf{k}}) = gr(b)z \tag{6.1}$$

for all $b \in D$. Since $H_{\mathbf{k}}$ and $U_{\mathbf{k}}$ are subalgebras of D they have an induced filtration, and we have

$$grH_{\mathbf{k}} \cong \mathbb{C}[u, v] * \Gamma, \quad grU_{\mathbf{k}} \cong \mathbb{C}[u, v]^{\Gamma}.$$

Similarly there is a differential operator filtration $\{B_n(\mathbf{r}', \mathbf{r})\}_{n\geq 0}$ on $B(\mathbf{r}', \mathbf{r})$ and we obtain a filtration on R by setting $R_n = \oplus_{\mathbf{r}'\geq\mathbf{r}\geq\mathbf{k}} B_n(\mathbf{r}', \mathbf{r})$.

Recall the isomorphisms from equation (5.3). The key remaining step in the proof of the main theorem is the following.

Theorem 6.1. *Assume that* $\mathbf{k} \in \mathbb{C}^{n-1}$ *is dominant and* $\mathbf{r}' \geq \mathbf{r} \geq \mathbf{k}$ *with* $\mathbf{r}' = \mathbf{r} + F(\mathbf{b})$. *Then*

$$grB(\mathbf{r}', \mathbf{r}) = H^0(X, \mathcal{O}(D(\mathbf{b}))).$$

Proof of the Main Theorem.
(1) The equivalence of categories $R - \mathrm{qgr} \simeq U_{\mathbf{k}}$-mod follows from Theorem 5.5 and Lemma 4.2.

(2) Let $S = \bigoplus_{\mathbf{b}\in\mathbb{N}^{n-1}} H^0(X, \mathcal{O}(D(\mathbf{b})))$. By Theorem 3.1 and Lemma 3.5 we know the category $S-\mathrm{qgr}$ is equivalent to $Coh(Hilb_\Gamma \mathbb{C}^2)$. On the other hand by Theorem 6.1 the associated graded ring of the $\mathbb{Z}^{n-1}$-algebra $R(\mathbf{k})$ is

$$\bigoplus_{\mathbf{r}'\geq\mathbf{r}\geq\mathbf{k}} H^0(X, \mathcal{O}(D(F^{-1}(\mathbf{r}'-\mathbf{r})))),$$

and this is the $\mathbb{Z}^{n-1}$-algebra arising from the $\mathbb{N}^{n-1}$-graded algebra S. Hence the result follows from Lemma 4.1.

It is easy to check Theorem 6.1 for basic bimodules.

Lemma 6.2. *If $1 \leq p \leq n-1$, and $\mathbf{r}' = \mathbf{r} + \mathbf{w_p}$, we have*

$$gr(B(\mathbf{r}', \mathbf{r})) \cong H^0(X, \mathcal{O}(D(\mathbf{v_p}))$$

Proof. By (5.9) $grB(\mathbf{r}', \mathbf{r}) \cong C^\Gamma u^p v^p + C^\Gamma u^n = C^\Gamma x^p + C^\Gamma x^n z^{-1}$. On the other hand Lemma 2.4 implies that

$$H^0(X, \mathcal{O}(D(\mathbf{v_p})) = C^\Gamma + C^\Gamma x^{n-p} z^{-1}.$$

The isomorphism is multiplication by x^{-p}.

Lemma 6.3. *For $\mathbf{b} \in \mathbb{N}^{n-1}$, set $f(\mathbf{b}) = \sum_{j=1}^{n-1} jb_j$. There is an injective linear map*

$$H^0(X, \mathcal{O}(D(\mathbf{b}))) \longrightarrow gr\ B(\mathbf{r} + F(\mathbf{b}), \mathbf{r}) \tag{6.2}$$

given by multiplication by $x^{f(\mathbf{b})}$.

Proof. Using Corollary 2.2, induction and then [GS1, Lemma 6.7 (1)] we get

$$H^0(X, \mathcal{O}(D(\mathbf{b}+\mathbf{c})))x^{f(\mathbf{b}+\mathbf{c})} = H^0(X, \mathcal{O}(D(\mathbf{b})))x^{f(\mathbf{c})} H^0(X, \mathcal{O}(D(\mathbf{c})))x^{f(\mathbf{b})} \subseteq$$

$$gr\ B(\mathbf{r} + F(\mathbf{b}+\mathbf{c}), \mathbf{r} + F(\mathbf{c})) gr\ B(\mathbf{r} + F(\mathbf{c}), \mathbf{r}) \subseteq gr\ B(\mathbf{r} + F(\mathbf{b}+\mathbf{c}), \mathbf{r}).$$

From now on we assume that $\mathbf{k} \in \mathbb{C}^{n-1}$ is dominant and fix $\mathbf{r}' \geq \mathbf{r} \geq \mathbf{k}$ with $\mathbf{r}' = \mathbf{r} + F(\mathbf{b})$. Note that $\mathbf{r}'$ and $\mathbf{r}$ are also dominant. We write $B(\mathbf{b})$ and d instead of $B(\mathbf{r}', \mathbf{r})$ and $d_\mathbf{r}$ respectively We have

$$r'_i = r_i + n\sum_{j=i}^{n-1} b_j \tag{6.3}$$

for $1 \leq i \leq n-1$.

We explain how the proof of Theorem 6.1 reduces to a Poincaré series computation. Note that the torus T acts on both the domain and target of the map in equation (6.2). Set $P = gr\ B(\mathbf{b})$ and consider the weight space decomposition $P = \bigoplus_{\chi\in\mathbb{X}(T)} P(\chi)$. Since $gr\ B(\mathbf{b}) \subseteq \mathbb{C}[u, v]$ it follows that

$$\{\chi \in \mathbb{X}(T) | P(\chi) \neq 0\} \subseteq \mathbb{N}\chi_1 + \mathbb{N}\chi_2.$$

Because $z \in C[n\chi_2]$, this implies that $\cap_{m \geq 0} P z^m = 0$. Therefore by Nakayama's lemma if $\mathcal{B}$ is a subspace of P whose image in $\overline{P} = P \otimes_{\mathbb{C}[z]} \mathbb{C}$ is a basis, then $\mathcal{B}$ generates P as a $\mathbb{C}[z]$-module. We will show that $dim \overline{P}(\chi) < \infty$ for all $\chi \in \mathbb{X}(T)$. Then since P is a torsion-free $\mathbb{C}[z]$-module, we have

$$\mathcal{H}_{\overline{P}}(q,t) = (1-t^n)\mathcal{H}_P(q,t).$$

Now we view $\mathbb{C}^\times$ as the subtorus $\{(\lambda, \lambda^{-1}) | \lambda \in \mathbb{C}^\times\}$ of T. Then $\mathbb{X}(\mathbb{C}^\times) = \mathbb{Z}\eta$, where $\eta = \chi_1|_{\mathbb{C}^\times} = -\chi_2|_{\mathbb{C}^\times}$. For a $\mathbb{C}^\times$-module V we now have the weight space decomposition $V = \bigoplus_{m \in \mathbb{Z}} V[m]$, where

$$V[m] = \{v \in V | \tau \cdot v = \eta(\tau)^m v \text{ for all } \tau \in \mathbb{C}^\times\}.$$

If $dim V[m] < \infty$ for all m, we define the one variable Poincaré series

$$\mathbb{H}_V(s) = \sum_{m \in \mathbb{Z}} \dim V[m]\, s^m.$$

Clearly $\overline{P}[m] = \bigoplus \overline{P}(\chi)$ where the sum is over all $\chi \in \mathbb{X}(T)$ such that $\chi|_{\mathbb{C}^\times} = m\eta$. If we can show that $dim \overline{P}[m] < \infty$ for all $m \in \mathbb{Z}$, then $dim \overline{P}(\chi) < \infty$ for all $\chi \in \mathbb{X}(T)$ and we have

$$\mathcal{H}_{\overline{P}}(s, s^{-1}) = \mathbb{H}_{\overline{P}}(s).$$

Similarly if $\overline{N} = N(\mathbf{b}) \otimes_{\mathbb{C}[z]} \mathbb{C}$, then

$$\mathbb{H}_{\overline{N}}(s) = \mathcal{H}_{\overline{N}}(s, s^{-1}) = (1 - s^{-n})\mathcal{H}_N(s, s^{-1}).$$

Therefore by equation (2.10)

$$\mathbb{H}_{\overline{N}}(s) = \frac{\sum_{i=0}^{n-1} s^{n \sum_{j=n-i}^{n-1} b_j}}{(1-s^n)}. \tag{6.4}$$

On the other hand the map in equation (6.2) is equivariant for the action of $\mathbb{C}^\times$, since $x = uv \in C[0]$. Hence to prove Theorem 6.1 it suffices to show that the Poincaré series $\mathbb{H}_{\overline{P}}(s)$ is given by equation (6.4). We do this by developing some representation theory of the algebras $H_{\mathbf{r}}$ and $U_{\mathbf{r}}$, and the bimodules $B(\mathbf{b})$.

The algebra $H_{\mathbf{r}}$ has a $\mathbb{Z}$-grading in which the degrees of the generators satisfy

$$\deg y = 1, \quad \deg d = -1, \quad \deg \gamma = 0.$$

If $M = \oplus_{\alpha \in \mathbb{Z}} M_\alpha$ is a graded $H_{\mathbf{r}}$-module with $\dim M_\alpha < \infty$ for all α, we define the Poincaré series of M to be

$$p(M, s) = \sum_{\alpha \in \mathbb{Z}} (\dim M_\alpha) s^\alpha.$$

Lemma 6.4.

$$\mathbb{H}_{\overline{gr\ B(\mathbf{b})}}(s) = p(B(\mathbf{b}) \otimes_{\mathbb{C}[d^n]} \mathbb{C}, s).$$

Proof. Note that $u = gr\ y \in \mathbb{C}[u,v][1]$, and $v = gr\ d \in \mathbb{C}[u,v][-1]$. Since $\mathbb{Z} \cong \mathbb{Z}\eta$ is the character group of $\mathbb{C}^\times$, this gives an action of $\mathbb{C}^\times$ on $\mathbb{C}[y^{\pm 1}, \partial] * \Gamma$ such that the map gr is $\mathbb{C}^\times$-equivariant. Therefore as $\mathbb{C}^\times$-modules

$$B(\mathbf{b}) \cong gr\ B(\mathbf{b}). \tag{6.5}$$

Furthermore by equation (6.1)

$$B(\mathbf{b})d^n \cong gr\ B(\mathbf{b})z. \tag{6.6}$$

The result follows since $B(\mathbf{b}) \otimes_{\mathbb{C}[d^n]} \mathbb{C}$ is the quotient of the left side of (6.5) by the left side of (6.6), and $\overline{gr\ B(\mathbf{b})} = (gr\ B(\mathbf{b})) \otimes_{\mathbb{C}[z]} \mathbb{C}$ is the corresponding quotient of the right sides.

Let $\mathcal{O}_\mathbf{r}$ denote the category whose objects are finitely generated $H_\mathbf{r}$-modules on which the action of d is locally nilpotent. As in [GS1, Section 6.12] and [GGOR, Section 2.4] we use a graded version $\widetilde{\mathcal{O}}_\mathbf{r}$ of the category $\mathcal{O}_\mathbf{r}$. Objects in $\widetilde{\mathcal{O}}_\mathbf{r}$ are $\mathbb{Z}$-graded $H_\mathbf{r}$-modules which are also objects in $\mathcal{O}_\mathbf{r}$. Morphisms in $\mathcal{O}_\mathbf{r}$ (resp. $\widetilde{\mathcal{O}}_\mathbf{r}$) are $H_\mathbf{r}$-module homomorphisms (resp. $H_\mathbf{r}$-module homomorphisms which are homogeneous of degree zero). We write Let $\mathcal{O}^U_\mathbf{r}$ for the category of finitely generated $U_\mathbf{r}$-modules on which the action of d^n is locally nilpotent, and let $\widetilde{\mathcal{O}}^U_\mathbf{r}$ denote the corresponding category of graded $U_\mathbf{r}$-modules.

If $M = \oplus_{\alpha \in \mathbb{Z}} M_\alpha$ is a module in $\widetilde{\mathcal{O}}_\mathbf{r}$ it follows from the local nilpotence of d and finite generation that $\dim M_\alpha < \infty$ for all α, so $p(M,s)$ is defined. For $\beta \in \mathbb{Z}$, the shift functor $[\beta]$ in $\widetilde{\mathcal{O}}_\mathbf{r}$ is defined by $(M[\beta])_\alpha = M_{\alpha-\beta}$. We have

$$p(M[\beta], s) = s^\beta p(M,s).$$

The algebra $H_\mathbf{r}$ has a triangular decomposition

$$H_\mathbf{r} = \mathbb{C}[y] \otimes \mathbb{C}[\Gamma] \otimes \mathbb{C}[d].$$

For $i = 0, \ldots, n-1$ let $\mathbb{C}\varepsilon_i$ denote the one-dimensional $\mathbb{C}\Gamma$-module on which e_i acts as the identity and make $\mathbb{C}\varepsilon_i$ into a $\mathbb{C}[d] * \Gamma$-module with $d\varepsilon_i = 0$. Then we define the *standard module* $M_\mathbf{r}(\varepsilon_i)$ by

$$M_\mathbf{r}(\varepsilon_i) = H_\mathbf{r} \otimes_{\mathbb{C}[d]*\Gamma} \mathbb{C}\varepsilon_i.$$

Note that θ acts on the subspace $1 \otimes \mathbb{C}\varepsilon_i$ of $M_\mathbf{r}(\varepsilon_i)$ as multiplication by the scalar r_i. We also define the *graded standard module* $\widetilde{M}_\mathbf{r}(\varepsilon_i)$ to be an isomorphic copy of $M_\mathbf{r}(\varepsilon_i)$ as an $H_\mathbf{r}$-module, with grading defined by $\deg(1 \otimes \varepsilon_i) = 0$.

To prove the main theorem we compute the Poincaré series of $B(\mathbf{b}) \otimes_{\mathbb{C}[d^n]} \mathbb{C}$ and show that it equals (6.4). To do this we observe that

$$B(\mathbf{b}) \otimes_{\mathbb{C}[d^n]} \mathbb{C} \cong B(\mathbf{b}) \otimes_{eH_\mathbf{r}e} eH_\mathbf{r}e \otimes_{\mathbb{C}[d^n]} \mathbb{C}.$$

So we begin our analysis with the left $H_\mathbf{r}$-module $G = H_\mathbf{r}e \otimes_{\mathbb{C}[d^n]} \mathbb{C}$. This module inherits a grading from $H_\mathbf{r}$, in which $\deg(e \otimes 1) = 0$.

Lemma 6.5. *In the Grothendieck group of the category $\widetilde{\mathcal{O}}_{\mathbf{r}}$, we have*

$$[G] = [\widetilde{M}_{\mathbf{r}}(\epsilon_0)] + \sum_{i=1}^{n-1}[\widetilde{M}_{\mathbf{r}}(\varepsilon_{n-i})[-i]]. \tag{6.7}$$

Proof. For $0 \leq i \leq n-1$ set $u_i = d^i e \otimes 1 \in G$, $N_i = H_{\mathbf{r}} u_i$ and $N_n = 0$. It is easy to see that $G = \sum_{i=0}^{n-1} \mathbb{C}[y]u_i$, a free $\mathbb{C}[y]$-module of rank n. Thus $N_i/N_{i+1} \cong \widetilde{M}_{\mathbf{r}}(\varepsilon_{n-i})[-i]$ for $0 \leq i \leq n-1$, and the result follows from this.

Key Lemma. *Assume that $\mathbf{r}$ is dominant. Then as objects of the category $\widetilde{\mathcal{O}}^U_{\mathbf{r}'}$ we have, for $1 \leq i \leq n-1$*

$$B(\mathbf{b}) \otimes e\widetilde{M}_{\mathbf{r}}(\varepsilon_i) \cong e\widetilde{M}_{\mathbf{r}'}(\varepsilon_i)[r'_i - r_i]. \tag{6.8}$$

To prove this we need several preliminary results. First we state an easy characterization of the $U_{\mathbf{r}'}$-modules $eM_{\mathbf{r}'}(\varepsilon_i)$.

Lemma 6.6. *Suppose that $M = \mathbb{C}[y^n]v$ is a left $U_{\mathbf{r}'}$-module which is generated by an element v such that $d^n_{\mathbf{r}'}v = 0$, $\theta v = (n - i + r'_i)v$, and M is a free $\mathbb{C}[y^n]$-module, then $M \cong eM_{\mathbf{r}'}(\varepsilon_i)$.*

In the next lemma we assume that $\mathcal{R}$ and $\mathcal{S}$ are subrings of a $\mathbb{C}$-algebra $\mathcal{Q}$ and that $\mathcal{B}$ is an $\mathcal{R} - \mathcal{S}$ bimodule and $\mathcal{C}$ an $\mathcal{S} - \mathcal{R}$-bimodule such that the functors

$$\mathcal{B} \otimes \_ : \mathcal{S}\text{−mod} \longrightarrow \mathcal{R}\text{−mod}$$

and

$$\mathcal{C} \otimes \_ : \mathcal{R}\text{−mod} \longrightarrow \mathcal{S}\text{−mod}$$

are inverse equivalences of categories. Suppose that $\mathcal{T}$ is a multiplicatively closed subset of both $\mathcal{R}$ and $\mathcal{S}$ and that $\mathcal{T}$ is an Ore set in $\mathcal{R}$. Assume also that $\mathcal{C}$ satisfies an Ore condition with respect to $\mathcal{T}$: given $t \in \mathcal{T}$ and $c \in \mathcal{C}$ there exist $t' \in \mathcal{T}$ and $c' \in \mathcal{C}$ such that

$$c't = t'c. \tag{6.9}$$

In the following "torsion" means torsion with respect to $\mathcal{T}$.

Lemma 6.7. *Let M be an $\mathcal{S}$-module, and N an $\mathcal{R}$-module.*
(1) If N is torsion so is $\mathcal{C} \otimes N$.
(2) If M is torsion-free so is $\mathcal{B} \otimes M$.

Proof. (1) Suppose that $n \in N, t \in \mathcal{T}$ and $tn = 0$. If $c \in \mathcal{C}$, we find c' and t' as in (6.9). Then $t'(c \otimes n) = c' \otimes tn = 0$.
(2) Since $\mathcal{T}$ is Ore in $\mathcal{R}$ the set N of torsion elements of $\mathcal{B} \otimes M$ forms a submodule. By (1), $\mathcal{C} \otimes N$ is a torsion submodule of $\mathcal{C} \otimes \mathcal{B} \otimes M$, but by assumption $\mathcal{C} \otimes \mathcal{B} \otimes M \cong M$ which is torsion free.

Corollary 6.8. *$B_p(\mathbf{r}) \otimes eM_{\mathbf{r}}(\varepsilon_i)$ is a torsion-free $\mathbb{C}[y^n]$-module.*

Proof. We apply Lemma 6.7 with $\mathcal{R} = U_{\mathbf{r}'}, \mathcal{S} = U_{\mathbf{r}}, \mathcal{B} = B_p(\mathbf{r})$ and $\mathcal{T} = \mathbb{C}[y^n] \setminus \{0\}$. The Ore conditions follow from the existence of the localizations. Indeed, it is easy to see that

$$\begin{aligned} H_{\mathbf{r}} \otimes_{\mathbb{C}[y^n]} \mathbb{C}(y^n) &= \mathbb{C}[y^{\pm 1}, \partial] * \Gamma \\ U_r \otimes_{\mathbb{C}[y^n]} \mathbb{C}(y^n) &= e\mathbb{C}[y^{\pm 1}, \partial] * \Gamma e \end{aligned}$$

and

$$\begin{aligned} B_p(\mathbf{r}) \otimes_{\mathbb{C}[y^n]} \mathbb{C}(y^n) &= e\mathbb{C}[y^{\pm 1}, \partial] * \Gamma e_p y^p \\ &= \mathbb{C}(y^n) \otimes_{\mathbb{C}[y^n]} B_p(\mathbf{r}). \end{aligned}$$

Lemma 6.9. *Assume that* $\mathbf{r}$ *is dominant and set* $B = B_p(\mathbf{r})$*. Consider the elements* $v = ey^n \otimes y^{n-i}\varepsilon_i$ *and* $w = ed^p_{\mathbf{r}'}y^p \otimes y^{n-i}\varepsilon_i$ *of* $B \otimes eM_{\mathbf{r}}(\varepsilon_i)$*. Then*
(1) $d^n_{\mathbf{r}'}w = 0$.
(2) If $1 \le i \le p$*, then* $d^n_{\mathbf{r}'}v = w = 0$.
(3) If $p+1 \le i \le n-1$*, then* $y^n w = \kappa v$ *for some nonzero* $\kappa \in \mathbb{C}$.
(4) $B \otimes eM_{\mathbf{r}}(\varepsilon_i) = U_{\mathbf{r}'}v + U_{\mathbf{r}'}w$.

Proof. We use the following identity repeatedly

$$(\theta + j - r'_j)y^{n-i}\varepsilon_i = (r_i + n - i + j - r'_j)y^{n-i}\varepsilon_i. \tag{6.10}$$

Equation (6.10) holds because both sides are equal to $y^{n-i}(\theta + n - i + j - r'_j)\varepsilon_i$.
(1) Multiplying (5.6) on the left by $d^p_{\mathbf{r}'}$ gives $ed^n_{\mathbf{r}'} \cdot d^p_{\mathbf{r}'}y^p = ed^p_{\mathbf{r}'}y^p \cdot d^n_{\mathbf{r}}$. Therefore

$$\begin{aligned} d^n_{\mathbf{r}'}w &= ed^n_{\mathbf{r}'} \cdot d^p_{\mathbf{r}'}y^p \otimes y^{n-i}\varepsilon_i \\ &= ed^p_{\mathbf{r}'}y^p \otimes d^n_{\mathbf{r}}y^{n-i}\varepsilon_i = 0. \end{aligned}$$

(2) We first show that $y^n w = 0$. Indeed by (5.2) and (6.10),

$$\begin{aligned} y^n w &= ey^n \cdot d^p_{\mathbf{r}'}y^p \otimes y^{n-i}\varepsilon_i \\ &= ey^n \otimes \prod_{j=1}^{p}(\theta + j - r'_j)y^{n-i}\varepsilon_i \\ &= ey^n \otimes y^{n-i}\prod_{j=1}^{p}(r_i + n - i + j - r'_j)\varepsilon_i = 0, \end{aligned}$$

since the term in the product with $j = i$ is zero.

It follows from Corollary 6.8 that $w = 0$. Now by equations (5.2) and (6.10),

$$d^n_{\mathbf{r}'}v = ed^n_{\mathbf{r}'}y^n \otimes y^{n-i}\varepsilon_i = e\prod_{j=1}^{n}(\theta + j - r'_j) \otimes y^{n-i}\varepsilon_i$$

$$= ed^p_{\mathbf{r}'} y^p \otimes \prod_{j=p+1}^{n} (\theta + j - r'_j) y^{n-i} \varepsilon_i$$

$$= ed^p_{\mathbf{r}'} y^p \otimes y^{n-i} \prod_{j=p+1}^{n} (r_i + n - i + j - r'_j) \varepsilon_i.$$

The result follows since this is a multiple of w.
(3) As in the proof of (2),

$$y^n w = ey^n \otimes y^{n-i} \prod_{j=1}^{p} (r_i + n - i + j - r'_j) \varepsilon_i,$$

but now the terms $(r_i + n - i + j - r'_j)$ are nonzero, since $i \geq p+1 > j$ and $\mathbf{r}$ is dominant.
(4) Since $B \otimes e\widetilde{M}_{\mathbf{r}}(\varepsilon_i) = B \otimes y^{n-i}\mathbb{C}[y^n]\varepsilon_i = B \otimes y^{n-i}\varepsilon_i$, the result follows from equation (5.9).

Proof of the Key Lemma. We can assume that $\mathbf{r}' = \mathbf{r} + \mathbf{w_p}$. Set $B = B_p(\mathbf{r})$. Suppose that $p+1 \leq i \leq n-1$. By (1), (3) and (4) in Lemma 6.9 $B \otimes e\widetilde{M}_{\mathbf{r}}(\varepsilon_i) = U_{\mathbf{r}'} w$, and $d^n_{\mathbf{r}'} w = 0$. Hence $B \otimes eM_{\mathbf{r}}(\varepsilon_i) = \mathbb{C}[y^n]w$ and this is a free $\mathbb{C}[y^n]$-module, since it is torsion free by Corollary 6.8. It is easy to check that $\theta w = (n - i + r_i)w$ and that w has the same degree as $y^{n-i}\varepsilon_i$. So the result in this case follows from Lemma 6.6. The proof for the case where $1 \leq i \leq p$, is similar using v instead of w.

Proof of Theorem 6.1. By Lemma 6.3 there is an inclusion $H^0(X, \mathcal{O}(D(\mathbf{b}))) \subseteq grB(\mathbf{b})$. To show the reverse inclusion, it suffices to show that

$$grB(\mathbf{b}) \otimes_{\mathbb{C}[z]} \mathbb{C}$$

and

$$H^o(X, \mathcal{O}(D(\mathbf{b}))) \otimes_{\mathbb{C}[z]} \mathbb{C}$$

have the same Poincaré series.

By equations (6.7) and (6.8) we have in the Grothendieck group of the category $\widetilde{\mathcal{O}}_{\mathbf{r}'}$

$$[B(\mathbf{b}) \otimes eG] = [e\widetilde{M}_{\mathbf{r}'}(\epsilon_0)] + \sum_{i=1}^{n-1} [e\widetilde{M}_{\mathbf{r}'}(\varepsilon_{n-i})[-i - r_{n-i} + r'_{n-i}]]. \tag{6.11}$$

Since $ey^i = y^i e_{n-i}$ it follows that for $1 \leq i \leq n-1$

$$e\widetilde{M}_{\mathbf{r}'}(\varepsilon_{n-i}) = y^i \mathbb{C}[y^n]\varepsilon_{n-i}$$

and so

$$p(e\widetilde{M}_{\mathbf{r}'}(\varepsilon_{n-i}), s) = s^i (1 - s^n)^{-1}. \tag{6.12}$$

In addition we have

$$p(e\widetilde{M}_{\mathbf{r}'}(\varepsilon_0), s) = (1 - s^n)^{-1}. \tag{6.13}$$

We combine (6.12) and (6.13) with equation (6.11) and then use equation (6.3) to obtain,

$$\begin{aligned} p(B(\mathbf{b}) \otimes eG, s) &= \sum_{i=0}^{n-1} p(e\widetilde{M}_{\mathbf{r}'}(\varepsilon_{n-i}), s) s^{-i-r_{n-i}+r'_{n-i}} \\ &= \sum_{i=1}^{n} \frac{s^{r'_i - r_i}}{(1 - s^n)} \\ &= \frac{\sum_{i=1}^{n} s^{n \sum_{j=i}^{n-1} b_j}}{1 - s^n}. \end{aligned}$$

Since this is the same as equation (6.4), the proof is complete.

7 Concluding remarks.

We relate our work to that of Hodges, [H] and Crawley-Boevey and Holland [CBH]. As in [H] we fix a monic polynomial $v(x) \in \mathbb{C}[x]$ and let $T(v)$ be the algebra generated by the elements h, a, b such that

$$ha - ah = a, \quad hb - bh = -b, \quad ba = v(h), \quad ab = v(h-1).$$

Fix $\mathbf{k} = (k_1, \dots, k_{n-1}) \in \mathbb{C}^{n-1}$, and let $\mathbf{a} = (a_1, \dots, a_{n-1}) \in \mathbb{C}^{n-1}$ be as equation (5.8), and $a_n = 0$.

Lemma 7.1. *Set $v(x) = \prod_{i=1}^{n}(x - a_i)$. Then $T(v) \cong U_{\mathbf{k}}$.*

Proof. The algebra $U_{\mathbf{k}}$ is generated by $A = ey^n$, $B = e(d_{\mathbf{k}}/n)^n$ and $H = e(\theta + n)/n$. We have $HA - AH = A$, and $HB - BH = -B$. Using equation (5.2) we see that

$$BA = e \prod_{i=1}^{n}((\theta + i - k_i)/n) = v(H)$$

and similarly $AB = v(H-1)$. Hence there is a surjective ring homomorphism from $T(v)$ to $U_{\mathbf{k}}$ sending a, b, h to A, B, H respectively. The map is injective since $T(v)$ and $U_{\mathbf{k}}$ have the same associated graded ring.

Now define an action of the symmetric group S_{n-1} on $\mathbb{C}^{n-1}$ by $s(x)_i = x_{s^{-1}i}$. Define the dot action of S_{n-1} on $\mathbb{C}^{n-1}$ by $s \cdot x = s(x + \rho) - \rho$.

Corollary 7.2. *For all $s \in S_{n-1}, U_{\mathbf{k}} \cong U_{s \cdot \mathbf{k}}$.*

Now suppose that $v(x) = w(x)u(x)$ is a factorization of v with u, w monic. Hodges considers the $T(v)$-module

$$P_w = T(v)a + T(v)w(h). \tag{7.1}$$

The next result is [H, Lemma 2.4].

Theorem 7.3. *The left $T(v)$ module P_w is projective if $u(x)$ and $w(x)$ are relatively prime, and is a generator if $u(x)$ and $w(x+1)$ are relatively prime.*

We compare P_w to the $U_{\mathbf{k}'} - U_{\mathbf{k}}$-bimodules $B_p(\mathbf{k})$. To do this we now assume that let $v(x) = \prod_{i=1}^{n}(x - a_i')$, where $a_i' = (n - i + k_i')/n$ for $1 \leq i \leq n-1$, and $a_n' = 0$ so that $T(v) \cong U_{\mathbf{k}'}$. We identify $T(v)$ with $U_{\mathbf{k}'}$ using this isomorphism. By Corollary 7.2 we may assume that $w(x) = \prod_{i=1}^{p}(x - a_i')$. Then by equation (5.2) $e(d_{\mathbf{k}'}/p)^n y^n = w(H)$. So comparing equations (5.9) and (7.1) we see that P_w is identified with $B_p(\mathbf{k})$. Now it is easy to see that Theorem 5.2 is equivalent to Theorem 7.3.

For a finite subgroup Γ of $SL_2\mathbb{C}$, and λ a central element in $\mathbb{C}\Gamma$, Crawley-Boevey and Holland define the algebras

$$S^\lambda = (\mathbb{C} < x, y > \Gamma)/(xy - yx - \lambda),$$

and $\mathcal{O}^\lambda = eS^\lambda e$, [CBH]. For Γ cyclic of order n, we compare the S^λ to the algebras $H_{\mathbf{k}}$. By equation (5.1)

$$d_{\mathbf{k}}y - yd_{\mathbf{k}} = 1 + \sum_{j=0}^{n-1}(k_j - k_{j+1})e_j = \sum_{j=0}^{n-1}\lambda_j e_j, \tag{7.2}$$

where $\lambda_j = (1/n) + k_j - k_{1+j}$. Note that the trace of $\lambda = \sum_{j=0}^{n-1}\lambda_j e_j$ on the regular representation of Γ equals 1. Also if we are given $\lambda \in \mathbb{C}\Gamma$ with trace 1, there is a unique solution to equation (7.2) with $k_0 = k_n = 0$ and $\mathbf{k} \in \mathbb{C}^{n-1}$. Clearly we have $S^\lambda \cong H_{\mathbf{k}}$, and $\mathcal{O}^\lambda \cong U_{\mathbf{k}}$.

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