# Hopf Down-up Algebras 

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## 1 Introduction

Given a field $k$ and elements $\alpha, \beta, \gamma \in k$, Benkart and Roby $[\mathbf{B R}]$ defined the down-up algebra $A(\alpha, \beta, \gamma)$ to be the algebra generated by two generators $u$ and $d$ subject to the two relations:

$$
\begin{aligned}
& d^{2} u=\alpha d u d+\beta u d^{2}+\gamma d, \\
& d u^{2}=\alpha u d u+\beta u^{2} d+\gamma u .
\end{aligned}
$$

By [KMP] $A(\alpha, \beta, \gamma)$ is a Noetherian ring if and only if $\beta \neq 0$. When $\gamma \neq 0$, the downup algebra $A(\alpha, \beta, \gamma)$ is isomorphic to the down-up algebra $A(\alpha, \beta, 1)$. Throughout let $A=A(\alpha, \beta, \gamma)$ be a Noetherian down-up algebra over an algebraically closed field of characteristic zero.

The down-up algebras $A(-2,-1, \gamma)$ are isomorphic to the enveloping algebra of the 3 dimensional Heisenberg Lie algebra when $\gamma=0$, or to the enveloping algebra of $\mathfrak{s l}_{2}$ when $\gamma \neq 0$, and hence they are Hopf algebras. Since general down-up algebras possess many properties of the down-up algebra $U\left(\mathfrak{s l}_{2}\right)$, it might be reasonable to expect they have Hopf structures. In $[\mathbf{B R}]$ the question of determining when a down-up algebra is a Hopf algebra is raised, and in $[\mathbf{B W}]$ it is shown that when $\gamma=0$ there is a group $G$ of automorphisms of $A$ such that the twisted group ring $A * G$ is a Hopf algebra. The question of determining when a down-up algebra has a Hopf structure can be considered as a case of the following more general test problem in noncommutative algebras: given an algebra $A$, what do ring theoretic and representation theoretic properties of the ring $A$ tell us about possible bialgebra or Hopf structures on $A$ ? An analog of this problem for commutative rings is to investigate the structure of algebraic groups of small dimension.

In this paper our aim is to determine whether there are other down-up algebras that possess Hopf structures. Our main result is the following theorem:

[^0]Theorem 1.1. Let $A=A(\alpha, \beta, \gamma)$ be a Noetherian down-up algebra that is a Hopf algebra; then $\alpha+\beta=1$. If $\gamma=0$ then $(\alpha, \beta)=(2,-1)$ and, as algebras, $A \cong U(\mathfrak{h})$, the universal enveloping algebra of the three dimensional Heisenberg Lie algebra. If $\gamma \neq 0$ then $-\beta$ is not an nth root of unity for $n \geq 3$.

The class of down-up algebras with $\alpha+\beta=1$ and $\gamma \neq 0$ has been studied previously. In $[\mathbf{C M}]$ this class of algebras has been parameterized as $A_{-\beta}=A(1+\beta,-\beta, 1)$, and in $[\mathbf{C M}] 5.4$ it is shown that the down-up algebra $A_{-\beta}$ has the property that all finite dimensional modules are completely reducible if and only if $-\beta$ is a root of unity or $-\beta$ is not a root of the polynomials

$$
f_{(n, m)}(x)=n\left(x^{m}-1\right)-m\left(x^{n}-1\right)
$$

for any $m \neq n$; hence all but possibly countably many algebras $A_{-\beta}$ have all finite dimensional modules completely reducible. DeConcini and Procesi have given sufficient conditions (see Theorem 3.2) for a Hopf algebra to have the property that all finite dimensional modules are completely reducible; these conditions do not apply to the algebras $A_{-\beta}$ when $-\beta$ is not a root of unity, since in that case the center of $A$ is $k$.

Our main theorem leaves unanswered the question of whether a Noetherian downup algebra is a Hopf algebra in two cases: (1). $A(0,1,1)$, and (2). $A(1+\beta,-\beta, 1)$, when $-\beta$ is not a root of unity. In case (1) the down-up algebra $A(0,1,1)$ is isomorphic to the enveloping algebra of the Lie superalgebra $\mathfrak{o s p}(1,2)$ and hence we know it has a graded Hopf structure, but we do not know if it has a Hopf structure.

In the third section we consider some localizations of down-up algebras that were considered by D. Jordan $[\mathbf{J}]$, and we use the techniques developed for down-up algebras and a theorem of DeConcini and Procesi to show that some of these localizations of down-up algebras are not Hopf algebras.

Throughout this paper let $H$ be a $k$-algebra which is a bialgebra with coproduct $\Delta$ and counit $\epsilon$. When $H$ is a Hopf algebra we denote the antipode by $S$. We begin with a general result.

Lemma 1.2. Let $H$ be a bialgebra with coproduct $\Delta$, and let

$$
I=\bigcap\{J: J \text { is an ideal of } H \text { and } H / J \text { is commutative }\}
$$

be the smallest ideal $I$ of $H$ such that $H / I$ is commutative. Then $I$ is a bi-ideal of $H$. If $H$ is a Hopf algebra with antipode $S$, then $I$ is a Hopf ideal of $H$.

Proof. Consider the natural map

$$
f: H \otimes H \rightarrow \frac{H}{I} \otimes \frac{H}{I}
$$

and let $K$ be the kernel of the composition of maps:

$$
H \stackrel{\Delta}{\rightarrow} H \otimes H \xrightarrow{f} \frac{H}{I} \otimes \frac{H}{I} \cong \frac{H \otimes H}{I \otimes H+H \otimes I} .
$$

Then $K=\Delta^{-1}(I \otimes H+H \otimes I)$. Since $H / K$ is a commutative ring, $I \subseteq K$; since $H / \operatorname{ker} \epsilon$ is a commutative ring, $I \subseteq \operatorname{ker} \epsilon$, and hence $I$ is a bi-ideal. Now set $J=\{a \in H \mid S(a) \in I\}$. Then $a+J \rightarrow S(a)+I$ defines an injective ring homomorphism $(H / J)^{\mathrm{op}} \rightarrow H / I$. Since $H / I$ is commutative, so is $H / J$ and we have $I \subseteq J$. Thus $S(I) \subseteq I$.

Throughout this paper we will denote by $I$ the ideal described in Lemma 1.2. For down-up algebras the ideal $I$ was described in $[\mathbf{C M}] 4.2$, the analysis depending on the four cases that were first described by Benkart and Roby in considering the isomorphism problem for down-up algebras. We will prove the main theorem in each of these four cases in the indicated paragraph of section 2 :
Case (a): $\gamma=0, \alpha+\beta=1$ (section 2.7),
Case (b): $\gamma=0, \alpha+\beta \neq 1$ (section 2.6),
Case (c): $\gamma \neq 0, \alpha+\beta \neq 1$ (section 2.5), and
Case (d): $\gamma \neq 0, \alpha+\beta=1$ (section 2.11).
This work was begun when both authors were members of the Mathematical Sciences Research Institute; we appreciate the institute's hospitality.

## 2 Proof of Main Theorem

In this section we will prove Theorem 1.1.
2.1. Recall the standard actions by which the tensor product $V \otimes_{k} W$ and $\operatorname{Hom}_{k}(V, W)$ of (left) $H$-modules $V$ and $W$ become $H$-modules:

$$
h \cdot(v \otimes w)=\sum_{(h)} h_{(1)} \cdot v \otimes h_{(2)} \cdot w, \quad(h \cdot f)(v)=\sum_{(h)} h_{(1)}\left[f\left(S\left(h_{(2)}\right) \cdot v\right)\right] .
$$

It is a standard fact $([\mathbf{K}]$ Proposition III.5.2) that for any (left) $H$ modules $V$ and $W$ the natural map $\theta: V \otimes_{k} W^{*} \rightarrow \operatorname{Hom}_{k}(W, V)$ given by $\theta(v \otimes f)(w)=f(w) v$, is an $H-$ module map, and when $V$ and $W$ are finite dimensional it is bijective. Furthermore, if $k=H /$ ker $\epsilon$ is the trivial module, then the "coevaluation map" $\delta: k \rightarrow V \otimes_{k} V^{*}$ given by $\delta(\alpha)=\sum_{i} \alpha v_{i} \otimes_{k} w_{i}$ (where $\left\{v_{i}\right\}$ is a basis of $V$ and $\left\{w_{i}\right\}$ is the corresponding dual basis of $V^{*}$ ) is an $H$-module homomorphism that is a monomorphism (see $[\mathbf{K}]$, Proposition III.5.3(b) or [BG] Proposition 1.6); when $V$ is a module of dimension one, it is an isomorphism since $V \otimes_{k} V^{*}$ has dimension one.

We will use the fact, following from the coassociative property of a Hopf algebra, that $(U \otimes V) \otimes W \cong U \otimes(V \otimes W)$ as $H$-modules.

We will also use the following homological property of Hopf algebras:
Proposition 2.2. ([BG] Proposition 1.3.) If $H$ is a Hopf algebra, then for all left $H$-modules $W, V$, and $X$ and all $i$ we have

$$
\operatorname{Ext}_{H}^{i}\left(W \otimes_{k} V, X\right) \cong \operatorname{Ext}_{H}^{i}\left(W, \operatorname{Hom}_{k}(V, X)\right)
$$

as $H$-modules.
It follows from this proposition that (two-sided) ideals of $H$ having codimension 1 have the following uniform homological behavior.

Proposition 2.3. Let $H$ be a Hopf algebra and let $M_{g}$ and $M_{h}$ be two ideals of codimension 1 and let $V_{g}=H / M_{g}$ and $V_{h}=H / M_{h}$ be the corresponding modules of dimension one. Then for all $i$ we have $\operatorname{Ext}_{H}^{i}\left(V_{g}, V_{g}\right) \cong \operatorname{Ext}_{H}^{i}\left(V_{h}, V_{h}\right)$ as $H$-modules.

Proof. Recall that in a Hopf algebra $H$ the set of algebra homomorphisms $\operatorname{Alghom}(H, k)$ is the set of grouplike elements in the Hopf dual $H^{\circ}$, and that the set of grouplike elements forms a group. Thus the set of isomorphism classes of one dimensional modules forms a group. Let $X=V_{g}, W=V_{h}$, and choose $V$ so that $W \otimes_{k} V \cong X$. Since $H$ is a Hopf algebra by Proposition 2.2 we have the following isomorphism (as $H$-modules):

$$
\operatorname{Ext}_{H}^{i}\left(V_{g}, V_{g}\right) \cong \operatorname{Ext}_{H}^{i}\left(V_{h}, \operatorname{Hom}_{k}\left(V, V_{g}\right)\right)
$$

and by 2.1 and coassociativity we have

$$
\begin{gathered}
\operatorname{Hom}_{k}\left(V, V_{g}\right) \cong \operatorname{Hom}_{k}\left(V, V_{h} \otimes_{k} V\right) \cong\left(V_{h} \otimes_{k} V\right) \otimes_{k} V^{*} \\
\cong V_{h} \otimes_{k}\left(V \otimes_{k} V^{*}\right) \cong V_{h} \otimes_{k} k \cong V_{h},
\end{gathered}
$$

giving the result.

Corollary 2.4. If $R$ is an algebra having both an idempotent ideal $M_{h}$ of codimension 1 and an ideal $M_{g}$ of codimension 1 which is not idempotent, then $R$ is not a Hopf algebra.

Proof. Let $V_{g}=R / M_{g}$ and $V_{h}=R / M_{h}$. Since $M_{h}$ is idempotent $\operatorname{Ext}_{R}^{1}\left(V_{h}, V_{h}\right)=$ 0 , and since $M_{g}$ is not idempotent $\operatorname{Ext}_{R}^{1}\left(V_{g}, V_{g}\right) \neq 0$. Hence $R$ cannot be a Hopf algebra.
2.5. We now prove Theorem 1.1 in case (c), where $A=A(\alpha, \beta, \gamma)$ is a down-up algebra with $\gamma \neq 0$ and $\alpha+\beta \neq 1$. We will show that in this case $A$ is never a Hopf algebra. By $[\mathbf{C M}] 4.4, A$ has an ideal $M_{h}$ of codimension 1 which is idempotent (namely $M_{h}=\langle u, d\rangle$ ), and an ideal $M_{g}$ of codimension 1 which is not idempotent (in fact $A$ has a one parameter family of such ideals). Hence by Corollary $2.4, A$ is not a Hopf algebra.
2.6. We now prove Theorem 1.1 in case (b), where $\gamma=0$ and $\alpha+\beta \neq 1$; we will show that $A$ is never a Hopf algebra. By $[\mathbf{C M}] 4.2(\mathrm{~b}) \bar{A}=A / I \cong k[a, b] /\left(a^{2} b, a b^{2}\right)$ where $a, b$ are the images of $d, u$, respectively. If $A$ were a Hopf algebra then Proposition 1.2 would imply that $\bar{A}$ is a Hopf algebra. One family of ideals of $\bar{A}$ with codimension 1 is $M_{t}=(a-t, b)$ for $t \in k$, giving the family $V_{t}=\bar{A} / M_{t}$ of $\bar{A}$-modules with dimension 1. We claim that the (vector space) $\operatorname{dim} \operatorname{Ext} \frac{1}{A}\left(V_{t}, V_{t}\right)$ is 2 when $t=0$ and 1 when $t \neq 0$. It then will follow from Proposition 2.3 that $\bar{A}$ is not a Hopf algebra. For the maximal ideal $M_{t}$ in the commutative $\operatorname{ring} \bar{A}$ we have $\operatorname{dim} \operatorname{Ext} \frac{1}{A}\left(\bar{A} / M_{t}, \bar{A} / M_{t}\right)=$ $\operatorname{dim} M_{t} / M_{t}^{2}$, and this easily gives the result.
2.7. We now prove Theorem 1.1 in case (a), where $\gamma=0$ and $\alpha+\beta=1$. We will show in 2.10 that if $A$ is a Hopf algebra then $\alpha=2$ and $\beta=-1$, and hence as algebras $A \cong U(\mathfrak{h})$, the enveloping algebra of the 3 dimensional Heisenberg Lie algebra.

By $[\mathbf{C M}] 4.2(\mathrm{a}) I=\omega A$, where $\omega=(d u-u d)$ is a normal element of $A$, and $\bar{A}=A / I \cong K[a, b]$, a commutative polynomial ring. If $A$ is a Hopf algebra, then by Proposition 1.2, $\bar{A}=A / I$ is a Hopf algebra. Next we consider the induced structure of $\bar{A}$.

Lemma 2.8. Suppose that $H$ is a finitely generated commutative Hopf algebra over an algebraically closed field $k$ of characteristic zero such that:

1. $H$ is a domain,
2. H has Krull dimension 2, and
3. the only units of $H$ belong to $k$.

Then (as a Hopf algebra) $H \cong \mathcal{O}(\mathrm{G})$, the coordinate ring of the algebraic group $\mathrm{G}=G_{a} \oplus G_{a}$, where $G_{a}$ is the additive group of the field $k$.

Proof. Since $H$ is a finitely generated reduced commutative $k$-algebra, by [A], p. 163, the pair $\left(G\left(H^{\circ}\right), H\right)$ is an affine algebraic group in the sense of $[\mathbf{H}]$; hence $H \cong \mathcal{O}(\mathrm{G})$, the coordinate ring of the algebraic group G . Note first that G is connected by 1., next by 2 . and [Bo], Corollary $11.6, \mathrm{G}$ is a solvable group, and finally by [Bo], Theorem 10.6, $\mathrm{G}=T \cdot G_{u}$ is the semidirect product of the unipotent radical $G_{u}$
by the maximal torus $T$. However 3 . implies that G has no nonconstant characters, so $\mathrm{G}=G_{u}$. By $[\mathbf{H}]$ Theorem XVI.4.2 Lie G is nilpotent and it suffices to show that Lie G is abelian, but this follows [Ja], p. 11.

Remark: The assumption that characteristic $k=0$ is essential in the lemma above, see $[\mathbf{H}]$ p.92, exercise 2 .

In the proof of case (a) we will use the following Lemma, which follows easily by induction.

Lemma 2.9. If $J$ is a biideal in a bialgebra $H$ then the $J$-adic filtration is a bialgebra filtration on $H$; that is

$$
\Delta J^{n} \subseteq \sum_{i=0}^{n} J^{i} \otimes J^{n-i}
$$

If $J$ is a Hopf ideal then $\left\{J^{n}\right\}$ is a Hopf algebra filtration.
2.10. To conclude the proof of Theorem 1.1 in case (a), let $A$ be a down-up algebra with $\gamma=0$ and $\alpha+\beta=1$. Since $\gamma=0, A=\oplus_{m \geq 0} A(m)$ is a graded algebra, where $A(1)=\operatorname{span}\{d, u\}$. Set $J=(d, u)$ then $J^{n}=\oplus_{m \geq n}^{\geq} A(m)$. We have $\bar{A}=A / I \cong \mathcal{O}(\mathrm{G})$, with G as in Lemma 2.8 , where $\bar{A}$ is a Hopf algebra with the usual maps $\Delta, \epsilon$, and $S$ for $\mathcal{O}(\mathrm{G})$. We will show that $\beta=-1$. Since $\bar{A}=A / I \cong \mathcal{O}(\mathrm{G})$

$$
\Delta d=d \otimes 1+1 \otimes d \bmod (I \otimes A+A \otimes I)
$$

Since $I \subseteq J$ it follows that $\Delta(d) \in J \otimes A+A \otimes J$, and similarly $\Delta(u) \in J \otimes A+A \otimes J$. In $A / I$ we have $\epsilon(d)=0 \bmod I($ similarly for $u)$, so $J / I$ is a biideal of $A / I$. We have $S(d)=-d \bmod I($ and similarly for $u)$ so $S(J) \subseteq J+I=J$ and so $J / I$ is a Hopf ideal of $A / I$. Then by Lemma 2.9 and $[\mathbf{S w}]$ (page 238, Exercise 3) the associated graded algebra $\operatorname{gr}(A)=\oplus J^{n} / J^{n+1}$ is a Hopf algebra. Note that as $k$-algebras $A \cong \operatorname{gr} A$. The images of $d$ and $u$ are primitive in $\operatorname{gr}(A)$, so it follows that $\omega=[d, u]$ is also primitive. Since $d \omega=-\beta \omega d$, expanding $\Delta(d) \Delta(\omega)=-\beta \Delta(\omega) \Delta(d)$ and comparing terms shows that $\beta=-1$ and then $\alpha=2$, and hence $A \cong U(\mathfrak{h})$, the enveloping algebra of the 3 dimensional Heisenberg Lie algebra.

Remark: Since the isomorphism $A \cong$ grA in the proof is an isomorphism only of algebras, it does not follow that $A \cong U(\mathfrak{h})$ as bialgebras. In fact there are nonstandard bialgebra structures on $U(\mathfrak{h})$ by $[\mathbf{C G}]$.
2.11. We now prove Theorem 1.1 in case (d), when $\gamma \neq 0$ and $\alpha+\beta=1$. We will show that any $A_{-\beta}=A(1+\beta,-\beta, 1)$ that is a Hopf algebra and has the property
that all finite dimensional modules are completely reducible must also have the same Clebsch-Gordan decomposition for tensor products of finite dimensional modules as $U\left(\mathfrak{s l}_{2}\right)$. A corollary of this decomposition is that if $-\beta$ is an nth root of unity for $n \geq 3$ then $A_{-\beta}$ is not a Hopf algebra.

Let $H$ be a Hopf algebra with at most one irreducible module $V_{n}$ of dimension $n$. Since there is only one irreducible module in each dimension, each finite dimensional module is self dual. Furthermore, for irreducible finite dimensional modules $V$ and $W$ we have $\left[V \otimes_{k} W: V_{1}\right]$ is 0 when $V \neq W$ and 1 when $V \cong W$.

Theorem 2.12. Let $H$ be a Hopf algebra with exactly one irreducible module $V_{n}$ of dimension $n$ for $n=1$ and $n=2$, and at most one irreducible module $V_{n}$ of dimension $n$ for $n \geq 3$. If all finite dimensional $H$-modules are completely reducible then for all $n \geq m \geq 1$ we have

$$
V_{m} \otimes_{k} V_{n} \cong \oplus_{\ell=1}^{m} V_{n+m-2 \ell+1} \cong V_{n} \otimes_{k} V_{m}
$$

In particular $H$ has exactly one irreducible module of every dimension $\geq 1$.
Proof. The result is true for $m=1$, so consider the case where $m=2$, where we will induct on $n \geq 2$. Since the 4 dimensional module $V_{2} \otimes_{k} V_{2}$ has exactly one $V_{1}$ in its decomposition it must split as claimed. Inductively we fix $n \geq 2$ and assume that $V_{2} \otimes_{k} V_{i} \cong V_{i-1} \oplus V_{i+1} \cong V_{i} \otimes_{k} V_{2}$ for all $1 \leq i \leq n$, and we will prove the claimed decomposition for $V_{2} \otimes_{k} V_{n+1}$ (a similar argument works for $V_{n+1} \otimes_{k} V_{2}$ ). Clearly $V_{2} \otimes_{k} V_{n+1}$ has no copy of $V_{1}$, and for $2 \leq i \leq n$

$$
\begin{gathered}
V_{i} \otimes_{k}\left(V_{2} \otimes_{k} V_{n+1}\right) \cong\left(V_{i} \otimes_{k} V_{2}\right) \otimes_{k} V_{n+1} \cong\left(V_{i-1} \oplus V_{i+1}\right) \otimes_{k} V_{n+1} \\
\cong\left(V_{i-1} \otimes_{k} V_{n+1}\right) \oplus\left(V_{i+1} \otimes_{k} V_{n+1}\right) .
\end{gathered}
$$

Hence for $2 \leq i \leq n-1$ the module $V_{i} \otimes_{k}\left(V_{2} \otimes_{k} V_{n+1}\right)$ has no copy of $V_{1}$ in its decomposition, so $V_{2} \otimes_{k} V_{n+1}$ has no copy of $V_{i}$. Furthermore applying the above when $i=n$ shows that $V_{2} \otimes_{k} V_{n+1}$ has a unique copy of $V_{n}$. Since $n \geq 2$ the only way to write $V_{2} \otimes_{k} V_{n+1}$ as a sum involving $V_{n}$ and no modules of dimension $i$ for $1 \leq i \leq n-1$ is as $V_{2} \otimes_{k} V_{n+1}=V_{n} \oplus V_{n+2}$. This establishes the result for $m=2$ and any $n \geq 2$. It follows that $H$ has exactly one irreducible module of every dimension $\geq 1$.

Next we induct on $m$. Hence we assume

$$
V_{p} \otimes_{k} V_{n}=\oplus_{\ell=1}^{p} V_{n+p-2 \ell+1} \text { for } 2 \leq p<m \leq n,
$$

so that $V_{m-1} \otimes_{k} V_{n}=\oplus_{\ell=1}^{m-1} V_{n+m-2 \ell}$. It follows that

$$
\left(V_{2} \otimes_{k} V_{m-1}\right) \otimes_{k} V_{n}=V_{2} \otimes_{k}\left(V_{m-1} \otimes_{k} V_{n}\right)=\oplus_{\ell=1}^{m-1} V_{2} \otimes_{k} V_{n+m-2 \ell},
$$

so by the case $n=2$

$$
\left(V_{m-2} \otimes_{k} V_{n}\right) \oplus\left(V_{m} \otimes_{k} V_{n}\right)=\oplus_{\ell=1}^{m-1} V_{2} \otimes_{k} V_{n+m-2 \ell}=\oplus_{\ell=1}^{m-1} V_{n+m-2 \ell-1} \oplus_{\ell=1}^{m-1} V_{n+m-2 \ell+1}
$$

Since by induction

$$
V_{m-2} \otimes_{k} V_{n}=\oplus_{\ell=1}^{m-2} V_{n+m-2 \ell-1},
$$

it follows that

$$
V_{m} \otimes_{k} V_{n}=V_{n-m+1} \oplus_{\ell=1}^{m-1} V_{n+m-2 \ell+1}=\oplus_{\ell=1}^{m} V_{n+m-2 \ell+1},
$$

proving the result.
Corollary 2.13. If $-\beta$ is a primitive $n$th root of unity for $n \geq 3$, then $A=$ $A(1+\beta,-\beta, 1)$ is not a Hopf algebra.

Proof. By Theorem 2.12 if $A$ is a Hopf algebra then $A$ must have irreducible modules of each dimension. However it was shown in $[\mathbf{C M}] 2.5$ and $[\mathbf{J}]$ Proposition 5.3 that for $-\beta$ a primitive $n$th root of unity $A$ has no irreducible modules of dimension $n$.

We note that the missing modules of dimensions that are multiples of $n$ create no contradiction when $n=2$ and $A_{-1} \cong U(\mathfrak{o s p}(1,2))$. The enveloping algebra $U(\mathfrak{o s p}(1,2))$ has a graded Hopf structure, but no Hopf structure that we know.

## 3 Localizations

By $[\mathbf{K M P}]$ a down-up algebra is a generalized Weyl algebra of the form $A=k[x, y](x ; \sigma)$, for $x=u d$ and $y=d u$. David Jordan has noted that when $k[x, y]$ contains an eigenvector $t$ for $\sigma$ then $t$ will be a normal element in $A$, and it is natural to consider the localization $S$ of $A$ at the powers of $t$. There are situations where $S$ has all finite dimensional modules completely reducible, while $A$ does not. Examples of such algebras include the algebra defined by Woronowicz $[\mathbf{W}]$ and the $(q, r)$-differential algebras for $q$ not a root of 1 and $r \neq 0$, which initiated the study of down-up algebras (see $[\mathbf{B R}]$, and $[\mathbf{J}]$ Example 5.7). In these cases the localized rings $S$ have finite dimensional representations similar to the down-up algebras $A$ of case (d). It seems a natural question to determine if these localizations may produce some Hopf algebras. S. P. Smith [S] notes (p 170) that of certain deformations of $U\left(\mathfrak{s l}_{2}\right)$, the localization of the deformation of Woronowicz, $W_{v}\left[K^{-4}\right]$ in his notation, is the most like $U\left(\mathfrak{s l}_{2}\right)$ in terms of finite dimensional simple modules, although "it does not have a Hopf structure". In this section we will use the techniques described in the previous section to present cases in which we can prove that the ring $S$ is not a Hopf algebra; in Proposition 3.1 we shall prove that in cases (a), (b), and under certain conditions case (c), $S$ is not a

Hopf algebra. In Proposition 3.3 we shall use a theorem of DeConcini and Procesi to present other conditions in case (c) when we can prove that $S$ is not a Hopf algebra.

In $[\mathbf{J}] \S 5.4$ Jordan studied the representation theory of $S$ in the cases to which his earlier papers applied. Let $r$ and $s$ be roots of the equation $x^{2}-\alpha x-\beta=0$; note that $r s=-\beta$ and $r+s=\alpha$. Jordan's results are phased in terms of parameters $\mu_{1}=1 / r$, and $\mu_{2}=1 / s$. Following Jordan, let $H^{*}$ denote the multiplicative group generated by $r$ and $s$ and $\tau=s / r$.

Let $A / I$ (resp. $S / J$ ) be the largest commutative image of $A$ (resp. $S$ ). Using the techniques of the first section we next classify the one-dimensional modules of $S$ in cases (a), (b), and (c) (extending results of Jordan). We obtain as a corollary the fact that in cases (a) and (b) $S$ is not a Hopf algebra, and in case (c), if $S$ is a Hopf algebra, the counit $\epsilon$ must have kernel $\langle u, d\rangle$. By $[\mathbf{J}]$ Proposition 5.5 (iv) and (v). in case (c) when the rank of $H^{*}$ is at least one and $\mu_{2}$ is not a root of unity, there is an arithmetic condition that determines if all finite dimensional $S$-modules are semisimple: there are no positive integers $e$ and $d$ with $d \neq e, \tau^{d} \neq 1, \tau^{e} \neq 1$ and

$$
\begin{equation*}
\left(\mu_{1}^{e}-1\right)\left(\tau^{d}-1\right)=\left(\tau^{e}-1\right)\left(\mu_{1}^{d}-1\right) \tag{1}
\end{equation*}
$$

Proposition 3.1. Let $A$ be a down-up algebra, and let $s$ and $r$ be the roots of the equation $x^{2}-\alpha x-\beta=0$. In cases (a) and (b), there is an eigenvector $t$ in $k[x, y]$ so that the localization $S$ of $A$ at the powers of $t$ has no one-dimensional modules, and so $S$ is not a Hopf algebra. In case (c) there is an eigenvector t so that $S$ has exactly one one-dimensional module; if $\tau=s / r$ is an nth root of unity for $n \geq 3, s$ is not a root of unity, and if all finite dimensional modules are semisimple then $S$ is not a Hopf algebra.

Proof. In case (a), where $\alpha+\beta=1$ and $\gamma=0$, we set $t=x-y$. Then $I=(t)$, so inverting $t$ destroys $I$, so all one dimensional modules. In fact, when $\beta$ is not a root of unity, $S$ has no finite dimensional modules by $[\mathbf{J}]$ Proposition 5.9. Since $S$ has no one dimensional modules, it cannot be a Hopf algebra.

In case (b), where $\alpha+\beta \neq 1$ and $\gamma=0$, then both $r$ and $s$ are not 1 , and we may set $t=s x-y$ for one of the roots $s$ (it is possible that $s=r$ ). Then

$$
\frac{S}{J}=\frac{A}{I}\left[\bar{t}^{-1}\right]=\frac{k[a, b]}{\left(a^{2} b, a b^{2}\right)}[(s-1) a b]^{-1}=0
$$

since $(a b)^{2} \in I$ and $a b$ is a unit in $S / J$. Hence there are no one dimensional modules (c.f. [J] Propositions 5.5 (ii) and (iii) and 5.10 (ii) when the rank of $H^{*}$ is at least one), and $S$ cannot be a Hopf algebra.

In case ( c ), where $\alpha+\beta \neq 1$ and $\gamma=1$, we may chose $t=-r(s-1) x+(s-1) y+1$, an eigenvector for $s$, and abelianizing (where we take $a$ (resp. $b$ ) to be the image of
$d$ (resp. $u$ ) in $A / I)$ we find that the image of $t$ in $A / I$ is $(1-r)(s-1) a b+1$. Since in $A / I$ we have the following relations (as in $[\mathbf{C M}] 4.2$ )

$$
\begin{gathered}
a((r-1)(s-1) a b-1)=0, \text { and } \\
b((r-1)(s-1) a b-1)=0,
\end{gathered}
$$

it follows that $a=b=0$ in $S / J$. Therefore $S / J \cong k$ so there is a unique onedimensional module (c.f. [J] Proposition 5.5 in the case that the rank of $H^{*}$ is at least 1).

In case (c) with the further assumptions given, by [J] Proposition 5.5 (v) $S$ has exactly one $d$-dimensional simple module for all $d \geq 1$, unless $d$ is a multiple of $n$, in which case there is no $d$-dimensional simple module. Hence $S$ is not a Hopf algebra by Theorem 2.12.

To consider case (c) under different conditions we will use a result of DeConcini and Procesi. Let $H$ be a Hopf algebra with counit $\epsilon: H \rightarrow k$, where $k$ is a field in the center of $H$. Let $\omega$ be the kernel of $\epsilon$, and let $H / \omega \cong k$ be the trivial module. Hence $\omega$ is an ideal of $H$ of finite codimension; when all finite dimensional representations are semisimple, ideals of finite codimension must be idempotent.

Theorem 3.2. (DeConcini and Procesi) Let $H$ be a Hopf algebra satisfying the following two conditions:

1. The kernel $\omega=\operatorname{Ker}(\epsilon)$ of the counit satisfies $\omega^{2}=\omega$.
2. For each non-trivial finite dimensional irreducible representation, the central character is different from the trivial character.

Then all finite dimensional $H$-modules are semisimple.
Proof. This follows by making some minor changes to the proof in [DP], p.40-41.

Proposition 3.3. Let $A$ be a down-up algebra of type (c), and let $s$ and $r$ be the roots of the equation $x^{2}-\alpha x-\beta=0$. Assume that $r$ is a primitive nth root of unity and $s$ is not a root of unity. Let $S$ be the localization of $A$ at the powers of the eigenvector $t=-r(s-1) x+(s-1) y+1$ for $s$. If for all $d \geq 2$ and all $1 \leq j \leq n$ the equation

$$
\begin{equation*}
\left(s^{d}-r^{d}\right) \frac{\left(r^{j}-1\right)}{(r-1)}=\frac{\left(r^{d}-1\right)}{(r-1)}-\frac{\left(s^{d}-1\right)}{(s-1)} \tag{2}
\end{equation*}
$$

does not hold, then $S$ is not a Hopf algebra.

Proof. Since $r$ is a root of unity there are integers $d \neq e$ where both sides of equation (1) are 0 , so $S$ has finite dimensional modules that are not completely reducible. Without loss of generality, take $\gamma=1$. Since there is only one simple module of dimension 1, if the algebra $S$ is a Hopf algebra then this module must be the trivial module; furthermore $J=\langle u, d\rangle$ is an idempotent ideal when $\gamma \neq 0$. By [J] Proposition 5.5 (i) $S$ has at most one simple module in each dimension.

Let $z=-s(r-1) x+(r-1) y+1$, an eigenvector for $r$. One can check that $z^{n}$ is central in $A$ and $\epsilon\left(z^{n}\right)=1$; by $[\mathbf{H i}]$ or $[\mathbf{Z}] z^{n}$ generates the center of $A$. The ring $S$ has a $d$ dimensional simple if and only if $d$ is the minimal positive integer such that $\lambda_{d-1}=0$. It follows from our appendix (6) that $\lambda_{d-1}=0$ if $\lambda$ satisfies

$$
\begin{equation*}
\lambda\left(s^{d}-r^{d}\right)=\frac{\left(r^{d}-1\right)}{(r-1)}-\frac{\left(s^{d}-1\right)}{(s-1)} \tag{3}
\end{equation*}
$$

To determine the central character induced by the element $z^{n}$ on the $d$ dimensional simple module $L(\lambda)$ (if it exists), we can use the action of $z^{n}$ on the generator $v_{0} \in L(\lambda)$. The vector $z^{n}$ acts on $v_{0}$ as the scalar $\left.((r-1) \lambda+1)\right)^{n}$. The value of the central character at $z^{n}$ is the same as the trivial character if $\left.((r-1) \lambda+1)\right)^{n}=1$, i.e. if $(r-1) \lambda+1=r^{j}$ for some $j$. Solving this equation for $\lambda$, and substituting into equation (3) we get the condition (2). If this equation is not satisfied for all $d \geq 2$ and all $j$, then for each non-trivial finite dimensional irreducible representation, the central character is different from the trivial character. If $S$ were a Hopf algebra it would follow from Theorem 3.2 that all finite dimensional representations are completely reducible; since this is not the case, $S$ cannot be a Hopf algebra.

It can be shown, for example, that the conditions above hold when $r=i$ and $s=2($ and hence $\alpha=i+2$ and $\beta=-2 i)$.

There remain cases where the techniques we have developed do not apply, and we are unable to determine if $S$ has a Hopf structure. This concludes the analysis of the localizations considered by Jordan in $[\mathbf{J}]$.

## Appendix

Here we give proofs, which are shorter than those available in the literature, of some results that we use.

Fix $\lambda \in k$ and consider the recurrence relation

$$
\begin{equation*}
\lambda_{n}=\alpha \lambda_{n-1}+\beta \lambda_{n-2}+\gamma \tag{4}
\end{equation*}
$$

where $\lambda_{-1}=0, \lambda_{0}=\lambda$. This equation arises in connection with the structure of Verma modules $[\mathbf{B R}]$. The explicit solutions to (4) are given in $[\mathbf{B R}]$ Proposition 2.12. However often we are interested in knowing merely when $\lambda_{n-1}=0$, and this can be determined without explicitly solving (4).

Let $\left(\begin{array}{ll}0 & \beta \\ 1 & \alpha\end{array}\right), v_{-1}=(0, \gamma)$ and $v_{n}=\left(\lambda_{n-1}, \lambda_{n}\right)$ for $n \geq 0$. We can write (4) in the form

$$
v_{n}=v_{n-1} \sigma+v_{-1} .
$$

Now suppose that $P$ is an invertible matrix and $D=P^{-1} \sigma P$. We set $w_{n}=v_{n} P$. Then clearly $w_{n}=w_{n-1} D+v_{-1} P$ and hence by induction

$$
\begin{equation*}
w_{n}=w_{0} D^{n}+v_{-1} P\left(I+D+\cdots+D^{n-1}\right) \tag{5}
\end{equation*}
$$

For simplicity we assume that $\beta \neq 0$ and that $\sigma$ is diagonalizable. Here we will consider only the particular cases that arose in our analysis of cases (c) and (d). We leave the other cases to the interested reader.

Case (c) when $r \neq s$ : This case occurs in Case 1 of $[\mathbf{C M}]$ and in Case A of $[\mathbf{J}]$. Assume $\gamma=1 \neq \alpha+\beta$, and that $\sigma$ has distinct eigenvalues $r$ and $s$. Notice that in this case neither eigenvalue is 1 . To diagonalize $\sigma$ we take

$$
P=\left(\begin{array}{cc}
\beta / r & \beta / s \\
1 & 1
\end{array}\right) .
$$

By (5)

$$
w_{n}=\left(\lambda r^{n}, \lambda s^{n}\right)+\left(1+r+\cdots+r^{n-1}, 1+s+\cdots+s^{n-1}\right)
$$

On the other hand $w_{n}=v_{n} P=\lambda_{n-1}(\beta / r, \beta / s)+\lambda_{n}(1,1)$. Thus $\lambda_{n-1}=0$ if and only if $w_{n}$ has equal entries, and this happens if and only if

$$
\lambda\left(s^{n}-r^{n}\right)=\left(r^{n-1}+r^{n-2}+\cdots+r+1\right)-\left(s^{n-1}+s^{n-2}+\cdots+s+1\right)
$$

this means that

$$
\begin{equation*}
\lambda\left(s^{n}-r^{n}\right)=\left(\frac{r^{n}-1}{r-1}\right)-\left(\frac{s^{n}-1}{s-1}\right) \tag{6}
\end{equation*}
$$

Case (d) when $r \neq s$ : Assume that $\gamma=1=\alpha+\beta$ and that $\sigma$ has eigenvalues $r=$ $\eta=-\beta \neq 1$ and $s=1$. We take

$$
P=\left(\begin{array}{cc}
-1 & -\eta \\
1 & 1
\end{array}\right)
$$

Then by (5) we have

$$
w_{n}=\lambda\left(\eta^{n}, 1\right)+\left(\frac{\eta^{n}-1}{\eta-1}, n\right)
$$

As before we see that $\lambda_{n-1}=0$ if and only if

$$
\lambda\left(\eta^{n}-1\right)=n-\frac{\left(\eta^{n}-1\right)}{(\eta-1)}
$$

Thus $\eta^{n}=1$ implies that $n=0$, and if $n>0$ then $\lambda_{n-1}=0$ if and only if

$$
\lambda(\eta-1)=-\left(1-n\left(\sum_{i=0}^{n-1} \eta^{i}\right)^{-1}\right)
$$

This condition is $[\mathbf{C M}]$ Lemma 2.5. If $\lambda_{m-1}=\lambda_{n-1}=0$ for $0<n<m$ we easily obtain

$$
n\left(\eta^{m}-1\right)=m\left(\eta^{n}-1\right)
$$

c.f. $[\mathbf{C M}] 5.4$ and $[\mathbf{J}]$ equation (28).

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