Finite dimensional representations of invariant differential operators

Ian M. Musson*
Department of Mathematical Sciences
University of Wisconsin-Milwaukee, USA
E-mail: musson@uwm.edu

Sonia L. Rueda
Departamento de Matemáticas. E.T.S. Arquitectura
Universidad Politécnica de Madrid
Madrid, Spain
E-mail:srueda@aq.upm.es

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Abstract

Let k be an algebraically closed field of characteristic 0, $Y = k^r \times (k^\times)^s$ and let G be an algebraic torus acting diagonally on the ring of algebraic differential operators $\mathcal{D}(Y)$. We give necessary and sufficient conditions for $\mathcal{D}(Y)^G$ to have enough simple finite dimensional representations, in the sense that the intersection of the kernels of all the simple finite dimensional representations is zero. As an application we show that if $K \longrightarrow GL(V)$ is a representation of a reductive group K and if zero is not a weight of a maximal torus of K on V, then $\mathcal{D}(V)^K$ has enough finite dimensional representations. We also construct examples of FCR-algebras with any GK dimension ≥ 3.

1 Introduction

For a variety Y over k, we denote the ring of regular functions on Y by $\mathcal{O}(Y)$ and the ring of differential operators by $\mathcal{D}(Y)$. Recently there has been much interest in the study of the invariant ring $\mathcal{D}(Y)^G$ when G is a reductive group acting on a smooth affine variety Y, see for example [9], [11], [13], [14], [15], [17]. In this paper our primary focus is on the case where $Y = V \times W$ for a vector space V and a torus W, and G is a torus acting diagonally on Y. There is some motivation for the study of diagonal torus actions on Y, rather than on the vector space V, coming from the structure of differential operators on toric varieties, see [12],[13].

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We say that a k-algebra R has enough finite dimensional modules (resp. enough simple finite dimensional modules) if $\cap ann_R M = 0$, where the intersection is taken over all finite dimensional (resp. simple finite dimensional) R-modules.

Proposition A *If $\mathcal{D}(Y)^G$ has a finite dimensional module, then G acts transitively on W.*

Now assume that G acts transitively on W and let H be the stabilizer in G of $w \in W$. Note that the connected component H^o of the identity in H is a torus, but we may have $H \neq H^o$. The slice representation at w is isomorphic to (H, V), see § 3. We give necessary and sufficient conditions for $\mathcal{D}(Y)^G$ to have enough finite dimensional simple modules.

Theorem B *Assume that G acts transitively on W. The following conditions are equivalent.*

1. $V^{H^o} = 0$.
2. $\mathcal{D}(V)^{H^o}$ *has enough simple finite dimensional modules.*
3. $\mathcal{D}(Y)^G$ *has a nonzero finite dimensional module.*
4. $\mathcal{D}(Y)^G$ *has enough simple finite dimensional modules.*

Set $\mathfrak{h} = Lie(H) \subseteq \mathfrak{g} = Lie(G)$. For $\lambda \in \mathfrak{g}^*$, $\mu \in \mathfrak{h}^*$ we set

$$\mathcal{B}_\lambda(Y) = \mathcal{D}(Y)^G/(\mathfrak{g} - \lambda(\mathfrak{g}))\ , \qquad \mathcal{B}_\mu(V) = \mathcal{D}(V)^H/(\mathfrak{h} - \mu(\mathfrak{h})). \tag{1}$$

Here $(\mathfrak{g} - \lambda(\mathfrak{g}))$ is the ideal generated by all elements of the form $x - \lambda(x)$, with $x \in \mathfrak{g}$, and $(\mathfrak{h} - \mu(\mathfrak{h}))$ is defined similarly. The algebra $B_\lambda(Y)$ is studied in detail in [13]. Let $i^* : \mathfrak{g}^* \longrightarrow \mathfrak{h}^*$ be the map obtained from the inclusion $i : \mathfrak{h} \longrightarrow \mathfrak{g}$.

Proposition C

1. *There is an injective algebra homomorphism $\xi : \mathcal{D}(V)^H \longrightarrow \mathcal{D}(Y)^G$.*
2. *If $\lambda \in \mathfrak{g}^*$ and $\mu = i^*(\lambda)$, the above map induces an isomorphism $\mathcal{B}_\mu(V) \cong \mathcal{B}_\lambda(Y)$.*

Note that any simple $\mathcal{D}(Y)^G$-module is a $\mathcal{B}_\lambda(Y)$-module for some $\lambda \in \mathfrak{g}^*$. So Propositions A and C reduce the study of finite dimensional simple $\mathcal{D}(Y)^G$-modules to that of finite dimensional simple $\mathcal{D}(V)^H$-modules. Some other situations in which slice representations have been used to study invariant differential operators may be found in [14] and [17].

Now suppose that G is a maximal torus in a reductive subgroup K of $\mathrm{GL}(V)$. Then $\mathcal{D}(V)^K$ is a subring of $\mathcal{D}(V)^G$, so any $\mathcal{D}(V)^G$-module is also a $\mathcal{D}(V)^K$-module. Thus we obtain as an immediate corollary to Theorem B.

Theorem D *If* 0 *is not a weight of* G *on* V, *then* $\mathcal{D}(V)^K$ *has enough finite dimensional modules.*

In general, it seems rather difficult to say much about finite dimensional $\mathcal{D}(V)^K$-modules (or more generally primitive ideals in $\mathcal{D}(V)^K$) if K^o is not a torus. As far as we are aware if K is simple, the only cases where anything is known are the case of the adjoint representation of $SL(3)$ [15], and some representations arising from classical invariant theory [9].

Here is a brief outline of the paper. First we establish some notation and express the conditions in Proposition A and Theorem B in terms of the weights of the torus action. If either the action of G on W is not transitive or if $V^{H^o} \neq 0$, we show that $\mathcal{D}(Y)^G$ contains the fixed ring $A_1^{\mathbb{F}}$ of the first Weyl algebra A_1 under the action of a finite group $\mathbb{F}$. Since $A_1^{\mathbb{F}}$ is simple and infinite dimensional over k, it has no nonzero finite dimensional modules. On the other hand, if G acts transitively on W and $V^{H^o} = 0$, we show that $\mathcal{D}(Y)^G$ has enough finite dimensional modules by first analyzing the case $\mathcal{O}(Y)^G = k$, and then using Fourier transforms to reduce to this case.

Kraft and Small [6] call an algebra R an *FCR-algebra* if R has enough finite dimensional modules and every finite dimensional R-module is completely reducible. In the last section of the paper, we combine Theorem B with results from [13] to give examples of FCR-algebras with any given integer Gelfand-Kirillov dimension ≥ 3. We denote the Gelfand-Kirillov dimension of an algebra A by GK-*dim* A, see [8] for background.

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2 Notation

2.1 Actions of Tori

We assume that $V = k^r, W = (k^\times)^s$, and $Y = V \times W \subseteq k^n$, where $n = r + s$. A *diagonal action* of a torus G on Y is an action that extends to a diagonal action on k^n. Such an action is given by an embedding of G into the group T of diagonal matrices in $GL(n)$. Set $\mathbb{X}(T) = Hom(T, k^\times)$, $\mathbb{Y}(T) = Hom(k^\times, T)$, the groups of characters and one-parameter subgroups of T, respectively. There is a natural bilinear pairing

$$(\ , \ ) : \mathbb{X}(T) \times \mathbb{Y}(T) \longrightarrow \mathbb{Z}. \tag{2}$$

defined by the requirement that

$$(a \circ b)(\lambda) = \lambda^{(a,b)} \tag{3}$$

for all $a \in \mathbb{X}(T)$, $b \in \mathbb{Y}(T)$ and $\lambda \in k^\times$. Define the bilinear pairing

$$[\ , \ ] : \mathbb{X}(G) \times \mathbb{Y}(G) \longrightarrow \mathbb{Z} \tag{4}$$

in a similar way.

Let $\psi : \mathbb{X}(T) \longrightarrow \mathbb{X}(G)$ be the restriction map. Let $K = ker\psi$ and

$$K^{\perp} = \{\chi \in \mathbb{Y}(T) | (K, \chi) = 0\}. \tag{5}$$

There is a homomorphism

$$\omega : \mathbb{Y}(G) \longrightarrow K^{\perp} \tag{6}$$

defined by

$$(a, \omega(b)) = [\psi(a), b], \tag{7}$$

for $a \in \mathbb{X}(T)$ and $b \in \mathbb{Y}(G)$. A slightly different situation is described in [12].

By introducing bases we can describe these maps using matrices. Identify G with $(k^{\times})^m$ and for $1 \leq i \leq m$ define $\nu_i \in \mathbb{X}(G)$ by $\nu_i(g) = g_i$, for $g = (g_1, \ldots, g_m) \in G$. We call $\{\nu_i\}$ the *standard basis* for $\mathbb{X}(G)$. Similarly, identifying T with $(k^{\times})^n$, we obtain the standard basis $\{e_i\}$ for $\mathbb{X}(T)$. We write $\{\nu_i^*\}$, $\{e_i^*\}$ for the dual bases of $\mathbb{Y}(G)$ and $\mathbb{Y}(T)$. For matrix computations we identify $\mathbb{X}(T)$, $\mathbb{Y}(T)$, $\mathbb{X}(G)$, $\mathbb{Y}(G)$ as $\mathbb{Z}$-modules of column vectors using these bases. Let I_t denote the $t \times t$ identity matrix. If M is an abelian group, we set $M_{\mathbb{Q}} = M \otimes_{\mathbb{Z}} \mathbb{Q}$.

For $1 \leq i \leq n$ let $\eta_i = \psi(e_i)$. There is an $m \times n$ integer matrix $L = (l_{ij})$ such that

$$\eta_i = \sum_{j=1}^{m} l_{ji} \nu_j, \tag{8}$$

for $i = 1, \ldots, n$. From (7) it follows that

$$\omega(\nu_i^*) = \sum_{j=1}^{n} l_{ij} e_j^*. \tag{9}$$

Thus the maps $\mathbb{X}(T) \longrightarrow \mathbb{X}(G)$ and $\mathbb{Y}(G) \longrightarrow \mathbb{Y}(T)$ are given by multiplication by L and the transpose of L respectively. If $\beta = (\beta_1, \ldots, \beta_n)$ belongs to the submodule R of $\mathbb{Z}^n$ spanned by the rows of L, then

$$\beta(t) = (t^{\beta_1}, t^{\beta_2}, \ldots, t^{\beta_n}) \in T \tag{10}$$

determines the action of a one-parameter subgroup of G on Y. We summarize this data by saying that *G acts on Y by the matrix L*, or that *G acts on Y with weights* $\eta_1, \ldots, \eta_n$.

Note that Y is a toric variety with a dense torus $T = (k^{\times})^n \subseteq Y$. Write Q_i for the character e_i considered as a regular function on T. Then

$$\mathcal{O}(Y) = k[Q_1, \ldots, Q_r, Q_{r+1}^{\pm 1}, \ldots, Q_n^{\pm 1}]. \tag{11}$$

We consider the action of G on $\mathcal{O}(T)$ (or $\mathcal{O}(Y)$) given by *right* translation, that is

$$(g \centerdot f)(t) = f(tg), \tag{12}$$

for $g \in G$, $f \in \mathcal{O}(T)$, $t \in T$. This convention implies that Q_i has weight η_i as in [13], section 6.

Lemma 2.1. *If $\eta_{r+1}, \dots, \eta_n$ are linearly independent, there exist matrices $\Gamma \in GL_m(\mathbb{Z})$, $\Delta \in GL_n(\mathbb{Z})$ such that*

1. *$\Gamma L \Delta$ has the block matrix form*

$$\Gamma L \Delta = \begin{bmatrix} L_1 & 0 \\ L_2 & D \end{bmatrix}, \tag{13}$$

where D is a diagonal matrix with nonzero diagonal entries $d_1, \dots, d_s$.

2. *Δ has the block matrix form*

$$\Delta = \begin{bmatrix} I_r & 0 \\ 0 & \Delta_1 \end{bmatrix}, \tag{14}$$

with $\Delta_1 \in GL_s(\mathbb{Z})$.

Proof. Let R be the submodule of $\mathbb{Z}^n$ spanned by the rows of L, and let $\epsilon : \mathbb{Z}^n \longrightarrow \mathbb{Z}^s$ be the projection onto the last s coordinates. Since the submatrix of L obtained by deleting the first r columns has rank s, it follows that $\epsilon(R)$ is free abelian of rank s. Thus if $R' = \ker \epsilon$, the map $R \longrightarrow R/R'$ splits and there is a submodule R'' of R such that $R = R' \oplus R''$. By choosing bases for R' and R'', we find $\Gamma_1 \in GL_m(\mathbb{Z})$ such that

$$\Gamma_1 L = \begin{bmatrix} L_1 & 0 \\ L_2' & L_3 \end{bmatrix},$$

where L_1 has $m - s$ rows and r columns and the rows of $[L_1, 0]$ form a basis for R'. There exist matrices $\Gamma_2 \in GL_s(\mathbb{Z})$ and $\Delta_1 \in GL_s(\mathbb{Z})$ such that $D = \Gamma_2 L_3 \Delta_1$ has the desired form, and we set

$$\Gamma = \begin{bmatrix} I_{m-s} & 0 \\ 0 & \Gamma_2 \end{bmatrix} \cdot \Gamma_1, \qquad \Delta = \begin{bmatrix} I_r & 0 \\ 0 & \Delta_1 \end{bmatrix}.$$

□

Now suppose G acts on Y via the matrix L and set

$$\Sigma_L = \{\alpha \in \mathbb{N}^r \times \mathbb{Z}^s | L\alpha = 0\}. \tag{15}$$

It is often convenient to use exponential notation for elements of $\mathcal{O}(Y)$. If $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^r \times \mathbb{Z}^s$, we set $Q^\alpha = Q_1^{\alpha_1} \dots Q_n^{\alpha_n}$. Then $\mathcal{O}(Y)^G = span\{Q^\alpha \in \mathcal{O}(Y) | L\alpha = 0\} \cong k\Sigma_L$, the semigroup algebra on Σ_L. If Γ, Δ are as in the lemma and $L' = \Gamma L \Delta$, there is an isomorphism

$$\Sigma_L \longrightarrow \Sigma_{L'} \tag{16}$$

given by $x \mapsto \Delta^{-1}x$. Thus if $\eta_{r+1}, \dots, \eta_n$ are linearly independent, we assume henceforth that L has the special form

$$L = \begin{bmatrix} L_1 & 0 \\ L_2 & D \end{bmatrix}, \tag{17}$$

where D is a diagonal matrix with nonzero diagonal entries $d_1, \dots, d_s$.

2.2 Rings of differential operators

Let $P_i = \partial/\partial Q_i$,

$$\mathcal{D}(Y) = k[Q_1, \dots, Q_r, Q_{r+1}^{\pm 1}, \dots, Q_n^{\pm 1}, P_1, \dots, P_n]. \tag{18}$$

Note that $\mathcal{D}(Y)$ is a localization of the n^{th} Weyl algebra

$$A_n = k[Q_1, \dots, Q_n, P_1, \dots, P_n]. \tag{19}$$

We denote the generators of A_1 simply by Q and $P = \partial/\partial Q$.

The action of G on $\mathcal{O}(Y)$ extends to an action on $\mathcal{D}(Y)$ defined by

$$(g \centerdot d)(f) = g(d(g^{-1}f)) \tag{20}$$

for $g \in G$, $d \in \mathcal{D}(Y)$, $f \in \mathcal{O}(Y)$.

If $\mu = (\mu_1, \dots, \mu_n) \in \mathbb{N}^n$, set $P^\mu = P_1^{\mu_1} \dots P_n^{\mu_n}$. Then for $g = (g_1, \dots, g_m) \in G$

$$g \centerdot Q^\lambda = \prod_{i=1}^m g_i^{\sum_{j=1}^n \lambda_j l_{ij}} Q^\lambda, \qquad g \centerdot P^\mu = \prod_{i=1}^m g_i^{\sum_{j=1}^n -\mu_j l_{ij}} P^\mu. \tag{21}$$

The elements $Q^\lambda P^\mu \in \mathcal{D}(Y)$, with $L\lambda = L\mu$, form a basis of $\mathcal{D}(Y)^G$.

For $\alpha \in \mathbb{N}^r \times \mathbb{Z}^s$ we define

$$u_\alpha = Q_1^{(\alpha_1)} \cdots Q_r^{(\alpha_r)} Q_{r+1}^{\alpha_{r+1}} \cdots Q_n^{\alpha_n}, \tag{22}$$

where

$$Q_i^{(\alpha_i)} = \begin{cases} Q_i^{\alpha_i} \text{ if } \alpha_i \geq 0 \\ P_i^{-\alpha_i} \text{ if } \alpha_i < 0 \end{cases}, \qquad i \in \{1, \dots, r\}. \tag{23}$$

Then $\mathcal{D}(Y) = \oplus \mathcal{D}(Y)_\alpha$, where $\mathcal{D}(Y)_\alpha = \mathcal{A} u_\alpha$, $\alpha \in \mathbb{Z}^n$, with $\mathcal{A} = k[\Pi_1, \dots, \Pi_n]$. If $\Pi_i = Q_i P_i$, then $[\Pi_i, u_\alpha] = \alpha_i u_\alpha$. Using this it is easy to see that

$$\mathcal{D}(Y)^G = \oplus_{\alpha \in Supp \mathcal{D}(Y)^G} \mathcal{D}(Y)_\alpha, \tag{24}$$

with $Supp \mathcal{D}(Y)^G = \{\alpha \in \mathbb{Z}^n | L\alpha = 0\}$, see [11], Lemma 2.2.

Define $\Lambda \subseteq \mathbb{Z}^m$ by $\Lambda = \{L\alpha | \alpha \in \mathbb{N}^r \times \mathbb{Z}^s\}$. For $\chi \in \Lambda$ define

$$\mathcal{O}(Y)_\chi = span\{Q^\lambda \in \mathcal{O}(Y) | L\lambda = \chi\}. \tag{25}$$

It is easy to see that

$$\mathcal{O}(Y) = \oplus_{\chi \in \Lambda} \mathcal{O}(Y)_\chi. \tag{26}$$

We denote the canonical action of $\mathcal{D}(Y)$ on $\mathcal{O}(Y)$ by

$$(d, f) \to d \cdot f, \tag{27}$$

for $d \in \mathcal{D}(Y)$ and $f \in \mathcal{O}(Y)$.

Each $\mathcal{O}(Y)_\chi$ is a $\mathcal{D}(Y)^G$-module: for a given $Q^\lambda \in \mathcal{O}(Y)_\chi$ and $Q^\mu P^\nu \in \mathcal{D}(Y)^G$, $L\lambda = \chi$ and $L\mu = L\nu$. Hence $Q^\mu P^\nu \cdot Q^\lambda$ is a multiple of $Q^{\lambda + \mu - \nu}$ with $L(\lambda + \mu - \nu) = L\lambda = \chi$.

2.3 Some remarks on the algebra $\mathcal{D}(V)^H$

Note that $\mathcal{D}(V)^H$ is the fixed ring of $\mathcal{D}(V)^{H^o}$ under the action of the finite abelian group H/H^o. Thus $\mathcal{D}(V)^{H^o}$ is graded by the character group of H/H^o and $\mathcal{D}(V)^H$ is the identity component of this grading. Hence if I is any right ideal of $\mathcal{D}(V)^H$, we have

$$I\mathcal{D}(V)^{H^o} \cap \mathcal{D}(V)^H = I. \tag{28}$$

Now fix $\mu \in \mathfrak{h}^*$ and let $\mathfrak{m}$ be the ideal of $\mathcal{A}' = k[\Pi_1, \dots, \Pi_r]$ generated by $\mathfrak{h} - \mu(\mathfrak{h})$. We can apply the above remarks with $I = \mathfrak{m}\mathcal{D}(V)^H$. It follows that there is an embedding of algebras

$$\frac{\mathcal{D}(V)^H}{\mathfrak{m}\mathcal{D}(V)^H} \longrightarrow \frac{\mathcal{D}(V)^{H^o}}{\mathfrak{m}\mathcal{D}(V)^{H^o}}. \tag{29}$$

From now on we denote these algebras simply by $\mathcal{D}(V)^H/(\mathfrak{h}-\mu(\mathfrak{h}))$ and $\mathcal{D}(V)^{H^o}/(\mathfrak{h}-\mu(\mathfrak{h}))$. Note that by [8], Lemma 3.1 and [13], Theorem 8.2.1 we have

$$\text{GK-}dim\ \frac{\mathcal{D}(V)^H}{(\mathfrak{h}-\mu(\mathfrak{h}))} \leq 2(n-m). \tag{30}$$

Using a suitable filtration and passing to the associated graded ring as in [13], §8.2, it can be shown that equality holds.

3 Actions of tori and slice representations

In this section we express the hypotheses in Proposition A and Theorem B in terms of the weights of the action.

Lemma 3.1. *Suppose the torus $G = (k^\times)^m$ acts on $W = (k^\times)^s$ with weights $\eta_{r+1}, \dots, \eta_n$.*

1. *If $\eta_{r+1}, \dots, \eta_n$ are linearly independent, then G acts transitively on W.*

2. *If $\eta_{r+1}, \dots, \eta_n$ are linearly dependent, then any orbit of G on W has dimension less than dim W.*

Proof. 1. Note that $\eta := (\eta_{r+1}, \dots, \eta_n) : G \longrightarrow W$ is a homomorphism of tori. Since the η_i are linearly independent, the rank of $\eta(G)$ is s, hence η is onto W, and it follows that G acts transitively on W.

2. We can assume that G acts faithfully on W. Then $\cap_{i=r+1}^n \ker \eta_i = 1$. If $\eta_{r+1}, \dots, \eta_n$ are linearly dependent, this implies that $dim\ G = m < s$. Any element $w \in W$ has trivial stabilizer, so $dim\ Gw = dim\ G < dim\ W$.

□

Suppose that

$$\phi : G \times X \longrightarrow X \tag{31}$$

is an action of a reductive group G on a variety X and consider a point $u \in X$ with stabilizer $H = G_u$ and tangent space $T_u(X)$. The differential

$$d\phi : \mathfrak{g} \longrightarrow T_u(X) \tag{32}$$

has kernel $\mathfrak{h} = Lie(H)$. If H is reductive, we have

$$T_u(X) \cong \mathfrak{g}/\mathfrak{h} \oplus U, \tag{33}$$

for some H-module U. We call the pair (H, U) the slice representation at u, see [10], [16].

Now suppose $G = (k^\times)^m$ is a torus acting faithfully on $Y = V \times W$ with weights $\eta_1, \dots, \eta_n$. Suppose that G acts transitively on W, and let $w = (w_{r+1}, \dots, w_n)$ be an element of W. Then

$$H = G_w = \cap_{i=r+1}^{n} \ker\, \eta_i. \tag{34}$$

Lemma 3.2. *The slice representation at w is isomorphic to (H, V).*

Proof. Since the G-orbit of w in W is W, $\mathfrak{g}$ maps onto $T_w W$ and $T_w W \cong \mathfrak{g}/\mathfrak{h}$. Thus the H-invariant complement to $T_{0,w}(G \centerdot (0, w))$ in $T_{0,w}Y$ is V. $\square$

Finally, H^o is the subtorus of G generated by images of one parameter subgroups corresponding to rows of L_1. For $1 \leq i \leq r$, let ρ_i be the restriction of η_i to H. These characters can be thought of as columns of L_1.

We can easily see the following,

Lemma 3.3. $V^{H^o} = 0$ *if and only if* $\rho_i \neq 0$ *for all* $i = 1, \dots, r$.

4 The special case $\mathcal{O}(Y)^G = k$

Lemma 4.1. *1. If $\mathcal{O}(Y)^G = k$, then $\eta_{r+1}, \dots, \eta_n$ are linearly independent.*

2. Assume $\eta_{r+1}, \dots, \eta_n$ are linearly independent. Then the following conditions are equivalent.

(a) $\mathcal{O}(Y)^G = k$.

(b) For all $\chi \in \Lambda$, $\mathcal{O}(Y)_\chi$ is finite dimensional.

(c) There exists $\beta = (\beta_1, \dots, \beta_n) \in K^\perp$ such that $\beta_i > 0$ for $i = 1, \dots, r$, and $\beta_i = 0$ for $i = r+1, \dots, n$.

Proof. 1. If $\eta_{r+1}, \dots, \eta_n$ are linearly dependent, we can write

$$\sum_{i \in I} a_i \eta_i = \sum_{j \in J} b_j \eta_j, \tag{35}$$

where I, J are disjoint subsets of $\{r+1, \dots, n\}$, $I \neq \emptyset$ and the a_i, b_j are positive integers. Then

$$\Pi_{j \in J} Q_j^{b_j} / \Pi_{i \in I} Q_i^{a_i} \tag{36}$$

is a nonconstant element of $\mathcal{O}(Y)^G$.

2. (a) $\Longrightarrow$ (c) Suppose $\mathcal{O}(Y)^G = k$. For $1 \leq i \leq r$, set

$$C_i = \sum_{j=1, j \neq i}^{r} \mathbb{Q}_{\geq 0} \eta_j + \sum_{j=1}^{s} \mathbb{Q} \eta_{r+j} \subseteq \mathbb{X}(G)_{\mathbb{Q}} \tag{37}$$

and

$$C_i^{\vee} = \{\gamma \in \mathbb{Y}(G)_{\mathbb{Q}} | [C_i, \gamma] \geq 0\}. \tag{38}$$

If $\alpha \in K$ with $\alpha_i \geq 0$, for $i = 1, \dots r$, we have $\alpha = 0$ since $Q^\alpha \in \mathcal{O}(Y)^G$. It follows that $-\eta_i \notin C_i$, for $1 \leq i \leq r$. Hence by equation (*) on page 9 of [3], there exists $\gamma_i \in C_i^{\vee}$ such that $(e_i, \omega(\gamma_i)) = [\eta_i, \gamma_i] > 0$. If $\gamma = \sum_{i=1}^{r} \gamma_i$, then some integer multiple β of $\omega(\gamma)$ satisfies the condition in (c).

(c) $\Longrightarrow$ (b) We may assume that L has the form (17). Suppose that $\chi \in \Lambda$ and fix $\varphi \in \mathbb{N}^r \times \mathbb{Z}^s$ with $L\varphi = \chi$. If $\mathbb{Y}(T)$ is identified with $\mathbb{Z}^n$, we have

$$\mathcal{O}(Y)_\chi = span\{Q^\alpha | \alpha \in \mathbb{N}^r \times \mathbb{Z}^s, L(\alpha - \varphi) = 0\}. \tag{39}$$

For $\alpha \in \mathbb{N}^r \times \mathbb{Z}^s$, write $\alpha = \begin{pmatrix} \alpha' \\ \alpha'' \end{pmatrix}$, with $\alpha' \in \mathbb{N}^r$ and $\alpha'' \in \mathbb{Z}^s$. Define φ' and φ'' similarly. To show $\mathcal{O}(Y)_\chi$ is finite dimensional it suffices to show there are at most finitely many $\alpha \in \mathbb{N}^r \times \mathbb{Z}^s$ such that

$$L_1(\alpha' - \varphi') = 0, \tag{40}$$

$$L_2(\alpha' - \varphi') + D(\alpha'' - \varphi'') = 0. \tag{41}$$

Since D is invertible α'' is determined once we fix α'. (Note however that if D^{-1} does not have integer entries, equation (41) may impose additional conditions on α').

Thus we may assume that $r = n$ and $L_1 = L$. The condition $L(\alpha - \varphi) = 0$ is equivalent to $(\alpha - \varphi, K^\perp) = 0$. Hence given $\beta \in K^\perp$ as in (c) we have

$$\sum_{i=1}^{n} \beta_i \alpha_i = \sum_{i=1}^{n} \beta_i \varphi_i \tag{42}$$

and this equation has only finitely many solutions for $\alpha \in \mathbb{N}^n$, since all β_i are positive.

This completes the proof since obviously (b) $\Longrightarrow$ (a). $\square$

Remark 4.2. *If $k = \mathbb{C}$, there is a more geometric proof of the equivalence of (a) and (c) in the Lemma. Indeed if (a) holds then there is a unique closed orbit of G on Y, namely $O = 0 \times W$ where 0 is the zero subspace of V. On the other hand if $\beta \in K^\perp$ is a one-parameter subgroup of G as in (c), then for all $y \in Y$ we have $\lim_{t\to 0} \beta(t)y \in O$. Thus the equivalence of (a) and (c) follows from the Hilbert-Mumford criterion for tori [5], III 2.2.*

Let $Q_i^{(\alpha_i)}$ be as in (23). The following identities are easily proved:

$$\text{if } \alpha_i \geq 0, \text{ then } Q_i^{(\alpha_i)} \cdot Q_i^{\lambda_i} = Q_i^{\lambda_i+\alpha_i}, \tag{43}$$

$$\text{if } \alpha_i < 0, \text{ then } Q_i^{(\alpha_i)} \cdot Q_i^{\lambda_i} = \begin{cases} \frac{\lambda_i!}{(\lambda_i+\alpha_i)!} Q_i^{\lambda_i+\alpha_i} \text{ if } -\alpha_i \leq \lambda_i \\ 0 \text{ if } -\alpha_i > \lambda_i \end{cases}. \tag{44}$$

Lemma 4.3. *If $\mathcal{O}(Y)_\chi \neq 0$, then it is a simple $\mathcal{D}(Y)^G$-module.*

Proof. Let $Q^\lambda,\ Q^\mu \in \mathcal{O}(Y)_\chi$. Then $L\lambda = L\mu = \chi$. Hence $u_{\mu-\lambda} \in \mathcal{D}(Y)^G$. Set $\alpha = \mu - \lambda$. If $1 \leq i \leq r$, then $\mu_i \geq 0$, so from (43), (44) we get $u_{\mu-\lambda} \cdot Q^\lambda = c\, Q^\mu$, where c is a nonzero integer. Since all weight spaces of $\mathcal{O}(Y)_\chi$ are one dimensional, the result follows from this. □

Assume that $\mathcal{O}(Y)^G = k$. We assume the action of G on Y is defined via the matrix L in (17). The dimensions of the modules $\mathcal{O}(Y)_\chi$ can be calculated using the following result. To state it we require some notation. Let $\mathbb{F}$ be a product of cyclic groups of order $d_1, \ldots, d_s$ and $R = \mathbb{Z}[\mathbb{F}][[t_1, \ldots, t_{m-s}]]$ a ring of formal power series over the group ring $\mathbb{Z}[\mathbb{F}]$. For $1 \leq i \leq s$, let t_{m-s+i} be a generator for the cyclic subgroup of $\mathbb{F}$ of order d_i. For $\chi = (\chi_1, \cdots, \chi_m) \in \mathbb{Z}^m$, set $t^\chi = t_1^{\chi_1} t_2^{\chi_2} \cdots t_m^{\chi_m}$.

Proposition 4.4. *The dimensions of the modules $\mathcal{O}(Y)_\chi$ satisfy*

$$\sum_{\chi\in\Lambda} \dim\, \mathcal{O}(Y)_\chi\, t^\chi = \prod_{j=1}^{r} (1 - \prod_{i=1}^{m} t_i^{l_{ij}})^{-1}. \tag{45}$$

Proof. The coefficient of t^χ in the expansion of the right side equals the number of solutions for $\alpha \in \mathbb{N}^r$ to the equations

$$\chi_i = \sum_{j=1}^{r} l_{ij}\alpha_j, \tag{46}$$

for $1 \leq i \leq m - s$, and the congruences

$$\chi_{i+m-s} \equiv \sum_{j=1}^{r} l_{i+m-s,\, j}\alpha_j \mod d_i, \tag{47}$$

for $1 \leq i \leq s$. This is easily seen to equal the number of solutions to equations (40) and (41). □

5 Reduction to the special case

Suppose $I \subseteq \{1, \ldots, r\}$. For $1 \leq i \leq n$, set

$$\varsigma_i = \begin{cases} -\eta_i \text{ if } i \in I \\ \eta_i \text{ if } i \notin I \end{cases} . \tag{48}$$

Let L_I be the matrix with columns $\varsigma_1, \ldots, \varsigma_n$. Then L_I defines an action of a new torus G_I on Y. As before G_I can be thought of as a subtorus of T.

Lemma 5.1. *If $V^{H^o} = 0$, there is a subset I of $\{1, \ldots, r\}$ such that $\mathcal{O}(Y)^{G_I} = k$.*

Proof. Let R' be the submodule of $\mathbb{Z}^r$ spanned by the rows of L_1, and let $\epsilon_j : R'_{\mathbb{Q}} \longrightarrow \mathbb{Q}$ be the restriction of the projection onto the j^{th} coordinate. Since $V^{H^o} = 0$, it follows that $\epsilon_j(R'_{\mathbb{Q}}) \neq 0$, for all j so ker $\epsilon_j \neq R'_{\mathbb{Q}}$. Therefore $\cup_{j=1}^r \ker \epsilon_j \neq R'_{\mathbb{Q}}$ and we can find $\beta \in R'_{\mathbb{Q}}$ such that $\epsilon_j(\beta) \neq 0$, for $j = 1, \ldots, r$. Now set $I = \{j \in \{1, \ldots, r\} | \epsilon_j(\beta) < 0\}$. Using Lemma 4.1, it is easy to see that $\mathcal{O}(Y)^{G_I} = k$. □

Lemma 5.2. *Let I be a subset of $\{1, \ldots, r\}$. Then the map $\sigma_I : \mathcal{D}(Y) \rightarrow \mathcal{D}(Y)$ defined by*

$$\sigma_I(Q_i) = \begin{cases} -P_i \text{ if } i \in I \\ Q_i \text{ if } i \notin I \end{cases}, \qquad \sigma_I(P_i) = \begin{cases} Q_i \text{ if } i \in I \\ P_i \text{ if } i \notin I \end{cases} \tag{49}$$

$i = 1, \ldots, n$ induces an isomorphism between $\mathcal{D}(Y)^G$ and $\mathcal{D}(Y)^{G_I}$.

Proof. The map σ_I is an isomorphism. Therefore its restriction to $\mathcal{D}(Y)^G$ is one-to-one. Consider the $\mathbb{Z}^n$-grading of $\mathcal{D}(Y)^G$ given by (24). Since

$$\sigma_I(\Pi_i) = \begin{cases} -\Pi_i - 1 \text{ if } i \in I \\ \Pi_i \text{ if } i \notin I \end{cases},$$

we can easily see that $\sigma_I(\mathcal{A}) = \mathcal{A}$. We can also check that $\sigma_I(u_\alpha) = \pm u_{\alpha^I}$, with $\alpha^I = (\alpha_1^I, \ldots, \alpha_s^I)$, where

$$\alpha_i^I = \begin{cases} -\alpha_i \text{ if } i \in I \\ \alpha_i \text{ if } i \notin I \end{cases} . \tag{50}$$

From this we conclude that $\sigma_I(\mathcal{D}(Y)_\alpha) = \mathcal{D}(Y)_{\alpha^I}$. Therefore $\sigma_I(\mathcal{D}(Y)^G) = \mathcal{D}(Y)^{G_I}$. □

The automorphism σ_I can be thought of as a partial Fourier transform [2], Chapter 5. We consider the $\mathcal{D}(Y)^G$-module $\mathcal{O}(Y)^I$, which equals $\mathcal{O}(Y)$ as a vector space with a new action given by

$$a * u = \sigma_I(a) \cdot u, \tag{51}$$

for $a \in \mathcal{D}(Y)^G$, $u \in \mathcal{O}(Y)$. Since $\sigma(\mathcal{D}(Y)^G) = \mathcal{D}(Y)^{G_I}$, it follows from Lemma 5.2 that

$$\mathcal{O}(Y)^I = \oplus_{\chi \in \Lambda_I} \mathcal{O}(Y)_\chi^I, \tag{52}$$

where $\Lambda_I = \{L_I\alpha | \alpha \in \mathbb{N}^r \times \mathbb{Z}^s\}$ and

$$\mathcal{O}(Y)^I_\chi = span\{Q^\alpha | L_I\alpha = \chi\}. \tag{53}$$

Note in particular that

$$\mathcal{O}(Y)^I_0 = \mathcal{O}(Y)^{G_I}. \tag{54}$$

6 Main results

The following lemma can be easily proved.

Lemma 6.1. *Suppose R is a $\mathbb{Z}^s$-graded ring. There is a homomorphism*

$$\Phi : R \longrightarrow R[x_1^{\pm 1}, \dots, x_s^{\pm 1}]$$

defined by $\Phi(r) = rx^\alpha$ for $r \in R(\alpha)$, $\alpha \in \mathbb{Z}^s$.

Assume that $\eta_{r+1}, \dots, \eta_n$ are linearly independent. As in [16], H acts on $G \times S$ by the rule

$$h \cdot (g, s) = (gh^{-1}, hs) \tag{55}$$

for $h \in H$, $g \in G$, and $s \in S$. Since all points in $G \times S$ have trivial stabilizer, all orbits are closed, and the quotient by H, denoted $G \times^H S$ is geometric. We denote the H-orbit of (g, s) by $[g, s]$.

Part of the Luna slice theorem states that there is a closed H-stable subvariety S containing w and a G-equivariant étale map $G \times^H S \longrightarrow Y$. Furthermore there are étale maps $V//H \longleftarrow S//H \longrightarrow Y//G$, [10], [16]. Taking $S = V + w$ we can strengthen this statement in our case.

Theorem 6.2. *There is a G-equivariant isomorphism $\sigma : G \times^H S \longrightarrow Y$, and this map induces an isomorphism between $V//H$ and $Y//G$.*

Proof. The map σ is defined by

$$\sigma[g, s] = gs. \tag{56}$$

It is easy to check that σ is one to one and onto. Since the source and target of σ are affine, and Y is smooth, it follows from Richardson's Lemma, [5] II.3.4 that σ is an isomorphism. This proves the first statement and the second is an immediate consequence. ∎

The second statement in Theorem 6.2 asserts that $\mathcal{O}(V)^H \cong \mathcal{O}(Y)^G$. Because this isomorphism is used in the proof of Proposition C, we wish to make it more explicit.

Assume that G acts on Y by the matrix L as in (17) and set

$$T_1 = \{\beta \in \mathbb{Z}^r | L_1\beta = 0\}, \tag{57}$$
$$T_2 = \{\beta \in \mathbb{Z}^n | L\beta = 0\} \tag{58}$$

and

$$T_1' = \{\beta \in T_1 | L_2\beta \in d_1\mathbb{Z} \times \ldots \times d_s\mathbb{Z}\}. \tag{59}$$

For $\beta \in T_1'$, let $\kappa(\beta) \in \mathbb{Z}^s$ be the column vector with i^{th} entry $-\gamma_i/d_i$ where $\gamma = L_2\beta \in \mathbb{Z}^s$, and set $\epsilon(\beta) = \begin{pmatrix} \beta \\ \kappa(\beta) \end{pmatrix}$. Since $\mathcal{O}(V)^H = span\{Q^\beta | L_1\beta = 0,\ \beta \in \mathbb{N}^r \cap T_1'\}$ and $\mathcal{O}(Y)^G = span\{Q^\beta | \beta \in (\mathbb{N}^r \times \mathbb{Z}^s) \cap T_2\}$, there is an isomorphism between $\mathcal{O}(V)^H \cong \mathcal{O}(Y)^G$ sending Q^β to $Q^{\epsilon(\beta)}$.

Proof of Proposition C.

Using the notation of § 2 we have

$$\mathcal{R} = \mathcal{D}(V)^H = \oplus_{\alpha \in T_1'} \mathcal{A}' u_\alpha$$

and

$$\mathcal{D}(Y)^G = \oplus_{\alpha \in T_2} \mathcal{A} u_\alpha.$$

We regard $\mathcal{R}$ as a $\mathbb{Z}^s$-graded ring with $\deg(\mathcal{A}'u_\alpha) = \kappa(\alpha)$, and apply Lemma 6.1 to obtain a homomorphism

$$\mathcal{R} \longrightarrow \mathcal{R}[x_1^{\pm 1}, \ldots, x_s^{\pm 1}], \tag{60}$$

which is the identity on $\mathcal{R}$ and maps u_α to $u_\alpha x^{\kappa(\alpha)}$. Composition with the map specializing $x_i^{\pm 1}$ to $Q_{r+i}^{\pm 1}$, for $1 \le i \le s$, gives the map ξ. Note that $\xi(u_\alpha) = u_{\epsilon(\alpha)}$.

Since $(\mathfrak{h} - \mu(\mathfrak{h}))$ is in the kernel of the map from $\mathcal{D}(V)^H$ to $\mathcal{B}_\lambda(Y)$, ξ induces a map $\xi_\lambda : \mathcal{B}_\mu(V) \longrightarrow \mathcal{B}_\lambda(Y)$. To show that ξ_λ is surjective, it remains to show that the image of $\mathcal{A}$ in $\mathcal{B}_\lambda(Y)$ is contained in the image of ξ_λ. For $1 \le i \le s$, we have $d_i\Pi_{r+i} + \sum_{j=1}^r l_{m-s+i,j}\Pi_j \in \mathfrak{g}$. Hence modulo $(\mathfrak{g} - \lambda(\mathfrak{g}))$, Π_{r+i} is in the span of $\Pi_1, \ldots, \Pi_r$ and 1. On the other hand $\xi(\Pi_i) = \Pi_i$ for $1 \le i \le r$.

By [13], Theorem 8.2.1 and the remarks in § 2.3 we have

$$\text{GK-}dim(\mathcal{B}_\mu(V)) \le \text{GK-}dim(\mathcal{B}_\lambda(Y)) = 2(n-m). \tag{61}$$

The surjectivity of ξ_λ implies that we have equality here. Since $\mathcal{B}_\mu(V)$ is a domain, it follows from [8], Proposition 3.15 that ξ_λ is injective.

Remark 6.3. *The fact that there is an injective algebra homomorphism from $\mathcal{D}(V)^H$ to $\mathcal{D}(Y)^G$ is a general fact resulting from the isomorphisms $G \times^H S \simeq G \times^H V \simeq Y$. If $f \in \mathcal{O}(G \times^H V)$, then f is of the form $\sum_i u_i \otimes v_i$ where the u_i are in $\mathcal{O}(G)$, the v_i are in $\mathcal{O}(V)$ and the sum $\sum_i u_i \otimes v_i$ is H-invariant. If $P \in \mathcal{D}(V)^H$, then we can let P act on $\sum_i u_i \otimes v_i$ sending it to $\sum_i u_i \otimes P(v_i)$. One can check easily that this action gives an element of $\mathcal{D}(Y)^G$.*

Proof of Proposition A

By Lemma 3.1, it suffices to show that $\mathcal{D}(Y)^G$ has no finite dimensional modules if $\eta_{r+1}, \dots, \eta_n$ are linearly dependent. This follows from [13], Proposition 10.1.1(1), but we can give a direct proof as follows. There exist integers $c_{r+1}, \dots, c_n$ not all zero such that $\sum_{i=r+1}^{n} c_i\eta_i = 0$. We can assume that $c = c_n \neq 0$. Then $\mathcal{Q} = \prod_{i=r+1}^{n} Q_i^{c_i}$ and $\mathcal{P} = P_n^c \prod_{i=r+1}^{n-1} Q_i^{-c_i}$ belong to $\mathcal{D}(Y)^G$.

Let $\omega = e^{2\pi i/c}$ and consider the automorphism of $A_1 = k[P, Q]$ sending P to ωP and Q to $\omega^{-1}Q$. Let $\mathbb{F}$ be the subgroup of $Aut(A_1)$ generated by this automorphism. We have $A_1^{\mathbb{F}} = k[P^c, Q^c, PQ]$, and it is well known that $A_1^{\mathbb{F}}$ is a simple ring, [1]. Note that $A_1^{\mathbb{F}}$ is $\mathbb{Z}$-graded when we set $deg(Q^c) = 1, deg(P^c) = -1$, and $deg(PQ) = 0$. Applying Lemma 6.1, we see that there is a ring homomorphism $\Phi : A_1^{\mathbb{F}} \longrightarrow A_1^{\mathbb{F}}[x^{\pm 1}]$ with $\Phi(Q^c) = Q^c x$, $\Phi(PQ) = PQ$, $\Phi(P^c) = P^c x^{-1}$. In addition, there is a homomorphism $\Psi : A_1[x^{\pm 1}] \longrightarrow \mathcal{D}(Y)$ given by $\Psi(Q) = Q_n$, $\Psi(P) = P_n$, $\Psi(x^{\pm 1}) = (\prod_{i=r+1}^{n-1} Q_i^{c_i})^{\pm 1}$. The composite $\Psi \circ \Phi : A_1^{\mathbb{F}} \longrightarrow \mathcal{D}(Y)^G$ sends Q^c to $\mathcal{Q}$, P^c to $\mathcal{P}$ and PQ to P_nQ_n. Since $A_1^{\mathbb{F}}$ is simple, it follows that this map is injective and $\mathcal{D}(Y)^G$ has no finite dimensional modules.

Proof of Theorem B

(1) $\Longrightarrow$ (4) If $V^{H^o} = 0$, then by Lemma 5.1 there exists $I \subseteq \{1, \dots, r\}$ such that $\mathcal{O}(Y)^{G_I} = k$. Therefore, by Lemmas 4.1 and 5.2 the faithful $\mathcal{D}(Y)^G$-module $\mathcal{O}(Y)^I$ is a direct sum of finite dimensional simple modules $\mathcal{O}(Y)^I_\chi$. Hence $\mathcal{D}(Y)^G$ has enough finite dimensional simple modules.

(3) $\Longrightarrow$ (1) If $V^{H^o} \neq 0$, we claim that $\mathcal{D}(Y)^G$ has no finite dimensional representations. Using the notation introduced immediately before Lemma 3.3, we can assume that $\rho_1 = 0$. Then $A_1 \cong k[Q_1, P_1]$ is a subalgebra of $\mathcal{D}(V)^{H^o}$. Hence the invariants in A_1 under the action of the finite group H/H^o form a subalgebra of $\mathcal{D}(V)^H$. Since $\mathcal{D}(V)^H$ is a subalgebra of $\mathcal{D}(Y)^G$ by Proposition C the claim follows.

Since (4) obviously implies (3), this proves the equivalence of (1), (3), (4). The equivalence of (1) and (2) is a special case of the equivalence of (1) and (4).

7 More on the modules $\mathcal{O}(Y)^I$

We give some alternative descriptions of the modules $\mathcal{O}(Y)^I$ defined in §5. First, it is easy to see that as $\mathcal{D}(Y)$-modules

$$\mathcal{O}(Y)^I \cong \mathcal{D}(Y)/(\sum_{i\in I} \mathcal{D}(Y)Q_i + \sum_{i\notin I} \mathcal{D}(Y)P_i). \tag{62}$$

Next, for $I \subseteq \{1, \dots, r\}$ set $Q_I = \prod_{i\in I} Q_i$ and let

$$\mathcal{O}(Y)_I = \mathcal{O}(Y)[Q_I^{-1}]. \tag{63}$$

Note that $\mathcal{O}(Y)_I$ is a $\mathcal{D}(Y)$-module. Now let M_I be the sum of all submodules of $\mathcal{O}(Y)_I$ of the form $\mathcal{O}(Y)_J$, where J is a proper subset of I. Then set

$$N_I = \mathcal{O}(Y)_I / M_I. \tag{64}$$

Proposition 7.1. *As a $\mathcal{D}(Y)$-module (and hence also as a $\mathcal{D}(Y)^G$-module) $\mathcal{O}(Y)^I \cong N_I$.*

Proof. Let n_I be the image of Q_I^{-1} in N_I. It is easy to verify that $Q_i \cdot n_I = 0$ if $i \in I$ and $P_i \cdot n_I = 0$ if $i \notin I$. Since N_I is generated by n_I, it follows from (62) that there is a surjective map from $\mathcal{O}(Y)^I$ onto N_I. This map is an isomorphism since $\mathcal{O}(Y)^I$ is a simple $\mathcal{D}(Y)$-module. □

8 FCR-algebras

We give examples of Noetherian FCR-algebras of every given integer GK dimension ≥ 3. It is conjectured in [4] that FCR-algebras with GK dimension 2 do not exist. It was proved in [13], Theorem 8.2.1(4) that GK-*dim* $\mathcal{D}(Y)^G = 2n - m$.

1. **Odd GK dimension ≥ 3.** Assume *dim* $G = 1$. We can take $n \geq 2$. Let the action of G on the n^{th} Weyl algebra A_n be given by the matrix:

$$\begin{bmatrix} b_1 & \dots & b_n \end{bmatrix}, \tag{65}$$

   with $b_i \neq 0$ for all $i \in \{1, \dots, n\}$. By [13], Proposition 10.2.1(3), these are algebras with the reductive property, and by Theorem B they have enough simple finite dimensional modules. Then GK-$dim A_n^G = 2n - 1$. In this way we have many examples of FCR-algebras with odd GK-*dim* ≥ 3.

   If we take all the b_i equal to 1, so that $k^\times$ acts by scalar multiplication on k^n, we obtain an unpublished example of the first author and M. Van den Bergh which is also mentioned in [4], §3.

2. **Even GK dimension ≥ 6.** Assume *dim* $G = 2$ and $n \geq 4$. Then GK-$dim A_n^G = 2n - 2$. Let the action of G on A_n be given by the matrix:

$$\begin{bmatrix} 1 & 0 & 1 & 0 & \dots & 1 & 0 \\ 0 & 1 & 0 & 1 & \dots & 0 & 1 \end{bmatrix} \text{ or } \begin{bmatrix} 1 & 0 & 1 & 0 & \dots & 0 & 1 \\ 0 & 1 & 0 & 1 & \dots & 1 & 0 \end{bmatrix}. \tag{66}$$

   In this way we get examples of FCR-algebras with even GK-*dim* ≥ 6. By [13], Corollary 10.1.6 , A_n^G has the reductive property, and by Theorem B it has enough simple finite dimensional modules.

3. **GK dimension 4.** Take *dim* $G = 2$, $n = 3$, $s = 1$ and $r = 2$. Consider the algebra

$$A = k[Q_1, Q_2, Q_3^{\pm 1}, P_1, P_2, P_3].$$

Let the action of G on A be given by the matrix

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Then GK-*dim* $A^G = 4$. By [13], Corollary 10.1.6 and Theorem B, A^G is an FCR-algebra.

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