Dedicated to Don Passman on the occasion of his $65^{\text {th }}$ birthday.

# FAITHFUL CYCLIC MODULES FOR ENVELOPING 

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#### Abstract

Let $U$ be the enveloping algebra of a finite dimensional nonabelian Lie algebra $\mathfrak{g}$ over a field of characteristic zero. We show that there is an open nonempty open subset $X$ of $U_{1}=\mathfrak{g} \oplus K$ such that $U / U x$ is faithful for all $x \in X$. We prove similar results for homogenized enveloping algebras and for the three dimensional Sklyanin algebras at points of infinite order. It would be interesting to know if there is a common generalization of these results.


A prime Noetherian ring $U$ is bounded if every essential left ideal contains a nonzero two sided ideal. We say that $U$ is fully bounded Noetherian (FBN) if every prime image of $U$ is bounded. The main examples of FBN rings are Noetherian rings satisfying a polynomial identity (PI), see AmSm. If a filtered ring $U=\cup_{n \geq 0} U_{n}$ is not bounded, it is reasonable to ask if we can find a regular element $x \in U_{1}$ such that $U / U x$ is faithful. In addition we might hope that most elements $x \in U_{1}$ have this property. In this paper we show that this is the case for enveloping algebras of nonabelian Lie algebras in characteristic zero. We prove similar results for homogenized enveloping algebras and for the three dimensional Sklyanin algebras. Since these graded algebras are defined in rather different ways, we might expect that there is a more general underlying result. We remark that the proof for nonnilpotent Lie algebras is rather similar to the Sklyanin algebra case, but for nilpotent Lie algebras a different argument is required.

For more information and references on enveloping algebras satisfying a PI the reader may consult Chapter 4 of [BMPZ]. Group algebras satisfying a PI are discussed at length in the masterful book of Passman, $[\mathrm{P}]$.

My interest in the issues raised in this paper was stimulated by a question of Jorg Feldvoss at the conference in Madison. Jorg asked whether an enveloping algebra which is FBN must necessarily be commutative, in characteristic zero. I thank Jorg for his question. In addition I thank Michaela Vancliff for pointing me towards some relevant results in the literature.

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## 1. Enveloping Algebras.

We assume that $\mathfrak{g}$ is a finite dimensional Lie algebra over an algebraically closed field $K$ of characteristic zero. Let $\left\{U_{n}\right\}_{n \geq 0}$ be the filtration on $U=U(\mathfrak{g})$ given by $U_{0}=K, U_{1}=\mathfrak{g} \oplus K$ and $U_{n}=U_{1}^{n}$ for $n \geq 2$. We write $S=G r U=\oplus_{n \geq 0} U_{n} / U_{n-1}$ for the associated graded ring.

Theorem 1.1. If $\mathfrak{g}$ is nonabelian there is a nonempty open subset $X$ of $U_{1}$ such that $U / U x$ is faithful for all $x \in X$.

Corollary 1.2. If $U$ is bounded then $\mathfrak{g}$ is abelian.
Proof. This follows immediately from Theorem 1.1 .
The proof of Theorem 1.1 will be given as a series of lemmas. The subset $X$ in the Theorem has the form $X=X^{\prime} \oplus K$, where $X^{\prime}$ is a nonempty open subset of $\mathfrak{g}$. First we suppose that $\mathfrak{g}$ is not nilpotent. Then by Engel's Theorem the set of ad-nilpotent elements of $\mathfrak{g}$ is a proper closed subset, and we let $X^{\prime}$ be its complement.

If $x^{\prime} \in X^{\prime}$, then $a d x^{\prime}$ has a nonzero eigenvalue $\lambda$ on $\mathfrak{g}$, so there exists $y \in \mathfrak{g}$ such that $\left[x^{\prime}, y\right]=\lambda y$. Replacing $x^{\prime}$ by $\lambda^{-1} x^{\prime}$ we can assume that $\lambda=1$. If $\mu \in K$, and $x=x^{\prime}+\mu$, then for all $n \geq 0$ we have in $U$

$$
\begin{equation*}
(x-n) y^{n}=y^{n} x \tag{1.1}
\end{equation*}
$$

Using the PBW theorem we see that $y^{n} \notin U x$. If $m$ is an element of a left $U$-module $M$, we write $a n n_{U} m$ for the annihilator of $m$ in $U$.

Lemma 1.3. Let $M=U / U x$, and denote the image of $u \in U$ in $M$ by $\bar{u}$. Then

$$
a n n_{U} \bar{y}^{n}=U(x-n) .
$$

Proof. It follows from (1.1) that $U(x-n) \subseteq a n n_{U} \bar{y}^{n}$. If $u \in U_{m} \cap$ $a n n_{U} \bar{y}^{n}$, we show by induction on $m$ that $u \in U(x-n)$. Define a filtration $\left\{M_{p}\right\}$ on $M$ by setting

$$
M_{p}=\left(U_{p}+U x\right) / U x
$$

for $p \geq 0$. Then the associated graded module $G r M=\oplus_{p \geq 0} M_{p} / M_{p-1}$ is isomorphic to $S / S x^{\prime}$, so every nonzero element of $G r M$ has annihilator $S x^{\prime}$. Hence there exists $v \in U_{m-1}$ such that

$$
u^{\prime}=u-v(x-n) \in U_{m-1} .
$$

Since $u^{\prime} \bar{y}^{n}=0$ the result follows by induction.
Lemma 1.4. We have $\cap_{n \geq 0} U(x-n)=0$.

Proof. Let $x_{1}, \ldots, x_{m}, x^{\prime}$ be a basis for $\mathfrak{g}$, and let $W$ be the subspace of $U$ spanned by all elements of the form

$$
x_{1}^{a_{1}} \ldots x_{m}^{a_{m}}
$$

with $a_{1}, \ldots, a_{m} \in \mathbb{N}$. It follows easily from the PBW theorem that any element $u$ of $U$ can be written uniquely in the form

$$
u=\sum_{i=0}^{N} w_{i} x^{i}
$$

with $w_{0}, \ldots, w_{N} \in W$. If $u \in U(x-n)$, it is easy to see that $\sum_{i=0}^{N} n^{i} w_{i}=$ 0 . If $u \in \cap_{n \geq 0} U(x-n)$ we conclude from the nonvanishing of the Vandermonde determinant that all the $w_{i}$ are zero.

If $\mathfrak{g}$ is not nilpotent Theorem 1.1 follows from Lemmas 1.3 and 1.4 . Now suppose that $\mathfrak{g}$ is nilpotent with center $\mathfrak{z}$, and let $Z$ be the center of $U$. Theorem 1.1 in this case follows from the next result.

Lemma 1.5. Assume $\mathfrak{g}$ is nilpotent and nonabelian, and set $X^{\prime}=$ $\mathfrak{g} \backslash \mathfrak{z}, X=X^{\prime} \oplus K$. If $x \in X$, then $M=U / U x$ is a faithful $U$-module.

Proof. If $I=a n n_{U} M$ is nonzero then by [D], Proposition 4.7.1, $I \cap Z$ is nonzero. It suffices to show that $U x \cap Z=0$ since then $M$ contains the free $Z$-module $(Z+U x) / U x$.

Suppose that $x^{\prime} \in X, \mu \in K$ and $x=x^{\prime}+\mu$. Choose $y \in \mathfrak{g}$ such that $\left[y, x^{\prime}\right]=[y, x] \neq 0$. We extend ady to a locally nilpotent derivation of $U$. Assume $u \in U$ is nonzero, and let $i, j$ be the least integers such that

$$
(a d y)^{i+1}(u)=0=(a d y)^{j+1}(x) .
$$

Then $i \geq 0$ and $j \geq 1$. If $n=i+j$, then

$$
(a d y)^{n}(u x)=\binom{n}{i}(a d y)^{i}(u)(a d y)^{j}(x)
$$

and this is nonzero since $U$ is a domain. It follows that $u x \notin Z$.

## 2. Graded Algebras.

Now we suppose that $A$ is a 3 dimensional Sklyanin algebra, that is 3 -dimensional generic regular algebra of type $A$ with 3 generators and 3 quadratic relations, see ATV1, Section 4.13 or ArSc, (10.14). To construct $A$, consider a nonsingular cubic curve $E$ in $\mathbb{P}^{2}=\mathbb{P}\left(V^{*}\right)$. We use the symbols $\oplus, \ominus$ to addition or subtraction using the group law on $E$, since we also need to consider addition of divisors. We suppose the identity 0 in $E$ is a flex, so that $P \oplus Q \oplus R=0$ if and only if $P, Q$ and
$R$ are colinear. Fix a $p$ point on $E$ and let $\sigma$ be the an automorphism of $E$ defined by $\sigma(P)=P \ominus p$.

Given this data, we define $A$ as follows. Let $T(V)=\bigoplus_{n \geq 0} V^{\otimes n}$ be the tensor algebra on $V$. For $n>0$ let $E_{n}$ be the subset of $E \times \ldots \times E$ ( $n$ copies) consisting of all $n$-tuples $\left(e, \sigma(e), \ldots, \sigma^{n-1}(e)\right)$ with $e \in E$, and set

$$
\mathcal{R}_{n}=\left\{f \in V^{\otimes n} \mid f \text { vanishes on } E_{n}\right\} .
$$

For $n \geq 0$ let $I_{n}$ be the ideal of $T(V)$ generated by the subspaces $\mathcal{R}_{i}$ for $2 \leq i \leq n$. We set $A=T(V) / I_{2}$. Then $A=\bigoplus_{n \geq 0} A(n)$ where $A(n)$ is the image of $V^{\otimes n}$ in $A$. Let $L$ be the line bundle associated to the embedding of $E$ into $\mathbb{P}^{2}$. In ATV1 and ATV2 the algebra $A$ is denoted by $A(\mathcal{T})$ where $\mathcal{T}$ is the triple $(E, \sigma, L)$. By ATV1] Theorem 6.8, there is an element $g \in A(3)$ such that $g A=A g$ and $A / A g$ is isomorphic to the twisted homogeneous coordinate ring $B$, (denoted $B(\mathcal{T})$ in ATV1). It follows that $I_{n}=I_{3}$ for all $n \geq 3$. The algebra $A$ is a domain by ATV2] Section 3, and is finite module over its center if and only if $p$ is a point of finite order, ATV2 Theorem 7.1. Hence by Posner's theorem, [MR] Theorem 13.6.5, $A$ is a PI algebra if and only if $p$ is a point of finite order. It is well known that $B$ is a domain. To see this we can adapt the proof of [ V ], Lemma 3.5.

Theorem 2.1. If $p$ is a point of infinite order on $E$ then $A / A L$ is a faithful $A$-module for all $L \in A(1)$.

Again the proof will be given as a series of lemmas. Equation (2.1) and Lemmas 2.3 and 2.4 are analogs of equation (1.1) and Lemmas 1.3 and 1.4 respectively.

We refer to nonzero elements of $A(1)=V$ as lines. A divisor on $E$ is a finite formal sum of the form $\sum_{P \in E} n_{p} P$ with $n_{p} \in \mathbb{Z}$. If $D=$ $\sum_{P \in E} n_{p} P$ we set $\sigma_{*}(D)=\sum_{P \in E} n_{p} \sigma(P)$. The divisor $D$ is effective if $n_{p} \geq 0$ for all $P \in E$. If $D$ is effective we say that $P$ occurs in $E$ if $n_{p}>0$.

If $F \in V^{\otimes n}$ does not vanish on $E_{n}$ we write $\operatorname{Div}_{n}(F)$ for the divisor on $E$ cut out by $F$, that is

$$
\operatorname{Div}_{n}(F)=\sum\left\{e \in E \mid F\left(e, \sigma(e), \ldots, \sigma^{n-1}(e)\right)=0\right\}
$$

where zeroes of $F$ are counted with multiplicities. If $F \in V^{\otimes n}$ then $\operatorname{Div}_{n}(F)$ depends only on the image of $F$ modulo $\mathcal{R}_{n}$. Hence if $f \in$ $A(n)$, and $F \in V^{\otimes n}$ is a preimage of $f$ in $V^{\otimes n}$, we set $\operatorname{Div}_{n}(f)=$ $\operatorname{Div}_{n}(F)$. For $F \in V$ we write $\operatorname{Div}(F)$ in place of $\operatorname{Div}_{1}(F)$.

Note that if $F \in V^{\otimes(n-1)}$, and $L \in V$, then

$$
\operatorname{Div}_{n}(F \otimes L)=\operatorname{Div}_{n-1}(F)+\sigma_{*}^{1-n}(\operatorname{Div}(L))
$$

and

$$
\operatorname{Div}_{n}(L \otimes F)=\operatorname{Div}(L)+\sigma_{*}^{-1}\left(\operatorname{Div}_{n-1}(F)\right) .
$$

Let $L_{0}=L$ and suppose that $\operatorname{Div}\left(L_{0}\right)=P+Q+R$. Let $\mathbb{N}^{*}=\mathbb{N} \backslash\{0\}$ be the set of positive integers. Since $P, Q, R$ are colinear, and $p$ has infinite order, we cannot have $3 P, 3 Q, 3 R \in 3 \mathbb{N}^{*} p$, and we may assume that $3 R \notin 3 \mathbb{N}^{*} p$. For all $n>0$ fix a line $L_{n}$ through $P \oplus n p$ and $Q \oplus n p$ and $R \ominus 2 n p$.

Choose a line $M_{0}$ through $R$ such that $M_{0}$ is not a scalar multiple of $L_{0}$. Then $\operatorname{Div}\left(M_{0}\right)=R+S+T$, where $\{P, Q\} \cap\{S, T\}=\emptyset$.

Lemma 2.2. For all $n>0$ there are lines $M_{n}$ such that
(1) $\operatorname{Div}\left(M_{n}\right)=(R \ominus 2 n p)+(S \oplus n p)+(T \oplus n p)$.
(2) $L_{n} M_{n-1}=M_{n} L_{n-1}$ holds in $A(2)$.

Proof. Since the points $R \ominus 2 n p, S \oplus n p$ and $T \oplus n p$ sum to zero using the group law on $E$, there are lines $M_{n}$ satisfying (1). It suffices to show that (2) holds when $n=1$. Since $\operatorname{Div}_{2}\left(L_{1} \otimes M_{0}\right)$ and $\operatorname{Div}_{2}\left(M_{1} \otimes L_{0}\right)$ are both equal to

$$
(P \oplus p)+(Q \oplus p)+(R \ominus 2 p)+(R \oplus p)+(S \oplus p)+(T \oplus p)
$$

it follows that $L_{1} \otimes M_{0}-\lambda M_{1} \otimes L_{0} \in \mathcal{R}_{2}$ for some nonzero scalar $\lambda$. Replacing $M_{1}$ by $\lambda^{-1} M_{1}$ we obtain the result.

Now set $N_{1}=M_{0}, N_{1}^{\prime}=M_{1}$, and for $i>1$ define $N_{i}, N_{i}^{\prime}$ inductively by $N_{i}=M_{i-1} N_{i-1}$ and $N_{i}^{\prime}=M_{i} N_{i-1}^{\prime}$. By Lemma 2.2 and induction, we have for $n \geq 1$

$$
\begin{equation*}
L_{n} N_{n}=N_{n}^{\prime} L \tag{2.1}
\end{equation*}
$$

We write $G K(M)$ for the Gelfand-Kirillov dimension of the graded $A$-module $M$. This can be defined as the order of the pole of the Hilbert series of $M$, see ATV2] Section 2. We say that $M$ is critical if $G K(N)<$ $G K(M)$ for every proper graded factor module $N$ of $M$. Since the Hilbert series is additive on short exact sequences we see that if $M$ is critical then any nonzero graded submodule of $M$ has GK-dimension equal to $G K(M)$.
Lemma 2.3. If $\bar{N}_{n}$ is the image of $N_{n}$ in $A / A L$, then ann $\bar{N}_{A}=A L_{n}$.
Proof. We show first that $N_{n} \notin A L$. From Lemma 2.2, it follows that
$\operatorname{Div}\left(N_{n}\right)=n(S \oplus(n-1) p)+n(T \oplus(n-1) p)+\sum_{i=0}^{n-1} R \oplus(3 i-2(n-1)) p$.
On the other hand if $F \in A(n-1)$, then
$\operatorname{Div}_{n}(F L)=D i v_{n-1}(F)+(P \oplus(n-1) p)+(Q \oplus(n-1) p)+(R \oplus(n-1)) p$.

If $N_{n}=F L$, then since $\{P, Q\} \cap\{S, T\}=\emptyset$, it would follow that $P=R \ominus 3 i p$, and $Q=R \ominus 3 j p$, where $0<i, j \leq n-1$. Then since $P, Q$ and $R$ are colinear we would have $3 R=3(i+j) p \in 3 \mathbb{N}^{*} p$, a contradiction.

By equation 2.1), $A L_{n} \subseteq a n n_{A} \bar{N}_{n}$. It follows from ATV2, Proposition 6.1, that $A / A L$ is a critical $A$-module with GK-dimension 2. Therefore the submodule $A \bar{N}_{n}$ of $A / A L$ has GK-dimension 2 , and any proper factor module of $A / A L_{n}$ has GK-dimension less than 2 . Hence the natural map from $A / A L_{n}$ onto $A \bar{N}_{n}$ must be an isomorphism, and this implies the result.

Lemma 2.4. $\cap_{n \geq 0} A L_{n}=0$.
Proof. We first show that $\cap_{n \geq 0} A L_{n} \subseteq(g)$. Suppose that $f \in \cap_{n \geq 0} A L_{n}$. We can assume that $f \in A(m)$ for some $m$. Let $F \in V^{\otimes m}$ be a preimage of $f$ in $T(V)$. If $f \notin(g)$, then $F \notin \mathcal{R}_{m}$, so $F$ does not vanish on $E_{m}$ and we can consider the effective divisor $\operatorname{Div}_{m}(F)$. For all $n$ we can write $f=G_{n} L_{n}$ for some $G_{n} \in A(m-1)$ but then $\operatorname{Div}_{m}(f)$ contains $\sigma^{1-m}(P \oplus n p)=P \oplus(n+m-1) p$ for all $n \geq 0$. However the divisor $\operatorname{Div} v_{m}(f)$ cannot contain more than $3 m$ points. Since $p$ is a point of infinite order this is a contradiction.

Next we show that $\cap_{n \geq 0} A L_{n} \subseteq\left(g^{i}\right)$ for all $i \geq 1$. Any nonzero element of $\left(g^{i}\right)$ has degree at least $3 i$, so $\cap_{i \geq 1}\left(g^{i}\right)=0$ and this will finish the proof. Suppose that $x=a_{n} L_{n}$ with $a_{n} \in A$, for all $n \geq 0$. By induction we assume that $a_{n} \in\left(g^{i}\right)$ for all $n$. Then $x=g^{i} b_{n} L_{n}$, for some $b_{n} \in A$, and $y=b_{n} L_{n}$ is independent of $n$ since $A$ is a domain. By the first part of the proof $y \in(g)$. Since $B=A /(g)$ is a domain and $L_{n} \notin(g)$ by comparing degrees, we see that $b_{n} \in(g)$, and then $a_{n} \in\left(g^{i+1}\right)$ as required.

Theorem 2.1 follows at once from Lemmas 2.3 and 2.4 .

## 3. Concluding Remarks.

The enveloping algebra result can be formulated in terms of graded rings. In general if $\left\{U_{n}\right\}_{n \geq 0}$ is a filtration on a ring $U$, the Rees ring $\widetilde{U}$ of this filtration is the graded subring $\widetilde{U}=\bigoplus_{n \geq 0} \widetilde{U}(n)$ of the polynomial ring $U[z]$ with $\widetilde{U}(n)=U_{n} z^{n}$.

Lemma 3.1. Suppose that $x \in U_{n}$ and set $\widetilde{x}=x z^{n}$. Assume that $G r U$ is a domain. Then $U / U x$ is a faithful $U$-module if and only if $\widetilde{U} / \widetilde{U} \widetilde{x}$ is a faithful $\widetilde{U}$-module.

Proof. Assume that $U / U x$ is faithful. Since the annihilator of $\widetilde{U} / \widetilde{U} \widetilde{x}$ is a graded ideal, it is enough to show that if $u=u_{m} z^{m} \in \widetilde{U}(m)$, and $u \widetilde{U} \subseteq \widetilde{U} \widetilde{x}$, then $u=0$. However

$$
u \widetilde{U}(p) \subseteq \widetilde{U} \widetilde{x} \cap \widetilde{U}(p+m)=\widetilde{U}(p+m-n) \widetilde{x}
$$

implies that $u_{m} U_{p} \subseteq U_{p+m-n} x$. In particular $u_{m} U \subseteq U x$, so $u=0$.
Conversely, suppose that $\widetilde{U} / \widetilde{U} \widetilde{x}$ is faithful. If $a \in U_{m} \backslash U_{m-1}$ we set $\operatorname{deg} a=m$. Since $G r U$ is a domain

$$
\operatorname{deg}(a b)=\operatorname{deg} a+\operatorname{deg} b
$$

Since $x \in U_{n}$ we have $\operatorname{deg} x=n-i$ with $i \geq 0$. Suppose $a \in a n n_{U} U / U x$ and $\operatorname{deg} a=m$. If $b \in U$, and $\operatorname{deg} b=p$, then

$$
a b=c x
$$

for some $c \in U$ with $\operatorname{deg} c=m+p-n+i$. If $u=a z^{m+i}$ then

$$
u \cdot b z^{p}=c z^{p+m-n+i} \cdot x z^{n} \in \widetilde{U} \widetilde{x} .
$$

Thus, $u=0$ by hypothesis, so $a=0$.
When $U=U(\mathfrak{g})$ is an enveloping algebra with the standard filtration $\left\{U_{n}\right\}_{n \geq 0}$, the Rees ring is known as the homogenized enveloping algebra and denoted by $H(\mathfrak{g})$. We have $H(\mathfrak{g})(1)=K z \oplus \mathfrak{g} z$.
Corollary 3.2. Let $\mathfrak{g}$ be a finite dimensional nonabelian Lie algebra, and set $H=H(\mathfrak{g})$. Let $X$ be the open subset of $U_{1}$ given in Theorem 1.1. If $\widetilde{X}=X z \subseteq H(1)$ then $H / H L$ is a faithful $H$-module for all $L \in \widetilde{X}$.

Proof. Combine Theorem 1.1 and Lemma 3.1.
Our results prompt the following
Question 3.3. Suppose that $A$ is a connected graded algebra over a field and that $A$ is generated $A(1)$ which is finite dimensional. If $A$ is not $F B N$, is the set

$$
\{x \in A(1) \mid A / A x \text { is faithful }\}
$$

Zariski dense in $A(1)$ ?

To study this question the following result should be useful. We give $\operatorname{Spec} A$ has the Jacobson topology, so that the closed sets have the form

$$
V(I)=\{P \in \operatorname{Spec} A \mid I \subseteq P\}
$$

where $I$ is an ideal of $A$.

Theorem 3.4. Suppose that $A=\bigoplus_{n \geq 0} A(n)$ is a graded $K$-algebra which is a domain. Assume that $\operatorname{dim}_{K} \bar{A}(n)<\infty$ for all $n$, and that if $L \in A(1)$ the $A$-module $M=A / A L$ has prime annihilator. Then the map

$$
A(1) \longrightarrow S p e c A, L \mapsto \operatorname{ann}_{A} A / A L
$$

is continuous.
We remark that the hypothesis that $A / A L$ has prime annihilator for all $L \in A(1)$ holds for regular algebras of dimension 3 by ATV2] Propositions 2.30 and 6.1. The proof of the theorem is based on the next lemma.
Lemma 3.5. Keep the hypothesis of the theorem. If $F \in A(n)$ for $n \geq 1$ then the set $X=\{L \in A(1) \mid F \in A L\}$ is Zariski closed.

Proof. Let $U=A(n-1), V=A(1)$ and $W=A(n)$. If $u \in U$ is nonzero, let $[u] \in \mathbb{P}(U)$ denote the line through $u$. We use similar notation for elements of $V, W$. Since $A$ is a domain, there is a well defined morphism of varieties

$$
\pi: \mathbb{P}=\mathbb{P}(U) \times \mathbb{P}(V) \longrightarrow \mathbb{P}(W)
$$

given by

$$
([u],[v]) \longrightarrow[u v] .
$$

Thus the set $\pi^{-1}([F])$ is closed in $\mathbb{P}$. Now if $p: \mathbb{P} \longrightarrow \mathbb{P}(V)$ is the projection onto the second factor, it follows from [H], Theorem 3.12 that $Y=p\left(\pi^{-1}([F])\right)$ is closed in $\mathbb{P}(V)$. It is easy to see that

$$
X=\{L \in A(1) \mid[L] \in Y\}
$$

and it follows that $X$ is closed in $A(1)$.
Proof of Theorem 3.4. If $I$ is an ideal of $A$, then

$$
\begin{aligned}
\pi^{-1}(V(I)) & =\{L \in A(1) \mid \pi(L) \in V(I)\} \\
& =\{L \in A(1) \mid I \subseteq A L\} \\
& =\cap_{F \in I}\{L \in A(1) \mid F \in A L\}
\end{aligned}
$$

and this is an intersection of closed sets by the Lemma.
Question 3.6. For a group algebra analog of the problems considered in this paper set

$$
X=\{g \in G \mid K G / K G(g-1) \text { is faithful }\}
$$

What can be said about X?

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