DOWN-UP ALGEBRAS AND THEIR REPRESENTATION THEORY

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ABSTRACT. A class of algebras called down-up algebras was introduced by G. Benkart and T. Roby [5]. We classify the finite dimensional simple modules over Noetherian down-up algebras and show that in some cases every finite dimensional module is semisimple. We also study the question of when two down-up algebras are isomorphic.

1. INTRODUCTION

Given a field K and α, β, γ arbitrary elements of K, the associative algebra $A = A(\alpha, \beta, \gamma)$ over K with generators d, u and defining relations

$$(R1) \qquad d^2 u = \alpha dud + \beta u d^2 + \gamma d$$

$$(R2) \qquad du^2 = \alpha udu + \beta u^2 d + \gamma u$$

is a down-up algebra.

In [13] it is shown that $A(\alpha, \beta, \gamma)$ is Noetherian if and only if $\beta \neq 0$. Down-up algebras are also studied in [4], [5], [6], [9], [14] and [17]. In this paper we study the representation theory and the isomorphism problem for Noetherian down-up algebras.

By [13, 2.2] if $\beta \neq 0$ then $A = A(\alpha, \beta, \gamma)$ embeds in a skew group ring $S = R[z, z^{-1}; \sigma]$. Here $R = K[x, y]$, σ is the automorphism of R defined by $\sigma(x) = y$ and $\sigma(y) = \alpha y + \beta x + \gamma$. As a right R-module, S is free on the basis $\{z^n | n \in \mathbb{Z}\}$ and the multiplication in S is defined by $rz = z\sigma(r)$. The embedding $\theta : A \longrightarrow S$ is given by $\theta(d) = z^{-1}$, $\theta(u) = xz$, so that $\theta(ud) = x$ and $\theta(du) = y$.

Techniques involving skew group rings play an important role in this paper. We show that S as defined above is isomorphic to A_d the localization of A at $\{d^n | n \geq 0\}$. Similarly the localization A_u is a skew group ring.

In section 2 we determine the finite dimensional simple modules over down-up algebras. Here we use techniques of D. Jordan [8].

We say that a left A-module M is d-torsion (resp. u-torsion) if $A_d \otimes_A M = 0$ (resp. $A_u \otimes_A M = 0$). If A is a down-up algebra arising from a finite poset then the defining representation of A is both d and u torsion (see [5] for background and some examples). Clearly skew group ring methods can tell us nothing about such modules. In section 3 we study finitely generated modules which are both d-torsion and u-torsion. In particular we obtain necessary and sufficient conditions for all such modules to be finite dimensional.

The question of when two down-up algebras are isomorphic was raised by G. Benkart and T. Roby in [5]. They divided down-up into four types such that no two algebras of different types can be isomorphic. In section 4 we solve the isomorphism problem for Noetherian down-up algebras in three of their cases and in the last case for fields of characteristic 0.

A particularly interesting class of down-up algebras arises in the following way. For $\eta \neq 0$, let A_η be the algebra with generators h, e, f and relations

$$he - eh = e,$$

$$hf - fh = -f,$$

$$ef - \eta fe = h.$$

Then A_η is isomorphic to a down-up algebra $A(1 + \eta, -\eta, 1)$ and conversely any down-up algebra $A(\alpha, \beta, \gamma)$ with $\beta \neq 0 \neq \gamma$ and $\alpha + \beta = 1$ has the above form. For this and other reasons we write $\eta = -\beta$ throughout this paper.. Note that $A_1 \cong U(sl_2)$ and $A_{-1} \cong U(osp(1,2))$, the enveloping algebra of sl_2 and $osp(1,2)$, respectively. In section 5 we study the representation theory of the algebras A_η, for $\eta \neq 0$ in detail. In particular we give a necessary and sufficient conditions for every finite dimensional A_η-module to be semisimple.

Let $\mathbb{N}$ denote the set of positive integers and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

Throughout this paper we will assume that K is an algebraically closed field.

1.1. Note that if we define $deg(d) = 1$ and $deg(u) = -1$, then relations $(R1)$ and $(R2)$ are homogeneous of degree $1, -1$. It follows then that A is a $\mathbb{Z}$-graded ring

$$A = \oplus_{n \in \mathbb{Z}} A(n).$$

Moreover if $\beta \neq 0$ then using the embedding of A into S easily one sees that $A(0) = R = K[ud, du]$, $A(n) = d^n A(0) = A(0)d^n$ if $n \geq 0$ and $A(n) = u^{-n}A(0) = A(0)u^{-n}$ if $n \leq 0$.

1.2.   Unless otherwise stated, we will assume $\beta \neq 0$. Also write $\eta = -\beta$.

Lemma. *Let* $A = A(\alpha, \beta, \gamma)$ *be a down-up algebra. The sets* $\{d^n | n \in \mathbb{N}_0\}$ *and* $\{u^n | n \in \mathbb{N}_0\}$ *are Ore sets in* A.

Proof. Clearly $d = z^{-1}$ is a unit in S. Let

$$B = R[z^{-1}; \sigma] = \{\sum_{i=0}^{m} z^{-i} r_i | r_i \in R\}$$

so that $B \subseteq A \subseteq S = R[z, z^{-1}; \sigma]$. Also any element of S has the form $d^{-n} a = \sigma^{-n}(a) d^{-n}$, where $n \geq 0$ and $a \in B \subseteq A$. Thus $\{d^n | n \in \mathbb{N}_0\}$ is (left and right) Ore in A and $S = A_d$. By [5, §2] there is an antiautomorphism of A interchanging u and d and it follows that $\{u^n | n \in \mathbb{N}_0\}$ is Ore in A. $\square$

1.3.   **Lemma.** *Any unit in* A *belongs to* K^*.
Proof. It is well known that any unit in S has the form az^n with $a \in K^*$ and $n \in \mathbb{Z}$, [16, Proposition VI.1.6]. The result follows since z^n is not a unit in A unless $n = 0$. $\square$

1.4.   Let $R = K[x, y]$ and σ defined as before. Let $f(\lambda) = \lambda^2 - \alpha\lambda - \beta$ and r_1, r_2 its roots.

Note that σ stabilizes the subspace W of R spanned by 1, x and y. It is useful to find w_1, w_2 in R such that 1, w_1, w_2 is a basis for W and the matrix of σ with respect to this basis is in Jordan canonical form. We can take w_1, w_2 as follows:

Case 1: $\alpha^2 + 4\beta \neq 0$ and $\alpha + \beta \neq 1$. Then r_1, r_2 are distinct and both different from 1. We set

$$w_i = \beta(r_i - 1)x + r_i(r_i - 1)y + \gamma r_i,$$

for all $i \in \{1, 2\}$. Then $\sigma(w_i) = r_i w_i$ for all $i \in \{1, 2\}$.

Case 2: $\alpha^2 + 4\beta \neq 0$ and $\alpha + \beta = 1$. In this case $\alpha \neq 2$ and $f(\lambda)$ has roots 1 and η. Set

$$w_1 = \beta x + y$$

$$w_2 = -x + y + \gamma(\alpha - 2)^{-1}.$$

Then $\sigma(w_1) = w_1 + \gamma$ and $\sigma(w_2) = \eta w_2$.

Case 3: $\alpha^2 + 4\beta = 0$ and $\alpha + \beta \neq 1$. Then $f(\lambda)$ has a multiple root $\alpha/2$. Set

$$w_1 = (2\beta + \alpha)x + (\alpha - 2)y + 2\gamma$$

$$w_2 = 2y - 2x.$$

Then $\sigma(w_1) = (\alpha/2)w_1$ and $\sigma(w_2) = (\alpha/2)w_2 + w_1$.

Case 4: $\alpha^2 + 4\beta = 0$ and $\alpha + \beta = 1$. Then $(\alpha, \beta) = (2, -1)$ and 1 is a multiple root of $f(\lambda)$. Set

$$w_1 = -x + y + \gamma$$

$$w_2 = y.$$

Then $\sigma(w_1) = w_1 + \gamma$ and $\sigma(w_2) = w_2 + w_1$.

2. FINITE DIMENSIONAL SIMPLE MODULES OVER DOWN-UP ALGEBRAS

2.1.   We need a construction which is similar to the one given by D. Jordan in [8, 3.1]. Suppose P is a maximal ideal of R such that $\sigma^n(P) = P$ for some $n \in \mathbb{N}$ and suppose n is minimal with this property. Set

$$M_P = \bigoplus_{i=0}^{n-1} R/(\sigma^i(P)).$$

We can make M_P a left S-module by defining for each $i \in \{0, 1, \ldots, n-1\}$ and $r \in R$,

$$z.(r + \sigma^i(P)) = \sigma^{-1}(r) + \sigma^{i-1}(P).$$

Lemma. *Every finite dimensional d-torsion free simple left A-module is isomorphic to M_P for some maximal ideal P of R.*

Proof. Since $A_d \cong S$, M_P is a torsion free left A-module. It is easy to show that M_P is simple.

Conversely, assume that M is a finite dimensional d-torsion free simple left A-module. From the Euclidean algorithm and the fact that M is finite dimensional and d-torsion free it is easy to conclude that $dM = M$. Hence we can identify M with the A_d-module $A_d \otimes_A M$. As M is finite dimensional, M has a finite composition series as a $R = K[x, y]$-module with composition factors isomorphic to R/P_i, for some finite number of distinct maximal ideals $P_1, \ldots, P_n$ of R. As an R-module, we have that

$$M = \oplus_{i=1}^n M(P_i)$$

where $M(P_i) = \{m \in M | P_i^{k_i} m = 0\}$ for some $k_i \in \mathbb{N}$ is an R-submodule of M. As for each $i \in \{1, \ldots, n\}$, $zM(P_i) = M(\sigma^{-1}(P_i))$, we conclude that the maximal ideals $P_1, \ldots, P_n$ are all in a single orbit, and that this orbit is finite.

If $M_i = \{m \in M | P_i m = 0\}$, then $\oplus_{i=1}^n M_i$ is an A_d-submodule of M. Hence we have $M = \oplus_{i=1}^n M_i$, and the result follows. $\square$

2.2.   We investigate when the σ-orbit of a given maximal ideal is finite. The proof of the following lemma will be omitted as it is straightforward.

Lemma. *Let $A = A(\alpha, \beta, \gamma)$ be a down-up algebra and $P = (w_1 - a_1, w_2 - a_2)$ for $(a_1, a_2) \in K^2$. Then there is $n > 0$ such that $\sigma^n(P) = P$ if and only if one of the following holds:*

   i) *In case 1, $(r_i^n - 1)a_i = 0$, for $i = 1, 2$;*
   ii) *In case 2, $n\gamma = (\eta^n - 1)a_2 = 0$;*
   iii) *In case 3, $na_1 = 0$ and $[(\alpha/2)^n - 1]a_i = 0$ for $i = 1, 2$;*

iv) *In case 4, char(K) divides n or $\gamma = a_1 = 0$.*

Following the notation in [5], we denote by $V(\lambda)$ the Verma module of highest weight λ. Let $\lambda_{-1} = 0, \lambda_0 = \lambda$ and define for each $n \in \mathbb{N}$, $\lambda_n = \alpha\lambda_{n-1} + \beta\lambda_{n-2} + \gamma$. The Verma module $V(\lambda)$ has basis $\{v_n | n \in \mathbb{N}_0\}$. The action of $A = A(\alpha, \beta, \gamma)$ is defined as follows, see [5, Proposition 2.2]

$$d.v_0 = 0$$

$$d.v_n = \lambda_{n-1}v_{n-1}, \text{for all } n \geq 1$$

$$u.v_n = v_{n+1}.$$

In [5, Proposition 2.4] it is shown that $V(\lambda)$ is simple if and only if $\lambda_n \neq 0$ for all n. Furthermore if m is minimal with $\lambda_m = 0$, then $M(\lambda) = span_K\{v_j | j \geq m + 1\}$ is a maximal submodule of $V(\lambda)$ and we set $L(\lambda) = V(\lambda)/M(\lambda)$.

Let $\mathfrak{h} = Kud \oplus Kdu$. We say that an A-module V is a weight module if $V = \sum_{\nu \in \mathfrak{h}^*} V_\nu$, where $V_\nu = \{v \in V | h.v = \nu(h)v \text{ for all } h \in \mathfrak{h}\}$, and the sum is over elements in the dual space $\mathfrak{h}^*$ of $\mathfrak{h}$. If $V_\nu \neq 0$, then ν is a weight and V_ν is the corresponding weight space. Each weight ν is determined by a pair of elements (ν', ν'') of K where $\nu' = \nu(du)$ and $\nu'' = \nu(ud)$, and we will identify ν with (ν', ν'').

An $A(\alpha, \beta, \gamma)$-module V is said to be a highest weight module of weight λ if V has a vector v such that $d.v = 0$, $du.v = \lambda v$ and $V = A(\alpha, \beta, \gamma)v$. The vector v is said to be a highest weight vector of V.

2.3. The next Lemma is easily proved by induction, taking into account the recursive construction of the λ_i.

Lemma. 1) *For all $n \in \mathbb{N}$, $\sigma^{-1}(x - \lambda_n, y - \lambda_{n+1}) = (x - \lambda_{n+1}, y - \lambda_{n+2})$.*
2) *For all $n \in \mathbb{N}$, $\sigma^{-n}(x, y - \lambda) = (x - \lambda_{n-1}, y - \lambda_n)$.*

We remark that, if $\{v_n | n = 0, 1, 2, ...\}$ is the basis of the Verma module, $V(\lambda)$, $Kv_i \cong R/\sigma^{-i}((x, y - \lambda))$ and we can write the Verma modules using the notation of [8], as

$$V(\lambda) \cong \oplus_{i \geq 0} R/\sigma^{-i}((x, y - \lambda)).$$

Also if $dim(L(\lambda)) = n$ and we set for a given maximal ideal $\underline{m}$ of $R = K[x, y]$, $L[\underline{m}] = \{a \in L(\lambda) | \underline{m}a = 0\}$, then

$$L(\lambda) = \oplus_{i=0}^{n-1} L[\sigma^{-i}((x, y - \lambda))].$$

If $L[\underline{m}] \neq 0$, we say that $\underline{m}$ is a weight of $L(\lambda)$.

2.4. **Proposition.** *Every finite dimensional simple A-module M is isomorphic to one of the following*

1) *a simple homomorphic image of a Verma module;*
2) *M_P where P is a maximal ideal of R which has a finite σ-orbit.*

Proof. Let M be any finite dimensional simple A-module. The A_d-module $A_d \otimes_A M$ is either simple or zero, by [7, Theorem 9.17].

If $A_d \otimes_A M = 0$, M is d-torsion. Let $M_0 = \{m \in M | dm = 0\}$. Then $(ud)M_0 = 0$ and as $d^2 um = \alpha dudm + \beta ud^2 m + \gamma dm = 0$, for all $m \in M_0$, we have $(du)M_0 \subseteq M_0$. Hence there is $m \in M_0$ such that $dum = \lambda m$ for $\lambda \in K$. Now Am is a highest weight module of weight λ, hence a homomorphic image of the Verma module $V(\lambda)$, [5, Proposition 2.8]. As M is a simple module we have $M = Am$.

If $A_d \otimes_A M \neq 0$, then as M is simple, M is torsion free and we can identify M with $A_d \otimes_A M$. The result follows now by Lemma 2.1. $\square$

Remarks.   i) *The simple homomorphic images of Verma modules are described in [5, Corollary 2.28], see also the addendum to [5].*
ii) *We mention that analogues of the category $\mathcal{O}$ of modules over a semisimple Lie algebra are introduced for down-up algebras in [5, Sections 4 and 5]. In addition the representation theory of a down-up algebra is related to its left spectrum in [14].*

Corollary. *Let $A(\alpha, \beta, \gamma)$ be a down-up algebra with the parameters α, β, γ satisfying case (2) of §1.4 and assume that $\gamma \neq 0$. Then any Verma module $V(\lambda)$ has a unique maximal submodule $M(\lambda)$. Also any finite dimensional simple A-module is d-torsion and isomorphic to $L(\lambda)$ for some λ.*

Proof. Necessary and sufficient conditions for the weight spaces of $V(\lambda)$ to be one dimensional are given in [5, Theorem 2.13], and in particular these conditions hold in case (2) of §1.4 when $\gamma \neq 0$. The statement about Verma modules now follows from [5, Proposition 2.23]. By Lemma 2.2 any finite dimensional simple A-module is d-torsion, and so has the form $L(\lambda)$ by the proof of Proposition 2.4. $\square$

2.5. Assume that K has characteristic zero. As mentioned in the introduction the algebras A_η are exactly the down-up algebras $A(\alpha, \beta, \gamma)$ with $\alpha + \beta = 1$ and $\gamma \neq 0$. The next result shows that the representation theory of these algebras has certain similarities with that of $U(osp(1,2))$ and $U(sl_2)$. We return to this topic in section 5.

Note that the algebra $U(sl_2)$ is the only down-up algebra with $\gamma \neq 0$ whose parameters satisfy case (4). We ignore this case below.

The recurrence relation for the λ_n is solved explicitly in [5, Proposition 2.12]. We use this result below.

Lemma. *Assume char(K)=0. Let $A = A_\eta$ with $\eta \neq 1$. Then*

    i) *if $\lambda_{n-1} = 0$ and $\eta^n = 1$ then $n = 0$;*

    ii) *$\lambda_{n-1} = 0$ if and only if*

$$\lambda(\eta - 1) = -\gamma(1 - n(\sum_{i=0}^{n-1} \eta^i)^{-1}).$$

Proof. By [5, Proposition 2.12 (i)], $\lambda_n = c_1 r_1^n + c_2 r_2^n + x_n$ where $r_1 = 1$, $r_2 = \eta$, $x_n = \gamma n(1 - \eta)^{-1}$, $c_1 = (\eta - 1)^{-1}[-\lambda - \gamma + \gamma(1 - \eta)^{-1}]$ and $c_2 = (\eta - 1)^{-1}[\eta\lambda + \gamma - \gamma(1 - \eta)^{-1}]$. Thus

$$\lambda_n = \frac{\eta^{n+1} - 1}{\eta - 1}\lambda + \frac{\eta^{n+1} - 1}{(\eta - 1)^2}\gamma - \frac{(n+1)\gamma}{\eta - 1}.$$

Therefore if $\lambda_{n-1} = 0$ and $\eta^n = 1$, then since $\gamma \neq 0$, it follows that

$$n(1 - \eta) = 0$$

so $n = 0$. This proves *i)* and *ii)* follows by multiplying the expression for λ_{n-1} by $(\eta - 1)^2/(\eta^n - 1)$. $\square$

Proposition. *Let $A = A_\eta$ with $\eta \neq 1$. The only finite dimensional simple modules of dimension $n \in \mathbb{N}$ are d-torsion modules of the form $L(\lambda)$ where n is the least positive integer such that λ satisfies $\lambda_{n-1} = 0$.*

Proof. Let A be a down-up algebra as in the statement of the proposition. By Corollary 2.4 we have that the only finite dimensional simple A-modules are d-torsion and of the form $L(\lambda)$.

By construction, the dimension of $L(\lambda)$ is n if and only if n is minimal with $\lambda_{n-1} = 0$, and the result follows. $\square$

It is well known that all simple modules over $U(osp(1, 2))$ have odd dimension. The next result gives a generalization of this fact.

Corollary. *Assume $char(K) = 0$. Let $A = A_\eta$ with $\eta \neq 1$ and η a primitive N^{th} root of unity. Then*

    i) *if n is a multiple of N there are no finite dimensional simple modules of dimension n.*

    ii) *if n is not a multiple of N there is a unique finite dimensional simple module of dimension n.*

Proof. Immediate from the Proposition and the Lemma. $\square$

2.6. By [5, §2], there is an antiautomorphism τ of A given by $\tau(u) = d$, $\tau(d) = u$. Let $\mathcal{C}$ be the category of finite dimensional left A-modules. If $M \in \mathcal{C}$, let M^* be the dual vector space to M. Then M^* is a left A-module via

$$(af)(m) = f(\tau(a)m)$$

for $a \in A, f \in M^*, m \in M$. We can now define a contragradient functor $(\ )^*$ on the category $\mathcal{C}$ as follows. If $M \in Obj(\mathcal{C})$, let M^* be the dual vector space to M and if $\psi : M \longrightarrow N$ is a map of left A-modules, then $\psi^* : N^* \longrightarrow M^*$ is the transpose of ψ.

Next suppose $L(\lambda)$ is a simple finite dimensional highest weight module with dimension n. Then $L(\lambda)$ has a basis $v_0, \dots, v_{n-1}$ such that

$$uv_i = v_{i+1} \qquad 0 \leq i \leq n-2, uv_{n-1} = 0,$$
$$dv_i = \lambda_{i-1}v_{i-1} \qquad 1 \leq i \leq n-1, dv_0 = 0.$$

Let $f_0, \dots, f_{n-1}$ be the basis of $L(\lambda)^*$ such that $f_i(v_j) = \delta_{ij}$ for all i, j. Then,

$$uf_i = \lambda_i f_{i+1} \qquad 0 \leq i \leq n-2, \qquad uf_{n-1} = 0,$$
$$df_i = f_{i-1} \qquad 1 \leq i \leq n-1, \qquad df_0 = 0.$$

In particular f_0 is a highest weight vector with weight λ and $L(\lambda)^* \cong L(\lambda)$.

Now suppose that $L(\lambda), L(\mu)$ are finite dimensional highest weight modules and that $Ext(L(\mu), L(\lambda)) \neq 0$. Then there is a nonsplit exact sequence

$$0 \longrightarrow L(\lambda) \longrightarrow M \longrightarrow L(\mu) \longrightarrow 0$$

of A-modules. Dualizing we obtain $Ext(L(\lambda), L(\mu)) \neq 0$.

Corollary. *Suppose that $L(\lambda), L(\mu)$ are finite dimensional. Then $Ext(L(\lambda), L(\mu)) = 0$ if and only if $Ext(L(\mu), L(\lambda)) = 0$.*

3. Down-up modules

3.1. In this subsection and 3.2, β is allowed to be zero.

Let $A = A(\alpha, \beta, \gamma)$ be a down-up algebra and M a left A-module. We define two filtrations on M and view d and u as operators which move down and up these filtrations (whence the title of this section). The filtrations need not to be exhaustive. For any r and s in $\mathbb{N}$, we define

$$M_r = \{m \in M | d^{r+1}m = 0\},$$
$$M^s = \{m \in M | u^{s+1}m = 0\},$$

and

$$M_r^s = M_r \cap M^s.$$

It is obvious that

$$M_0 \subseteq M_1 \subseteq M_2 \subseteq \dots$$

and that $M_r = \{m \in M | dm \in M_{r-1}\}$.

If $\beta \neq 0$ then $\cup M_r$ is the d-torsion submodule and $\cup M^s$ is the u-torsion submodule of M.

Lemma. *For any $r \in \mathbb{N}$, $uM_r \subseteq M_{r+1}$ and $dM^s \subseteq M^{s+1}$.*

Proof. Let $m \in M_0$. By $(R1)$ we have $d^2um = 0$ and so $uM_0 \subseteq M_1$. Suppose $uM_n \subseteq M_{n+1}$ for all $n < r$. Then $duM_n \subseteq M_n$ for all $n < r$. If $m \in M_r$, then since $dudm \in (du)M_{r-1} \subseteq M_{r-1}$, $ud^2m \in uM_{r-2} \subseteq M_{r-1}$ and $dm \in M_{r-1}$ we have

$$d^2um = (\alpha dud + \beta ud^2 + \gamma d)m \in M_{r-1}.$$

It follows that $uM_r \subseteq M_{r+1}$.

The other inclusion is proved in a similar way. $\square$

Corollary. *For any $r, s \in \mathbb{N}$, $dM_r^s \subseteq M_{r-1}^{s+1}$ and $uM_r^s \subseteq M_{r+1}^{s-1}$.*

Proof. Follows easily from Lemma 3.1. $\square$

3.2. Let $M^\infty = \cup M^s$, $M_\infty = \cup M_r$ and for each $t \in \mathbb{N}$ let $M(t) = \sum_{r+s=t} M_r^s$.

It follows easily that the sets M^∞, M_∞ and $M(t)$ are A-submodules of M.

Proposition. *If M is a Noetherian A-module such that $M = M^\infty = M_\infty$ then $M = M(t)$ for some $t \in \mathbb{N}$.*

Proof. Consider the chain of A-submodule of M, $M(1) \subseteq M(2) \subseteq \ldots$. Choose $t \in \mathbb{N}$ such that $M(t) = M(t+s)$ for all $s \in \mathbb{N}$. If $m \in M$ then there are $p, q \in \mathbb{N}$ such that $d^{p+1}m = u^{q+1}m = 0$ and this implies that $m \in M(t)$. $\square$

3.3. Given a module M as in Proposition in 3.2, we study conditions for M to be finite dimensional. From now on we will assume that $\beta \neq 0$.

We define two sequences of elements of $R = K[x, y]$

$$x_0 = 1, y_0 = 1$$

$$x_n = \sigma(x_{n-1}x) \quad \text{and} \quad y_n = x\sigma^{-1}(y_{n-1})$$

for any $n \in \mathbb{N}$. We claim that for any $n \in \mathbb{N}$, $d^n u^n = x_n$.

For $n = 0$ this is obvious. The induction step follows from

$$d^{n+1}u^{n+1} = dx_n u = z^{-1}x_n xz = \sigma(x_n x) = x_{n+1}.$$

Similarly we have for all $n \in \mathbb{N}$, $u^n d^n = y_n$.

Lemma.   i) *If $\sigma^n(x) \notin (x)$ for all $n \in \mathbb{N}$, then $x_t \notin (x)$, for all $t \in \mathbb{N}$;*

ii) *if $\sigma^{-n}(x) \notin (y)$ for all $n \in \mathbb{N}_0$, then $y_t \notin (y)$, for all $t \in \mathbb{N}_0$.*

Proof. For each $t \in \mathbb{N}$ write $x_t = \prod_{i=1}^{t} \sigma^i(x)$ and $y_t = \prod_{i=1}^{t} \sigma^{-(i-1)}(x)$. The result follows since (x), (y) are prime ideals of R. $\square$

Remark. We note that the conditions $\sigma^n(x) \notin (x)$ for all $n \geq 1$ and $\sigma^{-n}(x) \notin (y)$ for all $n \geq 0$ are equivalent. Indeed if there is $n \in \mathbb{N}$ such that $\sigma^n(x) \in (x)$ then $\sigma^n(x) = \lambda x$ for some $\lambda \in K^*$. So $y = \sigma(x) = \lambda \sigma^{1-n}(x)$ and $\sigma^{1-n}(x) \in (y)$. The converse follows by a similar argument.

3.4. We now state the main result of this section.

Theorem. *Assume that K is an algebraically closed field and that $\beta \neq 0$. Let M be a finitely generated left A-module and suppose that $\sigma^n(x) \notin (x)$ for all $n \in \mathbb{N}$. Then $A_d \otimes_A M = A_u \otimes_A M = 0$ if and only if M has a finite filtration whose factor modules are finite dimensional modules of the form $L(\lambda)$ for various $\lambda \in K$.*

Conversely if every finitely generated left A-module which is both d-torsion and u-torsion has finite dimension then $\sigma^n(x) \notin (x)$ for all $n \in \mathbb{N}$.

Proof. Obviously if M has a finite filtration whose factor modules are finite dimensional modules of the form $L(\lambda)$ then M is d-torsion and u-torsion.

Let M be a finitely generated left A-module such that $A_d \otimes_A M = A_u \otimes_A M = 0$. Then M is d-torsion and u-torsion or equivalently, $M = M^\infty = M_\infty$.

As M is a finitely generated left A-module and A is Noetherian, so is M and by Lemma 3.2 it follows that $M = M(t)$ for some t.

Assume that $\sigma^n(x) \notin (x)$ for all $n \in \mathbb{N}$. It is enough to show that M contains a finite dimensional submodule N which is a highest weight module. The only such modules which are u-torsion are those of the form $L(\lambda)$ so we can set $N = N_1$ and use the same argument to construct $0 \subset N_1 \subset N_2 \subset \ldots$ provided $(M/N_i) \neq 0$.

Pick $m \in M_0$ such that $m \neq 0$. Then $dm = 0$. Since $M = M(t)$ we have $u^{t+1}m = 0$. Hence

$$x_{t+1}m = d^{t+1}u^{t+1}m = 0.$$

By Lemma 3.3, $x_{t+1} \notin (x)$ so $\operatorname{ann}_A(m)$ contains a nonzero polynomial in du. Therefore du has a nonzero eigenvector in M_0 and the result follows.

Assume that every finitely generated d-torsion and u-torsion A-module has finite dimension. Fix $n \geq 1$ and set $I = Au^n + Ad$, $J = I \cap K[x,y]$ and $M = A/I$. Since $\{d^m | m \in \mathbb{N}\}$ and $\{u^m | m \in \mathbb{N}\}$ are Ore sets, M is d-torsion and u-torsion. Thus M has finite dimension.

Since $K[x,y]/J$ embeds in M, J has finite codimension in $K[x,y]$. Using the graded ring structure of A, 1.1, it is easily seen that J is the ideal of $K[x,y]$ generated by x and x_n. Hence $x_n \notin (x)$ and so $\sigma^n(x) \notin (x)$. $\square$

3.5. Next we give an example where the conclusions of Theorem 3.4 do not hold.

Example. *Let $A = A(0, \beta, 0)$ and consider the A-module M with a basis $\{m_i, n_i | i \in \mathbb{N}\}$ and such that $um_i = n_i$, $dn_i = m_{i+1}$, $un_i = 0$ and $dm_i = 0$.*

In this case $d^2 M = u^2 M = 0$ so the relations $d^2u = \beta ud^2$, $du^2 = \beta u^2 d$ are obviously satisfied, and M is generated as an A-module by m_1.

4. The Isomorphism problem for Down-up Algebras

Benkart and Roby divide down-up algebras into four classes such that no two algebras from different classes are isomorphic (see Proposition 4.2 below). Here we solve the isomorphism problem for three of the classes and make substantial progress on the fourth.

4.1. First we note the existence of certain isomorphisms and automorphisms.

Lemma. i) If $\beta \neq 0$, then $A(\alpha, \beta, \gamma) \cong A(-\alpha\beta^{-1}, \beta^{-1}, -\gamma\beta^{-1})$ via the map interchanging d and u.

ii) If $\gamma \neq 0$ then $A(\alpha, \beta, \gamma) \cong A(\alpha, \beta, 1)$.

iii) If $A = A(\alpha, \beta, \gamma)$, $A' = A(\alpha', \beta', \gamma')$ and $\Psi : A \longrightarrow A'$ is an isomorphism with $\Psi(d) = \lambda d'$, $\Psi(u) = \mu u'$ and $\lambda, \mu \in K^*$ then $\alpha' = \alpha$ and $\beta' = \beta$.

iv) If $\beta \neq 0$, $A = A(\alpha, \beta, \gamma)$, $A' = A(\alpha', \beta', \gamma')$ and $\Psi : A \longrightarrow A'$ is an isomorphism with $\Psi(d) = \lambda u'$, $\Psi(u) = \mu d'$ and $\lambda, \mu \in K^*$ then $\alpha' = -\alpha\beta^{-1}$ and $\beta' = \beta^{-1}$.

Proof. Straightforward $\square$

4.2. Next we consider commutative homomorphic images of $A = A(\alpha, \beta, \gamma)$. Let $I = \cap\{J | A/J \text{ is commutative}\}$. Note that $B = A/I$ is commutative since it is a subdirect product of commutative rings. Thus B is the largest commutative image of A and $Spec(B)$ should perhaps be thought of as the largest commutative subscheme of $Spec(A)$. The algebra B is given by adding the relations $du = ud$ to the defining relations for A. Note that the closed points of $Spec(B)$ correspond to the one-dimensional A-modules, so we recover [5, Theorem 6.1].

Similarly let $A' = A(\alpha', \beta', \gamma')$ be another down-up algebra and I' be the unique smallest ideal of A' such that A'/I' is commutative. Suppose there exists an isomorphism Ψ from A onto A'. It is easily seen that $\Psi(I) = I'$. Moreover

$$\Psi(\sum\{P | P \text{ minimal over } I\}) = \sum\{P' | P' \text{ minimal over } I'\}.$$

The images a, b of d, u in B satisfy

$$(R3) \qquad a(ab(1 - \alpha - \beta) - \gamma) = 0$$

$$(R4) \qquad b(ab(1 - \alpha - \beta) - \gamma) = 0.$$

Thus we obtain.

Proposition. *The largest commutative homomorphic image B of A is the factor ring of the commutative polynomial ring $K[a, b]$ defined by $(R3), (R4)$. In particular one of the following cases holds*

(a) $\gamma = 0$, $\alpha + \beta = 1$ and $B = K[a, b]$;

(b) $\gamma = 0$, $\alpha + \beta \neq 1$, $B = K[a,b]/(a^2 b, ab^2)$ *and the primes of A minimal over I are (d) and (u);*

(c) *If $\gamma \neq 0$, $\alpha + \beta \neq 1$ then the primes of A minimal over I have the form $M_0 = (d,u)$ and $P = \mathrm{Ker}\,(\Phi)$ where $\Phi : A \longrightarrow K[v,v^{-1}]$ is defined by $\Phi(u) = (1 - \alpha - \beta)^{-1} v$ and $\Phi(d) = \gamma v^{-1}$.*

(d) *If $\gamma \neq 0$, $\alpha + \beta = 1$ then $B = K$ and $I = (d,u)$.*

4.3.   We say that the down-up algebra $A = A(\alpha, \beta, \gamma)$ has type (a) (resp. (b), (c), (d)) if the parameters α, β, γ satisfy condition (a) (resp. (b), (c), (d)) of Proposition 4.2.

Let $A = A(\alpha, \beta, \gamma)$ and $A' = A(\alpha', \beta', \gamma')$ be down-up algebras of the same type and let d', u' be the generators of A'. If A and A' have type (a) or (d) and $\eta \neq 1 \neq \eta'$, let w_2 and w_2' be the elements constructed in case (2) of 1.4. If A and A' have type (c), let P be the ideal of A defined in Proposition 4.2 and P' the corresponding ideal of A'.

Corollary. *With the notation as above assume that A and A' are isomorphic via Ψ. We have*

i) *If A, A' have type (a), then $\Psi(w_2) = (w_2')$;*

ii) *If A, A' have type (b), then $\Psi(d,u) = (d',u')$;*

iii) *If A, A' have type (c), then $\Psi(P) = P'$.*

Proof. If A and A' have type (a) then using the decomposition in 1.1, we see that $w_2 = -x + y$ generates the ideal I. Hence (i) follows from the remarks before Proposition 4.2. The proofs of the remaining statements are similar. $\square$

4.4.   Before stating our main result on the isomorphism problem, it is worth commenting on the geometry of one-dimensional representations, for algebras of type (c). The maximal ideals of $K[v,v^{-1}]$ have the form $(v - \mu)$, $\mu \in K^*$, so we have homomorphisms $\Phi_\mu : A \longrightarrow K$ given by

$$\Phi_\mu(u) = (1 - \alpha - \beta)\mu, \;\; \Phi_\mu(d) = \gamma \mu^{-1}.$$

Set $M_\mu = Ker(\Phi_\mu)$. Then as in [5], the one dimensional modules are indexed by $K^* \cup \{0\} = K$. However from Proposition 4.2 we might expect the ideal M_0 to behave differently from the other M_μ. Indeed we have

Lemma. *For algebras of type (c), $M_\mu^2 = M_\mu$ if and only if $\mu = 0$.*

Proof. Since $M_0 = (d,u)$ and $\gamma \neq 0$, relations $(R1),(R2)$ imply that $M_0^3 = M_0$. On the other hand if $\mu \neq 0$ then $(v - \mu) \neq (v - \mu)^2$ in the commutative ring $K[v,v^{-1}]$, so it follows that $M_\mu \neq M_\mu^2$. $\square$

4.5. It follows from [5, Corollary 6.2] that if two down-up algebras are isomorphic then they have the same type. The next result gives a partial answer to [5, Problem (h)] and a partial converse of Lemma 4.1.

Theorem. *Suppose that $A = A(\alpha, \beta, \gamma)$ and $A' = A(\alpha', \beta', \gamma')$ are Noetherian down-up algebras of the same type. If both have type (d) assume also that $char(K) = 0$. Then $A \cong A'$ if and only if*

$$\textit{1)} \quad \gamma = 0 \textit{ if and only if } \gamma' = 0 \textit{ and}$$
$$\textit{either} \quad \textit{2)} \quad \alpha = \alpha', \ \beta = \beta'$$
$$\textit{or} \quad \textit{3)} \quad \alpha' = -\alpha\beta^{-1}, \ \beta' = \beta^{-1}.$$

4.6. Assume that $A = A(\alpha, \beta, \gamma)$ with $\alpha + \beta = 1$, and $\eta \neq 1$. Then case 2 of 1.4 holds and we set $w = w_2$, so that $\sigma(w) = \eta w$. Recall the definition of $A(n)$ from 1.1.

Lemma. *The set $\{a \in A | aw = \eta^m wa\}$ equals*

$$A(m) \qquad \textit{if} \quad \eta \textit{ is not a root of unity}$$
$$\bigoplus \{A(m') | m' \equiv m \ (mod \, n)\} \quad \textit{if} \quad \eta \textit{ is a primitive } n^{th} \textit{ root of unity.}$$

Proof. This is proved by computation using the decomposition in 1.1. $\square$

4.7. Now assume that A is a down-up algebra of type (a) or (b).

Lemma. *Let A and A' be down-up algebras both of type (a) with $\eta \neq 1 \neq \eta'$ or of type (b). Assume that A and A' are isomorphic via Ψ. Then $\Psi(d, u) \subseteq (d', u')$.*

Proof. In type (b) this follows directly from Corollary 4.3. Suppose that A, A' have type (a) and that $\eta \neq 1 \neq \eta'$. By Corollary 4.3 and the fact that A has only trivial units we have $\Psi(w) = \lambda w'$ for some $\lambda \in K^*$, where $w' \in A'$ is defined in a similar manner to w. Applying Ψ to the equation $dw = \eta wd$, we get by Lemma 4.6, $\Psi(d) \in \sum \{A'(m) | (\eta')^m = \eta\} \subseteq (d', u')$. Similarly $\Psi(u) \in (d', u')$. $\square$

4.8. **Proof of Theorem 4.5 for type (a) and type (b).**

Adopting some terminology from group theory we say that a subset X of $A = A(\alpha, \beta, \gamma)$ is *characteristic* if $\Psi(X) = X$ for all $\Psi \in Aut(A)$.

Assume that A and A' are down-up algebras both of type (a) or type (b). Assume as well that $\beta, \beta' \neq -1$ if A and A' are of type (a). Since $\gamma = 0$, $A = \oplus_{m \in \mathbb{N}_0} A[m]$ is a graded algebra with $A[0] = K$ and $A[1] = span\{d, u\}$ and similarly for A'. Set $A_n = \oplus_{m \geq n} A[m]$. Then $(d, u) = A_1$ and $(d, u)^n = A_1^n = A_n$. Hence if $\Psi : A \longrightarrow A'$ is an isomorphism we have $\Psi(A_n) \subseteq A'_n$ by Lemma 4.7. This means that Ψ induces an isomorphism of graded algebras $\oplus A_n/A_{n+1} \longrightarrow \oplus A'_n/A'_{n+1}$. However $A \cong \oplus A_n/A_{n+1}$ as graded algebras. Thus we may assume that Ψ is an isomorphism of graded algebras. As noted in [13, Theorem 4.1 and Lemma 4.2] A is Auslander regular of global dimension 3. Since A has GK-dimension 3 by [4, 4.2]

it follows from [10, Theorem 6.3] that A is Artin-Schelter regular. Therefore the isomorphism type of A as a graded algebra is determined in [1]. However instead of appealing to [1] we can now complete the proof with a short calculation. Since the relations are of degree three we work $mod A'_4$. Note that the algebra A/A_4 inherits the $\mathbb{Z}$-grading of 1.1. Write

$$\Psi(d) = bd' + cu'$$

$$\Psi(u) = rd' + su'$$

$mod A'_2$. Obviously $\Delta = bs - cr \neq 0$. Suppose first that A, A' have type (b), so $\alpha + \beta \neq 1$. Applying Ψ to the relation $d^2u - \alpha dud - \beta ud^2 = 0$ and looking at the terms of degree 3 and -3 we see that $br = cs = 0$. Since $\Delta \neq 0$ this gives two possibilities. Either $b \neq 0 \neq s$ and $c = r = 0$ or $c \neq 0 \neq r$ and $b = s = 0$. It is easily seen that we have one of the two statements of the Theorem.

Now suppose that A and A' have type (a) and $\beta, \beta' \neq -1$. Applying Ψ to the relation $d^2u - \alpha dud - \beta ud^2$ and cancelling Δ we obtain

$$0 = b[(d')^2u' - \alpha d'u'd' - \beta u'(d')^2] + c[\beta d'(u')^2 + \alpha u'd'u' - (u')^2 d']$$

$mod A'_4$. Comparing to the relations in A' gives the result.

Finally suppose that A, A' are down-up algebras of type (a), $\eta = 1$ and $\beta' \neq -1$. By Lemma 4.7, A' has a characteristic ideal of the form (d', u'). Since $\eta = 1$, $A = U(\mathfrak{h})$, the enveloping algebra of the Heisenberg algebra $\mathfrak{h}$. Now $\mathfrak{h}$ has basis $\{x, y, z\}$ such that $[x, y] = z$ is central in $\mathfrak{h}$. By [5, Theorem 6.1] any codimension one ideal in A has the form $(x - a, y - b, z)$. It is easy to see that $Aut(A)$ acts transitively on maximal ideals of codimension 1, hence there are no characteristic ideals of codimension 1. This proves Theorem 4.5 for type (a) and (b).

4.9. **Proof of Theorem 4.5 for type (c).**

To solve the isomorphism problem for down-up algebras of type (c) we consider the class of bimodules over $C = K[v, v^{-1}]$ which are free of rank n on the left. Let F be such a bimodule with basis $e_1, \ldots, e_n$. We can define a $\mathbb{Z}$-grading $\{F(m)|m \in \mathbb{Z}\}$ on F by $F(m) = \sum_{i=1}^{n} Kv^m e_i$. We assume that the right C-action preserves this grading, that is $e_i v \in F(1)$ for all i. Then we can write

$$e_i v = \sum_j p_{ij} v e_j$$

for some $n \times n$ matrix $P = (p_{ij})$ with $p_{ij} \in K$. If ϕ is an automorphism of F and

$$\phi(e_i) = \sum q_{ij} e_j = f_i$$

with $q_{ij} \in C$, then the right action of v on the basis $f_1, \ldots f_n$ is determined by the matrix QPQ^{-1}. Thus the isomorphism class of the bimodule F is determined by the conjugacy class of P under the action of $GL_n(C)$. An element t in a ring T is normal if $tT = Tt$.

Lemma. *Suppose $A = A(\alpha, \beta, \gamma)$ is a down-up algebra of type (c), let $\Phi : A \longrightarrow C = K[v, v^{-1}]$ be the homomorphism described in Proposition 4.2 and $P = \mathrm{Ker}\,(\Phi)$. Then P/P^2 is free of rank 2 as a left and right C-module. In addition one of the following holds:*

i) $\alpha^2 + 4\beta \neq 0$. Let r_1, r_2 be the roots of the polynomial $f(\lambda)$ and

$$w_i = \beta(r_i - 1)ud - r_i(r_i - 1)du + \gamma r_i.$$

Then w_1, w_2 are normal elements of A and $P = (w_1, w_2)$. The bimodule P/P^2 is free of rank two on the left and right as a C-module whose isomorphism class is determined by the conjugacy class of the matrix

$$\begin{bmatrix} r_1 & 0 \\ 0 & r_2 \end{bmatrix}$$

ii) $\alpha^2 + 4\beta = 0$. Then $\alpha/2$ is a double root of $f(\lambda)$. Let

$$w_1 = (2\beta + \alpha)ud + (\alpha - 2)du + 2\gamma$$

$$w_2 = 2(du - ud).$$

Then w_1 is normal in A, the image of w_2 is normal in $A/(w_1)$ and $P = (w_1, w_2)$. The bimodule P/P^2 is free of rank two on the left and right as a C-module whose isomorphism class is determined by the conjugacy class of the matrix

$$\begin{bmatrix} \alpha/2 & 1 \\ 0 & \alpha/2 \end{bmatrix}$$

Proof. We prove only part ii). The proof of part i) is similar. The fact that w_1 is normal in A and w_2 normal $mod(w_1)$ follows from 1.4. A short computation shows that $w_1, w_2 \in \mathrm{Ker}\,(\Phi)$. Using the decomposition of A as a $\mathbb{Z}$-graded ring in 1.1 we see that P/P^2 is free as a left and right C-module with basis $\overline{w}_i = w_i + P^2$, $i = 1, 2$. From 1.4, we obtain

$$\overline{w}_1 v = v(\alpha/2\overline{w}_1 + \overline{w}_2),$$

$$\overline{w}_2 v = (\alpha/2)v\overline{w}_2$$

and the result follows. $\square$

Before concluding the proof of Theorem 4.5 in this case we need another definition. Suppose that M is a C-bimodule and $\mu \in Aut(C)$. The C-bimodule twisted by μ has the same underlying vector space as M, and has bimodule structure maps $C \times M \to M$, $M \times C \to M$ given by $(c, m) \mapsto \mu(c)m$ and $(m, c) \mapsto m\mu(c)$.

Now suppose $\Psi : A \to A'$ is an isomorphism of down-up algebras of type (c) and let $\Phi : A \to C$, $\Phi' : A' \to C$ be the maps described in Proposition 4.2 (c). Let $P = \mathrm{Ker}\,(\Phi)$, $P' = \mathrm{Ker}\,(\Phi')$ and write $\phi : A/P \to C$, $\phi' : A'/P' \to C$ for the induced isomorphisms. There is an automorphism μ of C satisfying $\mu\Phi = \Phi'\Psi$. Since $\Psi(P) = P'$, by 4.3, Ψ induces a linear isomorphism from P/P^2 to $P'/(P')^2$.

If we regard P/P^2 as a C-bimodule via ϕ^{-1} and $P'/(P')^2$ as a C-bimodule via $(\phi')^{-1}\mu$ then the above map is an isomorphism of C-bimodules.

Now the isomorphism type of P/P^2 (resp. $P'/(P')^2$) is determined by the conjugacy classes of a matrix J, respectively J', as in the Lemma. It follows that these bimodules structures can be obtained from one another twisting by $\mu, \mu^{-1} \in Aut(C)$.

There are now two possibilities. If $\mu(v) = \lambda v$ for some $\lambda \in K^*$, then J, J' are conjugate and we have conclusion (2) of Theorem 4.5 while if $\mu(v) = \lambda v^{-1}$ for some $\lambda \in K^*$, then J^{-1} and J' are conjugate and we have conclusion (3).

4.10.   We want to establish an analogue of Corollary 4.3 (a) for down-up algebras of type (d).

Let A be a down-up algebra of type (d) and assume $\eta \neq 1$. Let $w_2 = du - ud + \gamma(\eta-1)^{-1}$. From section 1.4 we have $dw_2 = \eta w_2 d$ and $w_2 u = \eta u w_2$. In particular w_2 is a normal element. Clearly $A/(w_2)$ is isomorphic to the first Weyl algebra since $\gamma \neq 0$. Thus (w_2) is a completely prime ideal of A.

Lemma. *Let A be a down-up algebra of type (d) and assume that $char(K) = 0$ and $\eta \neq 1$. If P is a completely prime ideal of A such that A/P has infinite dimension over K, then $P = (w_2)$.*

Proof. Suppose P is a completely prime ideal such that A/P has infinite dimension over K. Then $(d, u) \not\subseteq P$ so assume that $d \notin P$ and localize at d. If $u \notin P$ we localize at u instead and use a similar argument. Then $Q = P_d$ is a nontrivial ideal of $S = A_d = R[z, z^{-1}; \sigma]$. If $Q \cap R = 0$ we can localize at $R\backslash\{0\}$ to obtain a nontrivial ideal in $F[z, z^{-1}; \sigma]$ where $F = Fract(R)$. However it follows from section 1.4 and the assumption that $char(F) = 0$, that σ has infinite order. Hence by [15, Theorem 1.8.5], $F[z, z^{-1}; \sigma]$ is a simple ring.

This contradiction shows that $I = Q \cap R \neq 0$. Now by 1.4, $R = K[w_1, w_2]$ where $\sigma(w_1) = w_1 + \gamma$ and $\sigma(w_2) = -\beta w_2$.

Now R/I embeds in S/Q which is a domain, so I is prime and clearly σ-invariant. By choosing a polynomial of least degree in w_2 with coefficients in $K[w_1]$ we see that $(w_2) \subseteq I$ and the lemma follows easily. $\square$

Corollary. *Assume that $char(K) = 0$ and $\eta \neq 1 \neq \eta'$. Let $A = A(\alpha, \beta, \gamma)$ and $A' = A(\alpha', \beta', \gamma')$ be isomorphic down-up algebras of type (d) via the isomorphism Ψ. Let w_2 be the element of A defined above and w_2' the corresponding element of A'. Then $\Psi(w_2) = (w_2')$.*

4.11.   **Proof of Theorem 4.5 for type (d) and char(K)=0.**

Suppose first that $\alpha^2 + 4\beta \neq 0$ and $(\alpha')^2 + 4\beta' \neq 0$. Using the notation of Corollary 4.10 we have

$$\Psi(w_2) = \lambda w_2'$$

for some $\lambda \in K^*$ since A' has no nontrivial units. By Corollary 2.5, η is an nth root of unity if and only if η' is an nth root of unity.

If η and η' are not roots of unity, applying Ψ to the equation $dw_2 = \eta w_2 d$ and using Lemma 4.6 gives $\Psi(d) \in \sum \{A'(m) | m \in \mathbb{Z}, (\eta')^m = \eta\}$. There can be at most one $m \in \mathbb{Z}$ such that $(\eta')^m = \eta$. If $m > 0$ then $\Psi(d) \in (d')^m R \subseteq d'A'$ and $\Psi(u) \in (u')^m R \subseteq u'A'$. If $m < 0$, $\Psi(d) \in (u')^{-m} R \subseteq u'A'$ and $\Psi(u) \in (d')^{-m} R \subseteq d'A'$. Similar remarks apply to the map Ψ^{-1}. As A has no nontrivial units, the result follows from Lemma 4.1.

Assume now that η, η' are primitive n^{th} roots of unity, $n > 1$ and that $\gamma = 1$. If $n = 2$, then $\eta = \eta' = -1$, so we can assume $n > 2$. By Proposition 2.5, there are unique simple modules L_1, L_2 of dimension $1, 2$ and these modules are weight modules with highest weights 0 and $-\alpha^{-1} = (\beta - 1)^{-1}$, respectively. Since $w_2 = du - ud - (\beta + 1)^{-1}$ we see that w_2 acts on the highest weight vectors of L_1, L_2 by the eigenvalues $-(\beta+1)^{-1}$ and $2(\beta^2 - 1)^{-1}$, respectively. Since w_2^n is central it acts as the nth power of these eigenvalues on L_1, L_2.

Now suppose $\Psi : A = A(\alpha, \beta, 1) \longrightarrow A' = A(\alpha', \beta', 1)$ is an isomorphism and define $w_2' \in A'$ analogously to the way $w_2 \in A$ is defined. By Corollary 4.10 $\Psi(w_2) = \epsilon w_2'$ for some $\epsilon \in K^*$. Let $\rho_i : A \longrightarrow End(L_i)$ (resp. $\rho_i' : A' \longrightarrow End(L_i)$) be the representation of A (resp. A') afforded by L_i for $i = 1, 2$. Then the representations ρ_i and $\rho_i' \Psi$ are equivalent for $i = 1, 2$. Hence

$$tr(\rho_i(w_2^n)) = tr(\rho_i' \Psi(w_2^n)) = \epsilon^n tr(\rho_i'((w_2')^n)).$$

Thus

$$\epsilon^n = \left(\frac{(\beta')^2 - 1}{\beta^2 - 1}\right)^n = \left(\frac{\beta' + 1}{\beta + 1}\right)^n.$$

Hence $(\beta - 1)^n = (\beta' - 1)^n$. This is an equation in $\mathbb{Q}(\beta, \beta')$ which we can identify with a subfield of $\mathbb{C}$. This implies that $|\beta - 1| = |\beta' - 1|$ and then by writing $\beta = e^{i\theta}$ that $\beta' = \beta$ or $\beta' = \beta^{-1}$.

It remains to rule out the possibility that A and A' are isomorphic down-up algebras of type (d) and $\alpha^2 + 4\beta = 0$, $(\alpha')^2 + 4\beta' \neq 0$. Hence $A \cong A(2, -1, 1) \cong U(sl_2)$. If η' is a root of unity we obtain a contradiction by Corollary 2.5. Thus η' is not a root of unity and by [14, Theorem 4.0.2] or [17, Theorem 1.3], the centre of A', $Z(A') = K \neq Z(A)$, so $A \not\cong A'$.

5. SEMISIMPLICITY

In this section we assume $char(K) = 0$ and we study down-up algebras of type (d). For brevity we use the notation A_η to refer to these algebras, since as observed in the introduction the A_η are exactly the down-up algebras of type (d). However we continue to use the generators d, u and relations $(R1)$, $(R2)$ rather than the generators h, e, f to preserve continuity. Recall that $\alpha = \eta + 1$, $\beta = -\eta$ and $\gamma \neq 0$.

Our main result is that all finitely generated A_η modules are semisimple if η is a root of unity.

Given any λ, we define the following maximal ideals of $R = K[x, y]$,

$$J_\lambda = (x, y - \lambda),$$

$$H_\lambda = (x - (\lambda - \gamma)\beta^{-1}, y).$$

It is trivial to confirm that $H_\lambda = \sigma(J_\lambda)$ and that $dJ_\lambda = H_\lambda d$.

5.1.   The proof of the next result is adapted from [8, Theorem 5.2].

Proposition. *Let $A = A_\eta$, $\lambda \neq \mu$ and*

$$0 \longrightarrow L(\lambda) \longrightarrow M \longrightarrow L(\mu) \longrightarrow 0$$

a nonsplit short exact sequence of finite dimensional A-modules. Then one of the following occurs

    i) *M is an epimorphic image of $V(\mu)$;*
    ii) *there is a nonzero $w \in L(\lambda)$ such that $H_\mu w = 0$.*

Proof. As in [8, Theorem 5.2] we can choose $v \in M \backslash L(\lambda)$ such that the image in $L(\mu)$ is a highest weight vector in $L(\mu)$ and that $J_\mu^2 v = 0$. Now $J_\mu v \subseteq L(\lambda)$ and we consider two cases.

If $dJ_\mu v \neq 0$. Then since $J_\mu^2 v = 0$ we get $0 = dJ_\mu^2 v = H_\mu(dJ_\mu v) = 0$ so $ii)$ holds.

If $dJ_\mu v = 0$, then $J_\mu v = 0$, since otherwise we would find $w \in J_\mu v$, $w \neq 0$ and then $J_\mu w = dw = 0$ would contradict $\lambda \neq \mu$. If also $dv = 0$ then $i)$ holds. If $J_\mu v = 0 \neq dv$, then since $H_\mu dv = dJ_\mu v = 0$, $ii)$ holds. $\square$

5.2.   Easy calculations show that there is no nonzero integer k such that $\sigma^k(J_\lambda) = J_\lambda$.

Lemma. *If there is $k \in \mathbb{Z}$ such that $\sigma^k(J_\lambda) = J_\lambda$, then $k = 0$.*

Proof. Suppose that $k \geq 0$ and $J_\lambda = \sigma^{-k}(J_\lambda)$. By Lemma 2.3

$$J_\lambda = (x - \lambda_{k-1}, y - \lambda_k).$$

Hence $\lambda_{-1} = \lambda_{k-1}$, $\lambda_0 = \lambda_k$, and by [5, Theorem 2.23] we conclude that $k = 0$. The result follows from this. $\square$

5.3. We prove that no nonsplit extensions of $L(\lambda)$ by itself ever occur for A_η.

Lemma. *Let $A = A_\eta$ such that $\eta \neq 1$. Assume that $dim(L(\lambda)) = n$. Then*

$$(x, \sigma^n(x)) = (x, y - \lambda).$$

Proof. Let $\{v_i\}$ be a basis for the Verma module $V(\lambda)$. As $dim(L(\lambda)) = n$, $\lambda_i \neq 0$ for all $i < n-1$ and $\lambda_{n-1} = 0$. We have that $J_\lambda = \operatorname{ann}_R(v_0)$ and $\sigma^{-n}(J_\lambda) = \operatorname{ann}(v_n)$. Since $\sigma^{-n}(J_\lambda) = (x, y - \lambda_n)$ by Lemma 2.3, $\sigma^n(x) \in J_\lambda$. Hence $(x, \sigma^n(x)) \subseteq (x, y - \lambda)$. Combined with the fact that σ stabilizes $span\{1, x, y\}$, this implies that we can write $\sigma^n(x) = ax + b(y - \lambda)$, for some $a, b \in K$. If $b = 0$, then $\sigma^n(x) = ax$. Thus a is either 1 or η^n and a short calculation shows that $x \in span\{w_2, 1\}$, a contradiction. Hence $y - \lambda \in (x, \sigma^n(x))$ $\square$

The proof of next result is adapted from [8, Theorem 5.4]

Proposition. *Let $A = A_\eta$ and $\eta \neq 1$. There are no nonsplit short exact sequences of A_η-modules of the form*

$$0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0$$

with $L \cong N \cong L(\lambda)$ finite dimensional.

Proof. Assume that there are sequences as above. First suppose that $dim(L(\lambda)) = n > 1$. As in [8, Theorem 5.2] choose $v \in M \backslash L$ such that the image in N is a highest weight vector and such that $J_\lambda^2 v = 0$. By Lemma 2.3 the weights of $L(\lambda)$ are the maximal ideals $\sigma^{-i}(J_\lambda)$ with $i = 0, \ldots, n - 1$. Thus by Lemma 5.2, $H_\lambda = \sigma(J_\lambda)$ and $\sigma^{-n}(J_\lambda)$ are not weights of $L(\lambda)$. On the other hand $H_\lambda^2 dv = dJ_\lambda^2 v = 0$, so $dv = 0$. Similarly since $dim(L(\lambda)) = n$ we have $u^n v \in L$, and $J_\lambda^2 v = 0$ implies that $0 = u^n J_\lambda^2 v = \sigma^{-n}(J_\lambda^2) u^n v$. Thus $u^n v = 0$.

Now $u^{n-1} \sigma^n(x) v = \sigma(x) u^{n-1} v = y u^{n-1} v = d u^n v = 0$ and also $d\sigma^n(x)v = \sigma^{n+1}(x) dv = 0$. Since $\sigma^n(x) \in J_\lambda$, $\sigma^n(x)v \in L$. Since $n > 1$ an easy computation shows that the only element of L annihilated by u^{n-1} and by d is the zero element. Hence $\sigma^n(x)v = 0$. Since $(x, \sigma^n(x)) = J_\lambda$, we have $J_\lambda v = 0$. Since L and M are simple, we have that $M = Av$ and hence M is a highest weight module of weight λ. As $u^n v = 0$, we conclude that $M \cong L(\lambda)$ and hence such a nonsplit exact sequence can not occur.

Finally suppose that $dim(L(\lambda)) = 1$, that is $\lambda = 0$. Since $L(0) \cong A/(d, u)$, we have $(d, u)^2 M = 0$. As in Lemma 4.3, we have $(d, u)^2 = (d, u)$. Hence M is a module over the field $A/(d, u)$, so the sequence splits, a contradiction. $\square$

5.4.   If A is a down-up algebra such that all finite dimensional A-modules are semisimple then A has type (d) since otherwise by Proposition 4.2 A has a commutative image which is not simple Artinian. Set

$$X_{m,n} = \{\eta \in K^* | \eta^m \neq 1 \neq \eta^n \text{ and } n(\eta^m - 1) = m(\eta^n - 1)\}$$

for $m, n \geq 1$.

Theorem. *Let $A = A_\eta$ with $\eta \neq 1$. Then the following are equivalent*

   i) *all finite dimensional A_η-modules are semisimple;*
   ii) *every Verma module for A_η has composition length ≤ 2;*
   iii) *$\eta \notin X_{m,n}$, for all $m \neq n$.*

Proof. We first show the equivalence of conditions $i)$ and $ii)$. By [5, Theorem 2.13 and Proposition 2.23] any submodule of $V(\lambda)$ has the form $N = span\{v_j | j \geq n\}$ for some $n > 0$. It follows that $V(\lambda)$ has length ≤ 2 if and only if every finite dimensional homomorphic image of $V(\lambda)$ is simple. Thus $i)$ implies $ii)$. Conversely suppose that $ii)$ holds and there is a nonsplit exact sequence

$$0 \longrightarrow L(\lambda) \longrightarrow M \longrightarrow L(\mu) \longrightarrow 0$$

with M finite dimensional. By Proposition 5.3 $\lambda \neq \mu$. Since all Verma modules have composition length ≤ 2, case (i) of Proposition 5.1 cannot occur. Thus $\sigma(J_\mu)$ is a weight of $L(\lambda)$, that is $\sigma^i(J_\mu) = J_\lambda$ for some $i > 0$. Similarly by dualizing the above exact sequence and applying Proposition 5.1 we get $\sigma^j(J_\lambda) = J_\mu$ for some $j > 0$. Hence $J_\lambda = \sigma^{i+j}(J_\lambda)$, but this contradicts Lemma 5.2.

Next we prove $iii)$ implies $ii)$ by showing that if $V(\lambda)$ is a Verma module over A_η of length > 2, then $\eta \in X_{m,n}$, for some $m \neq n$. By the description of the submodules of $V(\lambda)$ from [5] cited above we have $\lambda_{m-1} = \lambda_{n-1} = 0$ for some $m > n > 0$. Thus by Lemma 2.5, $\eta^m \neq 1 \neq \eta^n$ and

$$\lambda(\eta - 1) = -\gamma(1 - n(\sum_{i=0}^{n-1} \eta^i)^{-1})$$

$$\lambda(\eta - 1) = -\gamma(1 - m(\sum_{i=0}^{m-1} \eta^i)^{-1}).$$

Forming the difference between these equations we get

$$\gamma[m(\sum_{i=0}^{m-1} \eta^i)^{-1} - n(\sum_{i=0}^{n-1} \eta^i)^{-1}] = 0.$$

As $\gamma \neq 0$

$$(\eta^n - 1)m = (\eta^m - 1)n.$$

Thus $\eta \in X_{m,n}$.

Finally, if $\eta \in X_{m,n}$ and λ is chosen so that

$$\lambda(\eta - 1) = -\gamma(1 - m(\sum_{i=0}^{m-1} \eta^i)^{-1})$$

it follows that $\lambda_{m-1} = \lambda_{n-1} = 0$ and the Verma module $V(\lambda)$ has length > 2. $\square$

In the preliminary version of this paper, we proved semisimplicity of finite dimensional A_η-modules when η was a root of unity. The present version of Theorem 5.4 involves only minor changes to the earlier proof. We thank D. Jordan for pointing out the equivalence of conditions i) and iii), see also [9, Proposition 5.3].

5.5.   Finally we show that

Lemma. *If η is a root of unity or $K = \mathbb{C}$ and $|\eta| = 1$, then $\eta \in X_{m,n}$ implies $m = n$.*

Proof. Suppose $\eta^m \neq 1$, $\eta^n \neq 1$ and

$$(\eta^n - 1)m = (\eta^m - 1)n \qquad (*).$$

Again this is an equation in $\mathbb{Q}(\eta)$ which we can identify with a subfield of $\mathbb{C}$. Then $(\eta^m - 1)/(\eta^n - 1) = mn^{-1} \in \mathbb{R}$. Consideration of the imaginary part of this expression shows that

$$sin(m\theta)(cos(n\theta) - 1) = sin(n\theta)(cos(m\theta) - 1)$$

where $\eta = e^{i\theta}$. If $sin(m\theta) = 0$, then since $\eta^m \neq 1$ we get $m\theta + \pi \in 2\pi\mathbb{Z}$, and $cos(m\theta) = -1$. Thus $sin(n\theta) = 0$, $\eta^m = \eta^n = -1$ and $(*)$ forces $m = n$. Hence we can assume that $sin(m\theta) \neq 0 \neq sin(n\theta)$. Let $g(x) = (cos(x) - 1)/sin(x)$, so that $g(m\theta) = g(n\theta)$. Then $g(x)$ is decreasing on $(-\pi, \pi)$ so $(m - n)\theta \in 2\pi\mathbb{Z}$. Therefore $(*)$ forces $m = n$. $\square$

Proposition. *Any Verma module over A_η has length ≤ 3.*

Proof. If the result is false then from the description of the submodules of $V(\lambda)$, we can find positive integers $m < n < p$ such that $\lambda_{m-1} = \lambda_{n-1} = \lambda_{p-1} = 0$. As in the proof of Theorem 5.4 this means that $\eta^m \neq 1$, $\eta^n \neq 1$, $\eta^p \neq 1$,

$$m(\eta^n - 1) = n(\eta^m - 1)$$
$$\text{and } m(\eta^p - 1) = p(\eta^m - 1).$$

As before we identify $\mathbb{Q}(\eta)$ with a subfield of $\mathbb{C}$. By the previous Lemma $a = |\eta| \neq 1$. Now consider the function $h(x) = m(a^x - 1) - x(a^m - 1)$. Note that $h(m) = h(n) = h(p) = 0$. We obtain a contradiction since $h'(x)$ has at most one zero. $\square$

6. Concluding Remarks

6.1.   We apply the methods developed earlier in this paper to the case where $\beta = 0$. Let $A = A(\alpha, 0, \gamma)$. By [13, Lemma 4.3] A is not Noetherian. We show that there is usually a proper homomorphic image of A which is not Noetherian. The elements

$$w_1 = du - \alpha ud - \gamma$$

$$w_2 = (\alpha - 1)du + \gamma.$$

are analogous to those introduced in Section 1.4. We have $dw_1 = w_1 u = 0$. Also $dw_2 = \alpha w_2 d$ and $w_2 u = \alpha u w_2$. In particular if $\alpha \neq 0$ then $Aw_2 = w_2 A$ and $A/(w_2)$ is isomorphic to the algebra $B = k[x, y]$ generated by x, y subject to the relation $xy = \gamma$. It is well known that B is not Noetherian if $\gamma = 0$. If $\gamma \neq 0$ B is not von Neuman finite and in particular B is not Noetherian. Further results on down-up algebras $A(\alpha, \beta, \gamma)$ with $\beta = 0$ can be found in [9] and [12].

6.2.   We conclude with some remarks about homogenizations of down-up algebras. Assume $\beta \neq 0$. If $A = \cup A_n$ is a filtered algebra, the Rees algebra or homogenization of the filtration is the subalgebra $\oplus A_n T^n$ of $A[T]$.

A natural way to define a filtration on A is to take $A_0 = K$, A_1 a subspace of A, containing A_0 and a set of algebra generators for A, and set $A_n = (A_1)^n$ for $n \in \mathbb{N}$. When $A = A(\alpha, \beta, \gamma)$ is a down-up algebra an obvious choice for A_1 is $A_1 = span\{1, d, u\}$. We denote the Rees algebra of the filtration obtained in this way by $H_1 = H_1(\alpha, \beta, \gamma)$. Clearly H_1 is generated by $D = dT$, $U = uT$ and the central element T. Moreover we have

$$(R5) \qquad \begin{aligned} D^2 U &= d^2 u T^3 = (\alpha dud + \beta ud^2 + \gamma d)T^3 \\ &= \alpha DUD + \beta UD^2 + \gamma DT^2. \end{aligned}$$

This relation is a homogenization of relation $(R1)$. Similarly H_1 satisfies a homogenization of relation $(R2)$. Thus H_1 is the algebra refered to in [5, Question f)]. It is possible to show that H_1 has Hilbert series $[(1-t)^3(1-t^2)]^{-1}$, [3, Proposition 4.2.8].

However there is another set of generators for A which resembles the usual set of generators for $U(sl_2)$. Let λ, μ be the roots of the equation $x^2 - \alpha x - \beta = 0$. Thus $\lambda + \mu = \alpha$, $\lambda\mu = -\beta$ and set

$$(R6) \qquad h = du - \lambda ud.$$

Then

$$(R7) \qquad dh - \mu hd = \gamma d$$

and

$$(R8) \qquad hu - \mu uh = \gamma u.$$

By modifying the argument given in [13, §3.3], for the case $\gamma = 0$, we see that A is generated by h, u, d with relations $(R6), (R7), (R8)$.

Now set $A_1' = span\{1, h, u, d\}$ and let $A_n' = (A_1')^n$ to obtain a second filtration on A. The Rees algebra of this filtration is denoted $H_2 = H_2(\alpha, \beta, \gamma) = \oplus A_n' T^n$. The Hilbert series for H_2 is the same as that of a polynomial algebra in four variables.

Theorem. *The algebras H_1 and H_2 are Auslander-regular and Cohen-Macaulay with global dimension four.*

Proof. Note that $H_1/(T) \cong A(\alpha, \beta, 0)$ is Auslander regular of global dimension 3 and Cohen-Macaulay by [13, Lemma 4.2 and Theorem 4.1]. Using the Lemma in [11] and writing $H_2/(T)$ as an iterated Ore extension it follows that $H_2/(T)$ also satisfies these properties. From now on let H denote either H_1 or H_2. Note that H is graded and T is a homogeneous central element of positive degree in H which is not a zero divisor. Thus we can use a graded version of Nakayama's lemma and the proof of [15, Theorem 7.3.7] to show that $\mathrm{gl.dim}(H) = 4$. The result now follows from [10, Theorem 3.6]. $\square$

Remark. *Parts of the theorem have been obtained independently by Bauwens [3, Remark 4.2.9 and Proposition 4.4.1]. The noncommutative algebraic geometry arising from the graded algebras H_1, H_2 is studied in detail in [3]. In particular the point and line modules are obtained in the "generic" case.*

Finally we note that when $A = A(\alpha, \beta, 0)$, A is a graded algebra which is Auslander-regular of global dimension 3. In [2], the regular algebras with 2 generators and 2 defining relations of degree 3 are classified in terms of a divisor E in $\mathbb{P}^1 \times \mathbb{P}^1$, and an automorphism σ of E. It is easily checked that $E = \mathbb{P}^1 \times \mathbb{P}^1$ when $\alpha = 0$. If $\alpha \neq 0$, $A = A(\alpha, \beta, 0)$ is an algebra of type S_1 in [2, 4.13], see also [1, Table 3.9], that is $E = E_1 \cup E_2$ is the union of two curves of bidegree $(1, 1)$ and σ stabilizes each component. Furthermore we have $E_1 = E_2$ if and only the equation $x^2 - \alpha x - \beta = 0$ has multiple roots. This occurs for example for $A(2, -1, 0)$ which is the enveloping algebra of the Heisenberg Lie algebra and this case is worked out in detail in [2, pages 36-37].

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