Lie Superalgebras, Clifford Algebras, Induced Modules and Nilpotent Orbits

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Abstract

Let \mathfrak{g} be a classical simple Lie superalgebra. To every nilpotent orbit \mathcal{O} in \mathfrak{g}_0 we associate a Clifford algebra over the field of rational functions on \mathcal{O} . We find the rank, $k(\mathcal{O})$ of the bilinear form defining this Clifford algebra, and deduce a lower bound on the multiplicity of a $U(\mathfrak{g})$ -module with \mathcal{O} or an orbital subvariety of \mathcal{O} as associated variety. In some cases we obtain modules where the lower bound on multiplicity is attained using parabolic induction. The invariant $k(\mathcal{O})$ is in many cases, equal to the odd dimension of the orbit $G \cdot \mathcal{O}$ where G is a Lie supergroup with Lie superalgebra \mathfrak{g} .

1 Introduction

Completely prime primitive ideals play a central role in the study of the enveloping algebra of a semisimple Lie algebra. For example they are important in the determination of the scale factor in Goldie rank polynomials, and they are related to unitary representations, see [J3] for more details. On the other hand if \mathfrak{g} is a classical simple Lie superalgebra, there are very few completely

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prime ideals in $U(\mathfrak{g})$, see [M3, Lemma 1].

The results of this paper suggest that it may still be of interest to study primitive ideals of low Goldie rank in $U(\mathfrak{g})$, and their module theoretic analog, modules of low multiplicity.

To initiate this study we associate to any prime ideal q of $S(\mathfrak{g}_0)$, a Clifford algebra C_q over the field of fractions of $S(\mathfrak{g}_0)/q$. Let k(q) be the rank of the bilinear form defining this Clifford algebra. Given a finitely generated module M, we use some filtered-graded machinery along with an elementary result about Clifford algebras to obtain a lower bound on the multiplicity of M in terms of k(q), see Lemmas 2.1 and 5.1.

When \mathfrak{g}_0 is reductive and P is a primitive ideal in $U(\mathfrak{g}_0)$ the subvariety of \mathfrak{g}_0 defined by grP is the closure of a nilpotent orbit, [BB], [J1]. For this reason the most interesting primes in $S(\mathfrak{g}_0)$ are those defining nilpotent orbits or their orbital subvarieties. If \mathfrak{g} is classical simple and q is a prime ideal of $S(\mathfrak{g}_0)$ defining a nilpotent orbit we give a formula for k(q) in terms of a partition (or partitions) associated to the nilpotent orbit.

This work motivates the search for highest weight modules with given associated variety and low multiplicity. For $\mathfrak{g} = g\ell(m, n)$, $s\ell(m, n)$ or Q(n) we explain how to find examples of such modules using induction from parabolic subalgebras. For a precise statement, see Lemmas 5.5, 5.6 and Theorem 5.7. We also investigate the primitive ideals that arise as annihilators of these modules and the structure of the corresponding primitive factor algebras. We remark that the orbital varieties which occur in our examples have the simplest possible type, namely they are all linear subvarieties of the nilpotent orbit. One difficulty is that the closest analog for semisimple Lie algebras of the problem considered here is the quantization problem for orbital varieties, which is unsolved, see [Be],[J3]. It is worth noting also that the associated variety of a simple highest weight module is irreducible for sl(n), [Me]. This is not true in general [J2],[T]. We plan to return to the issues raised here in a subsequent paper. In particular we shall show that the modules we construct in this paper are quantizations of superorbital varieties.

Additional motivation for the study of the invariants k(q) comes from supergeometry. Suppose that \mathfrak{g} is classical simple, and that there is a nondegenerate even bilinear form on \mathfrak{g} . If $x \in \mathfrak{g}_0$, and m_x is the corresponding ideal of $S(\mathfrak{g}_0)$ then $k(m_x)$ is equal to the dimension of the centralizer of x in \mathfrak{g}_1 . If G is a Lie supergroup with Lie superalgebra \mathfrak{g} , this allows us to find the superdimension of the orbit $G \cdot x$, when x is nilpotent.

This paper is organized as follows. After some preliminaries in Section 2, we obtain our formulas for k(q) in Section 3. Although this is done on a case-by-case basis, the formulas in most cases depend on the same basic result (Lemma 2.4). Furthermore the exceptional algebras G(3) and F(4)can be treated using essentially the same method as the orthosymplectic algebras. In Section 5 we prove our main results about parabolically induced modules. We prove a result (Theorem 5.3) describing the structure of such modules as $U(\mathfrak{g}_0)$ -modules. This is used to derive analogs of several results on induced modules and their annihilators from [Ja2, Kapitel 15]. Several of the results in this section (for example Theorem 5.7 and Corollary 5.10) apply to the modules $F(\mu)$ constructed by Serganova in section 3 of [S2] for the Lie superalgebras $g\ell(m,n)$. In Section 4 we give some background on parabolic subalgebras needed in Section 5. Our results on nilpotent orbits may be found in Section 6. Nilpotent orbits do not seem to have been widely studied in the superalgebra case, see however [S1], so we spend some time developing the background.

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2 Preliminaries

2.1. Clifford Algebras. Let $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ be a finite dimensional Lie superalgebra over \mathbb{C} . The tensor algebra $T(\mathfrak{g})$ has a unique structure $T(\mathfrak{g}) = \bigoplus_{n \geq 0} T^n(\mathfrak{g})$ as a graded algebra such that $T^0(\mathfrak{g}) = \mathbb{C}, T^1(\mathfrak{g}) = \mathfrak{g}_1$ and $T^2(\mathfrak{g}) = \mathfrak{g}_0 + \mathfrak{g}_1 \otimes \mathfrak{g}_1$. Set $T_n = \bigoplus_{m \leq n} T^m(\mathfrak{g})$ and let U_n be the image of T_n in $U(\mathfrak{g})$. Then $\{U_n\}$ is a filtration on $U(\mathfrak{g})$ and we describe the associated graded ring $S = grU(\mathfrak{g})$. Observe that $R = S(\mathfrak{g}_0)$ is a central subalgebra of S and that the bracket [,] on \mathfrak{g}_1 extends to an R-bilinear form on $\mathfrak{g}_1 \otimes R$. The algebra S is isomorphic to the Clifford algebra of this bilinear form. If v_1, \ldots, v_n is a basis of \mathfrak{g}_1 over \mathbb{C} then the matrix of the bilinear form with respect to this basis is $M(\mathfrak{g}) = ([v_i, v_i])$. We do not refer to the basis in the notation for this matrix since we study only properties of the matrix which are independent of the basis.

We showed in [M2] that there is a homeomorphism

$$\pi: SpecR \longrightarrow GrSpecS.$$

where GrSpec() refers to the space of \mathbb{Z}_2 -graded prime ideals. Let us recall the details. Fix $q \in Spec(R)$ and let $\overline{S} = S/Sq$ and $\mathcal{C} = \mathcal{C}(q)$, the set of regular elements of R/q. Then $F_q = Fract(R/q)$ is a central subfield of the localization $T = \overline{S}_{\mathcal{C}}$. Moreover the Lie bracket on \mathfrak{g}_1 extends to a symmetric F_q -bilinear form on $\mathfrak{g}_1 \otimes F_q$. It is easy to see that T is the Clifford algebra of this form over F_q . The nilradical N of T is generated by the radical of the bilinear form on $\mathfrak{g}_1 \otimes F_q$, and T/N is the Clifford algebra of a nonsingular bilinear form. Then $\pi(q)$ is the kernel of the combined map

$$S = grU(\mathfrak{g}) \longrightarrow \overline{S} \longrightarrow T/N.$$

It follows that $\pi(q) = \sqrt{Sq}$ where $\sqrt{-}$ denotes the radical of an ideal. For $p \in GrSpecS, \pi^{-1}(p) = p \cap R$. Note that if $\pi(q) = p$ we have inclusions of rings

$$R/q \subseteq S/p \subseteq T/N.$$

Moreover T/N is obtained from S/p by inverting the nonzero elements of R/q. Hence S/p is an order in the Clifford algebra $C_q = T/N$. Let B_q be the bilinear form defining this Clifford algebra, δ_q the determinant of B_q and k(q) the rank of B_q . Thus

 $k(q) = \{ \max m | \text{some } m \times m \text{ minor of } M(\mathfrak{g}) \text{ is nonzero mod } q \}.$

A prime ideal q of $S(\mathfrak{g}_0)$ is homogeneous if $q = \bigoplus_{n \ge 0} (q \cap S^n(\mathfrak{g}_0))$ where $S(\mathfrak{g}_0) = \bigoplus_{n \ge 0} S^n(\mathfrak{g}_0)$ is the usual grading. All prime ideals q of $S(\mathfrak{g}_0)$ considered in this paper will be homogeneous. If q is homogeneous and k(q)is odd then δ_q is a rational function of odd degree and hence not a square in F_q . Therefore by [L, Theorems V.2.4 and V.2.5] C_q is a central simple algebra. Hence $C_q \cong M_{2^a}(D)$ for a division algebra D. Using the fact that $\dim_{F_q} C_q = 2^{k(q)}$ it is easy to prove the following result.

Lemma Let L be a simple C_q -module where q is a homogeneous prime ideal of $S(\mathfrak{g}_0)$.

(a) If k(q) is even then C_q is a central simple algebra over F_q and $\dim_{F_q} L \geq 1$

 $2^{k(q)/2}$. Equality holds if and only if $D = F_q$

(b) If k(q) is odd then C_q is a central simple algebra over $F_q(\sqrt{\delta_q})$ and $\dim_{F_q} L \geq 2^{(k(q)+1)/2}$. Equality holds if and only if $D = F_q(\sqrt{\delta_q})$.

We denote the greatest integer less than or equal to s by [s]. If $\dim_{F_q} L = 2^{[(k(q)+1)/2]}$, we say that C_q is *split*.

2.2.Evaluation of $M(\mathfrak{g})$. Let \mathfrak{g} be classical simple. Since \mathfrak{g}_0 is reductive there is a nondegenerate invariant bilinear form on \mathfrak{g}_0 . This allows us to identify \mathfrak{g}_0 with \mathfrak{g}_0^* and thus to view elements of $S(\mathfrak{g}_0)$ as functions on \mathfrak{g}_0 . If $\mathcal{O} \subseteq \mathfrak{g}_0$, and the ideal q of functions in $S(\mathfrak{g}_0)$ which vanish on \mathcal{O} is prime, we often write $k(\mathcal{O})$ in place of k(q). It is convenient to set $\ell(q) = [(k(q)+1)/2]$ and $\ell(\mathcal{O}) = [(k(\mathcal{O})+1)/2]$. We say that a closed subset X of \mathfrak{g}_0 is *conical* if $x \in X$ implies that $\mathbb{C}x \subseteq X$. For example closures of nilpotent orbits and their orbital subvarieties are conical. If X is a product of conical subvarieties of the simple summands of \mathfrak{g}_0 , then the defining ideal of X in $S(\mathfrak{g}_0)$ is independent of the choice of bilinear form, since any two nondegenerate invariant forms on a simple Lie algebra are proportional. Fix a nilpotent orbit \mathcal{O} , and suppose $q \in SpecS(\mathfrak{g}_0)$ is such that $V(q) = \mathcal{O}$. We want to compute k(q). For $x \in \mathcal{O}$, let M(x) be the evaluation of $M(\mathfrak{g})$ at x and let m_x be the maximal ideal of $S(\mathfrak{g}_0)$ corresponding to x. Since \mathcal{O} is dense in V(q) and the rank of $M(\mathfrak{g})$ is constant on \mathcal{O} we have

$$k(q) = rank(M(x)) = k(m_x) \quad \text{for all} \quad x \in \mathcal{O}.$$
(1)

Hence if X is an irreducible subvariety of \mathcal{O} we have $k(X) = k(\mathcal{O})$.

2.3. Matrix notation. We denote the $n \times n$ identity matrix by I_n , and the matrix with a 1 in row *i*, column *j* and zeroes elsewhere by e_{ij} . Let Υ_r be the $r \times r$ matrix with ones on the antidiagonal and zeros elsewhere. We write $M_{m,n}$ for the vector space of $m \times n$ complex matrices. The transpose of a matrix *A* is denoted by A^t . Since $M(\mathfrak{g})$ is a matrix over $S(\mathfrak{g}_0)$, and \mathfrak{g}_0 is often an algebra of matrices, we need an "external" version of the matrices e_{ij} . For clarity, a matrix *A* with entries in $S(\mathfrak{g}_0)$ will often be written in the form

$$A = \sum_{i,j} a_{i,j} \mathbf{e}_{\mathbf{i},\mathbf{j}}$$

meaning that $a_{i,j} \in S(\mathfrak{g}_0)$ is the entry in row *i* and column *j* of *A*.

Recall that if A and B are square matrices with rows and columns indexed by I, J respectively, then the Kronecker product $A \otimes B$ has rows and columns indexed by $I \times J$, and has entry in row (i, k), column (j, ℓ) equal to $a_{ij}b_{k\ell}$. To be more precise, we should also specify an ordering on the rows and columns of $A \otimes B$. If $I \subseteq \mathbb{Z}$ we give I the ordering inherited from \mathbb{Z} . If I, J are ordered sets then unless otherwise stated we give $I \times J$ the lexicographic order $<_{\ell ex}$ defined by

 $(i, j) <_{\ell ex} (k, \ell)$ if and only if i < k or i = k and $j < \ell$.

We need a twisted version of the Kronecker product. If A and B are as above, we define $A \widehat{\otimes} B$ to be the matrix with rows indexed by $I \times J$ and columns indexed by $J \times I$ such that the entry in row (i, k), column (ℓ, j) is equal to $a_{ij}b_{\ell k}$. Here we order $I \times J$ the lexicographically and order $J \times I$ so that (j, i) precedes (ℓ, k) if and only if $(i, j) <_{\ell ex} (k, \ell)$.

The definition of $A \otimes B$ might seem unnatural at first, but it is very convenient for the computation of $M(\mathfrak{g})$ when $\mathfrak{g} = g\ell(m, n)$. Note that if we relabel column (ℓ, j) of B as column (j, ℓ) , the rows and columns of $A \otimes B$ are then both indexed by $I \times J$ ordered lexicographically. It follows that $A \otimes B = A \otimes B^t$.

2.4. Partitions. If $\mu = (\mu_1 \ge \mu_2 \ge ...)$ is a partition of m we denote the nilpotent matrix with Jordan blocks of size $\mu_1, \mu_2, ...$ by J_{μ} . The dual partition μ' of m is defined by

$$\mu_i' = |\{j|\mu_j \ge i\}|$$

for all *i*. We set $\mu_i = 0$ for all $i > \mu'_1$. The set of all partitions of *m* is denoted $\mathbf{P}(m)$.

Lemma For $\mu \in \mathbf{P}(m)$ and $\nu \in \mathbf{P}(n)$ we have

$$rank(J_{\mu}\otimes I_n + I_m \otimes J_{\nu}) = mn - \sum_{i\geq 1} \mu'_i \nu'_i.$$

Proof. For $a \ge 1$, let L(a) be the simple $s\ell(2)$ -module of dimension a. If $e = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ we can choose bases for the modules $L(\mu_i)$ and $L(\nu_i)$ such that

 $E = J_{\mu} \otimes I_n + I_m \otimes J_{\nu}$ is the matrix representing the action of e on

$$\oplus_{i\geq 1}L(\mu_i)\otimes \oplus_{i\geq 1}L(\nu_i)$$

To compute rank E note that $L(a) \otimes L(b)$ is the direct sum of min(a, b) simple modules, and the rank of e acting on L(a) is a - 1. This implies

$$rankE = mn - \sum_{j,k} \min(\mu_j, \nu_k).$$

Now set

$$A_{i} = \{(j,k) | \min(\mu_{j}, \nu_{k}) = i\}$$
$$B_{i} = \{(j,k) | \min(\mu_{j}, \nu_{k}) \ge i\}.$$

Note that $|B_i| = \mu'_i \nu'_i$ and $|A_i| = |B_i| - |B_{i+1}|$. Thus

$$\sum_{j,k} \min(\mu_j, \nu_k) = \sum_i i |A_i| = \sum_i |B_i| = \sum_i \mu'_i \nu'_i.$$

Remark. Since $J_{\mu}\widehat{\otimes}I_n = J_{\mu} \otimes I_n$ and $I_m\widehat{\otimes}J_{\nu} = I_m \otimes J_{\nu}^t$ we also have a formula for $rank(J_{\mu}\widehat{\otimes}I_n + I_m\widehat{\otimes}J_{\nu})$.

2.5. Dimension and Multiplicity. Let $N = \bigoplus_{m \ge 0} N(m)$ be a finitely generated graded $S(\mathfrak{g}_0)$ -module and set $N_n = \bigoplus_{m=0}^n N(m)$. For n >> 0 we have

dim
$$N_n = a_d \begin{pmatrix} n \\ d \end{pmatrix} + a_{d-1} \begin{pmatrix} n \\ d-1 \end{pmatrix} + \ldots + a_0$$

for suitable constants a_0, \ldots, a_d with $a_d \neq 0$. We set d(N) = d and $e(N) = a_d$. We filter $U(\mathfrak{g})$ as in Section 2.1 and denote associated graded ring by $grU(\mathfrak{g})$. Let M be a finitely generated $U(\mathfrak{g})$ -module and equip M with a good filtration $\{M_n\}_{n\geq 0}$. Since N = grM is finitely generated over $grU(\mathfrak{g})$ and hence over $S(\mathfrak{g}_0)$, the above remarks apply and we set d(M) = d(N) and e(M) = e(N). It is not hard to show that d(M) and e(M) are independent of the good filtration and that d(M) is the Gelfand-Kirillov dimension of M calculated either as a $U(\mathfrak{g})$ -module or as a $U(\mathfrak{g}_0)$ -module. For details see [KL, Chapter 7]. If M is finite dimensional, we have d(M) = 0 and $e(M) = \dim_{\mathbb{C}} M$.

In Section 5 we use the following fact. Suppose q is a homogeneous prime ideal of $S(\mathfrak{g}_0)$ and N a finitely generated torsionfree graded module over $Z = S(\mathfrak{g}_0)/q$. If F = Fract(Z) then d(N) = d(Z) and $e(N) = e(Z) \cdot \dim_F Z^{-1}N$. This follows easily from [GW, Exercise 4L, Corollary 4.17 and Lemma 6.17]. If \mathcal{V} is the closed subset of \mathfrak{g}_0^* defined by q we set $e(\mathcal{V}) = e(Z)$.

A module M is homogeneous (resp. critical) if for any nonzero submodule M' we have d(M) = d(M') (resp. d(M) = d(M') and e(M) = e(M')).

2.6. Induced Modules. Let \mathfrak{p} be a subalgebra of the Lie superalgebra \mathfrak{g} and N a finitely generated $U(\mathfrak{p})$ -module. We write $Ind_{\mathfrak{p}}^{\mathfrak{g}} N$ for the induced module $U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} N$.

Lemma Suppose $M = Ind_{\mathbf{p}}^{\mathbf{g}} N$ and set $c_i = \dim \mathfrak{g}_i - \dim \mathfrak{p}_i$ for i = 0, 1. Then

$$d(M) = d(N) + c_0$$

and

$$e(M) = 2^{c_1} e(N).$$

Proof. This is easily adapted from the proof of [Ja2, Lemma 8.9].

2.7. Affiliated Series of a Module. Let N be a nonzero finitely generated module over a Noetherian ring S. An *affiliated submodule* of N is a submodule of the form $ann_N(P)$ where P is an ideal of S maximal among the annihilators of nonzero submodules of N, see [GW] for background. An *affiliated series* for N is a series of submodules

$$0 = N_0 \subset N_1 \subset \ldots \subset N_k = N$$

such that each N_i/N_{i-1} is an affiliated submodule of N/N_{i-1} . The prime ideals $P_i = ann_S(N_i/N_{i-1})$ are called the *affiliated primes* of the series.

2.8. Reductive Lie algebras. For the remainder of section 2, \mathfrak{g}_0 will be a reductive Lie algebra. Later we use the notation established here when \mathfrak{g}_0 is the even part of a classical simple Lie superalgebra. Let $\mathfrak{n}_0 \oplus \mathfrak{h}_0 \oplus \mathfrak{n}_0^+$ be a triangular decomposition of \mathfrak{g}_0 . So \mathfrak{h}_0 is a Cartan subalgebra and $\mathbf{b} = \mathfrak{h}_0 \oplus \mathfrak{n}_0^+$ a Borel subalgebra of \mathfrak{g}_0 . Let G be the adjoint algebraic group of \mathfrak{g}_0 . If α is a root of \mathfrak{g}_0 we denote the corresponding root space by \mathfrak{g}^{α} . There is a unique element $h_{\alpha} \in [\mathfrak{g}^{\alpha}, \mathfrak{g}^{-\alpha}]$ such that $\alpha(h_{\alpha}) = 2$. For $\lambda \in \mathfrak{h}_0^*$ we denote the Verma module with highest weight λ induced from **b** and its unique simple quotient by $M(\lambda)$ and $L(\lambda)$ respectively. We write (λ, α^{\vee}) in place of $\lambda(h_{\alpha})$.

2.9. Richardson Orbits. Let \mathfrak{p}_0 be a parabolic subalgebra of \mathfrak{g}_0 and suppose that $\mathfrak{p}_0 = \mathfrak{l}_0 \oplus \mathfrak{m}_0$ where \mathfrak{m}_0 is the nilradical of \mathfrak{p}_0 and \mathfrak{l}_0 is a Levi factor. Then $G\mathfrak{m}_0$ contains a unique dense orbit called the *Richardson orbit* induced from \mathfrak{l}_0 .

If L is a finite dimensional \mathfrak{l}_0 -module and $M = Ind_{\mathfrak{p}_0}^{\mathfrak{g}_0} L$ there are two prime ideals of $S(\mathfrak{g}_0)$ that we can associate to M. The first of these is $q' = \sqrt{grann_{U}(\mathfrak{g}_0)M}$ which is the defining ideal of the Richardson orbit \mathcal{O} induced from \mathfrak{l}_0 , [Ja2, 17.15]. On the other hand we can equip M with a good filtration and consider $q = \sqrt{ann_{S}(\mathfrak{g}_0)grM}$. Then $q = S(\mathfrak{g}_0)\mathfrak{p}_0$ is the defining ideal of $\mathfrak{m}_0 \subset \overline{\mathcal{O}}$, [Ja2, 17.12 (4)]. We have $2dim(\mathfrak{m}_0) = dim(\mathcal{O})$. However k(q) = k(q') since $\mathcal{O} \cap \mathfrak{m}_0$ is nonempty and by (1) in section 2.2 k(q) can be calculated by evaluating at any point of \mathcal{O} .

2.10. Orbital Varieties. Let \mathcal{O} be a nilpotent orbit in \mathfrak{g}_0 . The irreducible components of $\mathcal{O} \cap \mathfrak{n}_0^+$ are called *orbital varieties attached to* \mathcal{O} . If \mathcal{V} is such an orbital variety we have $k(\mathcal{O}) = k(\mathcal{V})$ as above. For example if \mathcal{O} is the Richardson orbit induced from \mathfrak{l}_0 and \mathfrak{m}_0 is as in section 2.9 then \mathfrak{m}_0 is an orbital variety in \mathcal{O} . In general however Richardson orbits contain many other orbital varieties, see [J3] for a recent survey.

2.11. The category O. We denote by **O** the category of $U(\mathfrak{g}_0)$ -modules defined in [Ja1, section 1.9]. For $M \in Ob \mathbf{O}$ we write [M] for the class of M in the Grothendieck group $G(\mathbf{O})$ of **O**. The group $G(\mathbf{O})$ is free abelian on the classes $[L(\lambda)]$ with $\lambda \in \mathfrak{h}_0^*$. For $M, M' \in Ob \mathbf{O}$ we have [M] = [M'] if and only if M and M' have the same character. We define a partial order \leq on $G(\mathbf{O})$ by the rule $\sum_{\lambda} a_{\lambda}[L(\lambda)] \leq \sum_{\lambda} b_{\lambda}[L(\lambda)]$ if and only if $a_{\lambda} \leq b_{\lambda}$ for all $\lambda \in \mathfrak{h}_0^*$.

3 Dimension Formulas

3.1. We describe the matrix $M(\mathfrak{g})$ explicitly when $\mathfrak{g} = g\ell(m, n)$. Let $\mathbf{I_1} = \{1, \ldots, m\}, \mathbf{I_2} = \{m + 1, \ldots, m + n\}, \mathbf{I} = \mathbf{I_1} \cup \mathbf{I_2}$ and consider the following matrices

$$N_1 = \sum_{i,j \in \mathbf{I}_1} e_{ij} \mathbf{e}_{\mathbf{i},\mathbf{j}}, \quad N_2 = \sum_{k,\ell \in \mathbf{I}_2} (e_{k,\ell}) \mathbf{e}_{\mathbf{k},\ell},$$

with entries in \mathfrak{g}_0 .

Lemma With a suitable choice of ordered basis for $\mathfrak{g}_1, M(\mathfrak{g})$ has block matrix form

$$\left[\begin{array}{cc} 0 & N\\ N^t & 0 \end{array}\right]$$

where $N = N_1 \widehat{\otimes} I_n + I_m \widehat{\otimes} N_2$.

Proof. Write $\mathfrak{g}_1^+ = span\{e_{ik}|(i,k) \in \mathbf{I_1} \times \mathbf{I_2}\}, \mathfrak{g}_1^- = span\{e_{\ell j}|(\ell, j) \in \mathbf{I_2} \times \mathbf{I_1}\},\$ so that $\mathfrak{g}_1 = \mathfrak{g}_1^+ \oplus \mathfrak{g}_1^-$. The rows and columns of $M(\mathfrak{g})$ are indexed by $\mathbf{I_1} \times \mathbf{I_2}$ ordered lexicographically followed by $\mathbf{I_2} \times \mathbf{I_1}$ ordered so that (j,i) precedes (ℓ,k) if and only if $(i,j) <_{\ell ex} (k,\ell)$.

The block matrix decomposition follows since $[\mathbf{g}_1^{\pm}, \mathbf{g}_1^{\pm}] = 0$ and $M(\mathbf{g})$ is symmetric. To compute N suppose $(i, k) \in \mathbf{I_1} \times \mathbf{I_2}$ and $(\ell, j) \in \mathbf{I_2} \times \mathbf{I_1}$, then $[e_{ik}, e_{\ell j}] = \delta_{k\ell} e_{ij} + \delta_{ij} e_{\ell k}$ and the result follows.

3.2. Let $N_1 = \sum_{i,j \in \mathbf{I}_1} e_{ij} \mathbf{e}_{\mathbf{i},\mathbf{j}}$ as above and $y = \sum_{k,\ell} y_{k\ell} e_{k\ell} \in g\ell(m)$. Using the bilinear form (A, B) = trace(AB) to evaluate N_1 at y we have that

$$N_1(y) = (y_{ji}) \in g\ell(m)$$

is the $m \times m$ matrix with i, j entry equal to y_{ji} . Thus $N_1(y)$ has the same Jordan form as y. Of course similar remarks apply to the evaluation of N_2 .

We denote the orbit of (J_{μ}, J_{ν}) in $\mathfrak{g}_0 = g\ell(m) \times g\ell(n)$ by $\mathcal{O}_{\mu,\nu}$.

Theorem For $\mu \in \mathbf{P}(m)$ and $\nu \in \mathbf{P}(n)$ we have

$$k(\mathcal{O}_{\mu,\nu}) = 2(mn - \sum_{i} \mu'_{i}\nu'_{i}).$$

Proof. This is immediate by Lemmas 2.4 and 3.1.

Remark. If $\mathfrak{g} = s\ell(m, n)$ then \mathfrak{g} has the same odd part as $g\ell(m, n)$ and the matrix $M(\mathfrak{g})$ can be calculated using Lemma 3.1. We can identify the nilpotent orbits in \mathfrak{g}_0 with those in the even part of $s\ell(m, n)$ and then Theorem 3.2 applies to \mathfrak{g} . Similar remarks apply to the Lie superalgebra $ps\ell(n, n)$.

3.3. If V is a vector space we write $\wedge^k V$ and $S^k V$ for the kth exterior and symmetric power of V respectively. For $v, w \in V$ we set $v \wedge w =$ $1/2(v \otimes w - w \otimes v) \in \wedge^2 V$, $v \circ w = 1/2(v \otimes w + w \otimes v) \in S^2 V$. The following description of the orthosymplectic Lie superalgebra algebra osp(m, n) can be found in [K, 2.1.2]. Let V_1 be an *m*-dimensional vector space with a nondegenerate symmetric bilinear form ψ_1 and V_2 an *n*-dimensional vector space with a nondegenerate skew-symmetric bilinear form ψ_2 .

Then we can realize $\mathfrak{g} = osp(m, n)$ by setting

$$\mathfrak{g}_0 = \wedge^2 V_1 \oplus S^2 V_2, \ \mathfrak{g}_1 = V_1 \otimes V_2.$$

The action of $\wedge^2 V_1$ on V_1 is given by

$$[a \wedge b, c] = \psi_1(a, c)b - \psi_1(b, c)a.$$

Similarly S^2V_2 acts on V_2 via

$$[a \circ b, c] = \psi_2(a, c)b + \psi_2(b, c)a.$$

The bilinear forms ψ_1 and ψ_2 are invariant under these actions, so $\wedge^2 V_1$ and $S^2 V_2$ identify with so(m) and sp(n) respectively. The product $\mathfrak{g}_1 \times \mathfrak{g}_1 \longrightarrow \mathfrak{g}_0$ is given by

$$[a \otimes c, b \otimes d] = \psi_1(a, b)(c \circ d) + \psi_2(c, d)(a \wedge b).$$

3.4. The following lemma applies to the computation of the matrix $M(\mathfrak{g})$

when \mathfrak{g}_0 is not simple, \mathfrak{g}_1 is an irreducible \mathfrak{g}_0 -module and \mathfrak{g} is not isomorphic to $\Gamma(\sigma_1, \sigma_2, \sigma_3)$. The discussion leading up to [Sch, page 143, equation (5.9)] allows us to make the following assumptions about the structure of \mathfrak{g} . Firstly $\mathfrak{g}_0 = \mathfrak{g}^1 \times \mathfrak{g}^2$ and $\mathfrak{g}_1 = V_1 \otimes V_2$ where the \mathfrak{g}^i are nonzero semisimple Lie algebras and the V_i are simple \mathfrak{g}^i -modules. Furthermore, for i = 1, 2 there are \mathfrak{g}^i -invariant bilinear maps

$$\pi_i: V_i \times V_i \longrightarrow \mathfrak{g}^i \quad , \quad \psi_i: V_i \times V_i \longrightarrow \mathbb{C}$$

such that

$$[u_1 \otimes u_2, v_1 \otimes v_2] = \psi_2(u_2, v_2)\pi_1(u_1, v_1) + \psi_1(u_1, v_1)\pi_2(u_2, v_2)$$
(2)

for $u_1, v_1 \in V_1; u_2, v_2 \in V_2$. In addition we can assume that π_2, ψ_1 are symmetric and π_1, ψ_2 are skew-symmetric.

We claim that if $\mathfrak{g} \neq \Gamma(\sigma_1, \sigma_2, \sigma_3)$ there are nonzero constants s_i such that the maps π_1, π_2 are given by

$$\pi_i(u, v)w = s_i(\psi_i(v, w)u - \psi_i(w, u)v) \tag{3}$$

for $u_i, v_i \in V_i$, cf. [Sch, page 144, equation (5.16)].

Indeed, from Section 3.3 equation (3) holds when $\mathfrak{g} = osp(m, n)$ with $m \geq 3, n \geq 2$. Also equation (3) defines \mathfrak{g}^i -invariant bilinear maps $\pi_i : V_i \times V_i \longrightarrow \mathfrak{g}^i$, so (3) holds whenever \mathfrak{g}^1 and \mathfrak{g}^2 are simple and the adjoint representation of \mathfrak{g}^1 resp. \mathfrak{g}^2 occurs with multiplicity one in $\wedge^2 V_1$ resp. $S^2 V_2$. This is the case for the Lie superalgebras G(3) and F(4). Note however that if $\mathfrak{g} = \Gamma(\sigma_1, \sigma_2, \sigma_3)$ then we can write \mathfrak{g}_0 as $\mathfrak{g}^1 \times \mathfrak{g}^2$ where $\mathfrak{g}^1 \cong so(4)$ and $\mathfrak{g}^2 \cong sl(2)$. In this case the map $\pi_1 : V_1 \times V_1 \longrightarrow \mathfrak{g}^1$ is not, in general given by (3). This exhausts all the classical simple Lie superalgebras \mathfrak{g} such that \mathfrak{g}_0 is not simple and \mathfrak{g}_1 is an irreducible \mathfrak{g}_0 -module.

Let $\{e_1, \ldots, e_m\}$ and $\{f_1, \ldots, f_n\}$ bases for V_1, V_2 respectively, and let J_1 , resp. J_2 be the matrix with entry in row *i* and column *j* equal to $\psi_1(e_i, e_j)$, resp. $\psi_2(f_i, f_j)$. We denote by $so(V_1), sp(V_2)$ the orthogonal and symplectic algebras preserving the forms ψ_1, ψ_2 respectively. Let $A_{ik} = \pi_1(e_i, e_k)$ and $B_{j\ell} = \pi_2(f_j, f_\ell)$. We evaluate matrices with entries in $so(V_1)$, and $sp(V_2)$ using the trace form $(a, b) \longrightarrow trace(ab)$ for $a, b \in so(V_1)$, or $a, b \in sp(V_2)$. **Lemma** (a) With respect to the basis $\{e_i \otimes f_j\}$ of \mathfrak{g}_1 , we have

$$M(\mathfrak{g}) = A \otimes J_2 + J_1 \otimes B$$

(b) For all $x \in so(V_1)$, $y \in sp(V_2)$, we have

$$(J_1^{-1}A)(x) = -2s_1x, \quad (J_2^{-1}B)(y) = 2s_2y.$$

Proof. (a) This follows easily from formula (2).

(b) We prove the statement about $so(V_1)$; the other part is similar. We assume that $s_1 = 1$ and write ψ, π and J in place of ψ_1, π_1 , and J_1 . Recall the notation for matrices with entries in $S(g\ell(V_1))$ from section 2.3. Write $K = J^{-1}$ and A in the form

$$K = \sum_{p,q} K_{p,q} \mathbf{e}_{\mathbf{p},\mathbf{q}}, \quad A = \sum_{i,j} A_{i,j} \mathbf{e}_{i,j}$$

Using equation (3) we compute that

 $trace(\pi(e_i, e_j)\pi(e_k, e_{\ell})) = 2(\psi(e_j, e_k)\psi(e_{\ell}, e_i) - \psi(e_{\ell}, e_j)\psi(e_k, e_i)).$

Hence if $x = \pi(e_k, e_\ell)$, we have

$$KA(x)e_r = 2\sum_{i,j,p,q} (\psi(e_j, e_k)\psi(e_\ell, e_i) - \psi(e_\ell, e_j)\psi(e_k, e_i))K_{p,q}\mathbf{e_{p,q}}\mathbf{e_{i,j}}e_r$$

$$= -2[\psi(e_\ell, e_r)e_k - \psi(e_r, e_k)e_\ell]$$

$$= -2\pi(e_k, e_\ell)e_r.$$

3.5. To apply Lemma 3.4 we need to consider three cases separately. Suppose first that $\mathfrak{g} = osp(m, n)$ with $m \geq 3, n \geq 2$. Then $\mathfrak{g}_0 = \mathfrak{g}^1 \times \mathfrak{g}^2$ where $\mathfrak{g}^1 = so(m), \mathfrak{g}^2 = sp(n)$. Also $\mathfrak{g}_1 = V_1 \otimes V_2$ where V_1 is the natural module for so(m) and V_2 is the natural module for sp(n). There are maps π_i, ψ_i for i = 1, 2 such that the product $\mathfrak{g}_1 \times \mathfrak{g}_1 \longrightarrow \mathfrak{g}_0$ is given by equation (2) in section 3.4.

We recall how nilpotent orbits in simple Lie algebras of types B, C and D can be described in terms of partitions. Let $\mathbf{P}_1(m)$ (resp. $\mathbf{P}_{-1}(m)$) be the set of partitions of m in which even (resp. odd) parts occur with even multiplicity. Then by [CM, Theorems 5.1.2 and 5.1.3], nilpotent orbits in

 $so(2r+1), r \geq 1$ (resp. $sp(2s), s \geq 1$) are in one-to-one correspondence with partitions in $\mathbf{P}_1(2r+1)$ (resp. $\mathbf{P}_{-1}(2s)$). We denote the orbit corresponding to a partition μ by \mathcal{O}_{μ} . We say that a partition is *very even* if it has only even parts, each with even multiplicity. By [CM, Theorem 5.1.4] any partition $\mu \in \mathbf{P}_1(2r)$ corresponds to a unique orbit \mathcal{O}_{μ} in $so(2r), r \geq 1$ unless μ is very even in which case μ corresponds to two orbits \mathcal{O}_{μ}^I and \mathcal{O}_{μ}^{II} .

From the proofs of [CM, Propositions 5.2.3, 5.2.5 and 5.2.8], we see that if a simple Lie algebra of type B, C or D is regarded as a subalgebra of $g\ell(N)$ using the defining representation then the Jordan form of a matrix in \mathcal{O}_{μ} (or $\mathcal{O}_{\mu}^{I}, \mathcal{O}_{\mu}^{II}$) corresponds to the partition μ .

If $\mu \in \mathbf{P}_1(m), \nu \in \mathbf{P}_{-1}(2s)$ and μ is not very even, we consider the orbits

$$\mathcal{O}_{\mu,
u} = \mathcal{O}_{\mu} imes \mathcal{O}_{
u}.$$

If m = 2r and μ is very even the existence of two orbits \mathcal{O}_{μ}^{I} and \mathcal{O}_{μ}^{II} causes some notational difficulties. The simplest solution is to abuse notation slightly and allow $\mathcal{O}_{\mu,\nu}$ to denote either of the orbits $\mathcal{O}_{\mu}^{I} \times \mathcal{O}_{\nu}$ or $\mathcal{O}_{\mu}^{II} \times \mathcal{O}_{\nu}$. Since the values of $k(\mathcal{O}_{\mu}^{I} \times \mathcal{O}_{\nu})$ and $k(\mathcal{O}_{\mu}^{II} \times \mathcal{O}_{\nu})$ turn out to be the same this does not create any problems.

3.6. Let $\mathfrak{g} = G(3)$, then $\mathfrak{g}_0 = \mathfrak{g}^1 \times \mathfrak{g}^2$ and $\mathfrak{g}_1 = V_1 \otimes V_2$ where $\mathfrak{g}^1 \cong \mathfrak{g}_2$, the 14 dimensional exceptional simple Lie algebra, $\mathfrak{g}^2 \cong s\ell(2), V_1$ is the 7dimensional simple \mathfrak{g}_2 -module and V_2 the 2-dimensional simple $s\ell(2)$ -module. There are invariant maps $\pi_1 : \wedge^2 V_1 \longrightarrow \mathfrak{g}_2, \pi_2 : S^2 V_2 \longrightarrow s\ell(2)$ and invariant bilinear forms ψ_1, ψ_2 such that the product $\mathfrak{g}_1 \times \mathfrak{g}_1 \longrightarrow \mathfrak{g}_0$ is given by equation (2). In particular since \mathfrak{g}_2 preserves ψ_1 it can be regarded as a subalgebra of $so(V_1) = so(7)$. If \mathcal{O} is a nilpotent orbit in \mathfrak{g}_2 we write $\mathcal{O} = \mathcal{O}_\mu$ where μ is the partition of 7 determined by the Jordan form of a representative element of \mathcal{O} when viewed as an element of $g\ell(V_1)$. These partitions, together with the usual Bala-Carter notation for orbits in \mathfrak{g}_2 [CM, page 128] and the dimension of the orbits are given in the table below.

$\mathcal{O}=\mathcal{O}_{\mu}$	0	A_1	\widetilde{A}_1	$G_2(a_1)$	G_2
μ	1^{7}	$2^2, 1^3$	$3, 2^2$	$3^2, 1$	7
$\dim \mathcal{O}$	0	6	8	10	12

For μ in the table and $\nu \in \mathbf{P}(2)$ set $\mathcal{O}_{\mu,\nu} = \mathcal{O}_{\mu} \times \mathcal{O}_{\nu}$.

In section 3.8 we apply Lemma 3.4 to calculate $k(\mathcal{O}_{\mu,\nu})$. However to do this we need to evaluate the matrix using an invariant bilinear form on \mathfrak{g}_2 , rather than on $g\ell(7)$ as was done in Lemma 3.4. Similar remarks apply when \mathfrak{g} is the Lie superalgebra F(4). Recall that any nonzero invariant form on a simple Lie algebra is proportional to the Killing form. Therefore since $\mathcal{O}_{\mu,\nu}$ is a product of conical subvarieties (see section 2.2), our method is justified by the following well-known lemma. Our proof is a modification of [LS, Lemma 2.5].

Lemma Suppose that $\mathfrak{k} \subseteq \mathfrak{l}$ are finite dimensional simple complex Lie algebras. Then the restriction of the Killing form B on \mathfrak{l} to \mathfrak{k} is nondegenerate.

Proof. There are connected, simply connected complex Lie groups K and L, unique up to isomorphism, such that $\mathfrak{k} = Lie(K)$ and $\mathfrak{l} = Lie(L)$. We can take K to be a subgroup of L since $\mathfrak{k} \subseteq \mathfrak{l}$.

Let K_0 denote a maximal compact subgroup of K. Then K_0 is contained in a maximal compact subgroup, L_0 , of L. Let \mathfrak{k}_0 (resp. \mathfrak{l}_0) denote the (real) Lie algebra of the compact Lie group K_0 (resp. L_0). We have $\mathfrak{l} = \mathfrak{l}_0 \oplus i\mathfrak{l}_0$ and $\mathfrak{k} = \mathfrak{k}_0 \oplus i\mathfrak{k}_0$.

Now *B* is negative definite when restricted to l_0 and hence it is negative definite on \mathfrak{k}_0 . Therefore the restriction *B'* of *B* to \mathfrak{k} is nonzero. However the radical of *B'* is an ideal of \mathfrak{k} , so *B'* is nondegenerate.

3.7. Now let $\mathfrak{g} = F(4)$. Then $\mathfrak{g}_0 = \mathfrak{g}^1 \times \mathfrak{g}^2$ and $\mathfrak{g}_1 = V_1 \otimes V_2$ where $\mathfrak{g}^1 \cong so(7), \mathfrak{g}^2 \cong s\ell(2), V_1$ is the spin representation of so(7) and V_2 is the 2-dimensional simple $s\ell(2)$ -module. We have the same analysis as for G(3) except that so(7) is now regarded as a subalgebra of $so(V_1) = so(8)$.

Nilpotent orbits in so(7) correspond to partitions $\eta \in \mathbf{P}_1(7)$. For $\eta \in \mathbf{P}_1(7)$ we write $\mu = \sigma(\eta)$ for where μ is the partition of 8 determined by the Jordan form of an element of the corresponding orbit when viewed as an element of $g\ell(V_1)$. We use μ to label the orbit. The map $\sigma : \mathbf{P}_1(7) \longrightarrow \mathbf{P}(8)$, together with the dimension of the orbits are given in the table below.

η	1^{7}	$2^2, 1^3$	$3, 1^4$	$3, 2^2$	$3^2, 1$	$5, 1^2$	7
$\mu = \sigma(\eta)$	1^{8}	$2^2, 1^4$	2^{4}	$3, 2^2, 1$	$3^2, 1^2$	4^{2}	7, 1
$\dim \mathcal{O}_{\mu}$	0	8	10	12	14	16	18

As before we set $\mathcal{O}_{\mu,\nu} = \mathcal{O}_{\mu} \times \mathcal{O}_{\nu}$ for μ in the table and $\nu \in \mathbf{P}(2)$.

3.8. Let $\mathfrak{g} = osp(m, n)$ $(m \geq 3)$, G(3) or F(4) and consider the nilpotent orbit $\mathcal{O}_{\mu,\nu}$ as defined in one of the three preceding subsections. Let dim $V_1 = m$ and dim $V_2 = n$.

Theorem We have

$$k(\mathcal{O}_{\mu,\nu}) = dim\mathfrak{g}_1 - \sum_i \mu'_i \nu'_i.$$

Proof. We use the notation from section 3.4. If $(x, y) \in \mathcal{O}_{\mu,\nu}$ then k(q) is the rank of the evaluation of $M(\mathfrak{g})$ at (x, y). This rank is the same as the rank of the evaluation of $(J_1 \otimes J_2)^{-1}M(\mathfrak{g}) = J_1^{-1}A \otimes I_n + I_m \otimes J_2^{-1}B$ at (x, y). Thus the result follows from Lemmas 2.4 and 3.4.

3.9. Theorem 3.8 does not apply to the Lie superalgebras $\mathfrak{g} = osp(m, 2r)$ when m = 1, 2. To handle these cases we use the description of osp(m, n) given in section 3.3.

If m = 1, we choose $e \in V_1$ such that $\psi_1(e, e) = 1$. Then for $v, w \in V_2$ we have

$$[e \otimes v, e \otimes w] = v \circ w. \tag{4}$$

If m = 2, we choose $e_{-}, e_{+} \in V_1$ such that

$$\psi_1(e_-, e_-) = \psi_1(e_+, e_+) = 0, \quad \psi_1(e_-, e_+) = 1.$$

Set $\mathfrak{g}_1^{\pm} = \mathbb{C}e_{\pm} \otimes V_2$, and $z = e_- \wedge e_+$. Then $\mathfrak{g}_0 = [\mathfrak{g}_0, \mathfrak{g}_0] \oplus \mathbb{C}z$, and $\mathfrak{g}_1 = \mathfrak{g}_1^+ \oplus \mathfrak{g}_1^-$, is a direct sum of \mathfrak{g}_0 -modules. Also $[\mathfrak{g}_1^{\pm}, \mathfrak{g}_1^{\pm}] = 0$ and for $v, w \in V_2$ we have

$$[e_{-} \otimes v, e_{+} \otimes w] = v \circ w + \psi_{2}(v, w)z.$$
(5)

If $\mathfrak{g} = osp(m, 2r)$ where m = 1, 2, then nilpotent orbits in \mathfrak{g}_0 are parameterized by partitions in $\mathbf{P}_{-1}(2r)$. We denote the orbit corresponding to a partition μ by \mathcal{O}_{μ} . Note that the rank of J_{μ} is $\sum_{i}(\mu_i - 1) = 2r - \mu'_1$.

Theorem (a) If $\mathfrak{g} = osp(1, 2r)$ and $\mu \in \mathbf{P}_{-1}(2r)$ we have

$$k(\mathcal{O}_{\mu}) = rank \ J_{\mu}.$$

(b) If $\mathfrak{g} = osp(2, 2r)$ and $\mu \in \mathbf{P}_{-1}(2r)$ we have

$$k(\mathcal{O}_{\mu}) = 2(rank \ J_{\mu}).$$

Proof. (a) Identify $\mathfrak{g}_1 = \mathbb{C}e \otimes V_2$ with V_2 via the map $e \otimes v \longrightarrow v$. Let e_1, \ldots, e_{2r} be a basis for \mathfrak{g}_1 and let J be the matrix of ψ on this basis.

The matrix $M(\mathfrak{g})$ equals $\sum_{i,j\in K} (e_i \circ e_j) \mathbf{e}_{\mathbf{i},\mathbf{j}}$, and as in the proof of Lemma 3.4 there is a nonzero constant λ such that

$$J^{-1}M(\mathfrak{g})(x) = \lambda x$$

for all $x \in \mathfrak{g}_0$. This easily gives the result.

(b) Let $\mathfrak{g} = osp(2, 2r)$, and $\mathfrak{k} = osp(1, 2r)$. By comparing equations (4) and (5), we see that with respect to a suitable ordered basis, $M(\mathfrak{g})$ has the block matrix form

$$\begin{bmatrix} 0 & M(\mathfrak{k}) \\ M(\mathfrak{k}) & 0 \end{bmatrix} \mod (z).$$

The result follows since z vanishes on any nilpotent orbit in \mathfrak{g}_0 .

3.10. Now let $\mathfrak{g} = \Gamma(\sigma_1, \sigma_2, \sigma_3)$ as in [Sch]. Then $\mathfrak{g}_0 = \mathfrak{g}^1 \times \mathfrak{g}^2 \times \mathfrak{g}^3$, $\mathfrak{g}_1 = V_1 \otimes V_2 \otimes V_3$ where $\mathfrak{g}^i \cong s\ell(2)$ and V_i is the 2-dimensional simple $s\ell(2)$ -module.

Let $\psi_i : V_i \times V_i \longrightarrow \mathbb{C}$ be a nonzero \mathfrak{g}^i -invariant skew-symmetric map and define a \mathfrak{g}^i -invariant symmetric map

$$\pi_i: V_i \times V_i \longrightarrow \mathfrak{g}^i$$

by

$$\pi_i(x,y)z = \psi_i(y,z)x - \psi_i(z,x)y$$

for $x, y, z \in V_i$. Then for $a_1 \otimes a_2 \otimes a_3, b_1 \otimes b_2 \otimes b_3 \in \mathfrak{g}_1$ we have

$$[a_1 \otimes a_2 \otimes a_3, b_1 \otimes b_2 \otimes b_3] = = \sum \sigma_k \psi_i(a_1, b_1) \psi_j(a_2, b_2) \pi_k(a_3, b_3)$$
(6)

where the sum is over all even permutations (i, j, k) of $\{1, 2, 3\}$. Let f, h, e be the basis of $s\ell(2)$ given by

$$f = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad h = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad e = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

and let $x = (1,0)^t$ and $y = (0,1)^t$ be basis vectors for the 2-dimensional $s\ell(2)$ -module. We write f_i, h_i, e_i (resp. x_i, y_i) for the corresponding elements of \mathfrak{g}^i (resp. V_i), and set $S_i = \{x_i, y_i\}$. Consider the matrices

$$\Psi = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad \Pi_i = \begin{bmatrix} 2e_i & -h_i \\ -h_i & -2f_i \end{bmatrix}$$

We assume that the matrix for each ψ_i on the ordered basis (x_i, y_i) for V_i is Ψ . Then the matrix for π_i on this basis is Π_i . We order the basis $\{a_1 \otimes a_2 \otimes a_3 | a_i \in S_i\}$ of \mathfrak{g}_1 lexicographically. It follows from equation (6) that the matrix $M(\mathfrak{g})$ is given by

$$M(\mathfrak{g}) = \sigma_3 \Psi \otimes \Psi \otimes \Pi_3 + \sigma_2 \Psi \otimes \Pi_2 \otimes \Psi + \sigma_1 \Pi_1 \otimes \Psi \otimes \Psi \tag{7}$$

This can also be deduced from Table I in [Z].

For $\mu, v, \eta \in \mathbf{P}(2)$ let $\mathcal{O}_{\{\mu,v,\eta\}}$ denote the orbit of $(J_{\mu}, J_{v}, J_{\eta})$ in \mathfrak{g}_{0} . Note that the evaluation of the matrix $\Psi^{-1}\Pi_{i}$ at any element x of \mathfrak{g}^{i} is a nonzero multiple of x. It follows from equation (7) that we can find $x \in \mathcal{O}_{\{\mu,v,\eta\}}$ such that the evaluation of $(\Psi \otimes \Psi \otimes \Psi)^{-1}M(\mathfrak{g})$ at x equals

$$J_{\mu} \otimes I_2 \otimes I_2 + I_2 \otimes J_{\nu} \otimes I_2 + I_2 \otimes I_2 \otimes J_{\eta}.$$

The values of dim $\mathcal{O}_{\{\mu,\upsilon,\eta\}}$ and $k(\mathcal{O}_{\{\mu,\upsilon,\eta\}})$ depend only on the set $\{\mu,\upsilon,\eta\}$. These values are given in the table below.

$[\mu, v, \eta]$	$\{2, 2, 2\}$	$\{2, 2, 1^2\}$	$\{2, 1^2, 1^2\}$	$\{1^2, 1^2, 1^2\}$
$\dim \mathcal{O}_{\{\mu, \upsilon, \eta\}}$	6	4	2	0
$k(\mathcal{O}_{\{\mu,\upsilon,\eta\}})$	5	4	4	0

We can view \mathfrak{g} as a deformation of D(2,1) = osp(4,2) and the values of $k(\mathcal{O})$ for \mathfrak{g} are the same as those for the corresponding orbits for D(2,1).

3.11. Let V_0, V_1 be vector spaces with bases e_1, \ldots, e_n and e'_1, \ldots, e'_n respectively, $V = V_0 \oplus V_1$, and let $\psi : V \longrightarrow V$ be the map sending e_i to e'_i and e'_i to $-e_i$. Let \mathfrak{g} denote the Lie superalgebra of all endomorphisms of V which supercommute with ψ . Then \mathfrak{g} is isomorphic to the Lie superalgebra of matrices of the form

$$\left[\begin{array}{rrr}a&b\\b&a\end{array}\right]$$

with $a, b \in g\ell(n)$. Thus $\mathfrak{g}_0 \cong g\ell(n)$ and $\mathfrak{g}_1 \cong \mathfrak{g}_0$ as a \mathfrak{g}_0 -module. The derived algebra \mathfrak{g}' consists of all matrices as above with $b \in s\ell(n)$. Also \mathfrak{g}' has a one-dimensional center $\mathfrak{z} = \mathbb{C}I_{2n}$. The factor algebra $\mathfrak{g}'/\mathfrak{z}$ is the simple Lie superalgebra denoted Q(n-1) in [K]. We assume that $n \geq 3$. Then the Lie superalgebra Q(n-1) is simple. As a Cartan subalgebra \mathfrak{h}_0 of \mathfrak{g}_0 we take all matrices of the above form with a diagonal and b = 0. We modify this in the obvious way to obtain Cartan subalgebras of \mathfrak{g}'_0 and $Q(n-1)_0$.

If \mathcal{O} is any nilpotent orbit in \mathfrak{g}'_0 then \mathfrak{z} vanishes on \mathcal{O} and \mathcal{O} may be regarded as a nilpotent orbit in $(\mathfrak{g}'/\mathfrak{z})_0$. All nilpotent orbits in $(\mathfrak{g}'/\mathfrak{z})_0$ arise in this way. Therefore it suffices to consider the Clifford algebras arising from \mathfrak{g} and \mathfrak{g}' .

If $\mu \in \mathbf{P}(n)$ let J_{μ} and \mathcal{O}_{μ} denote the corresponding Jordan matrix and nilpotent orbit. Set

$$\epsilon(\mu) = 1$$
 if all parts of μ are even
0 otherwise.

Note that $\mathfrak{g}'_0 = \mathfrak{g}_0$. If $q \in SpecS(\mathfrak{g}_0)$ let k(q) (resp. k'(q)) be the rank of the bilinear form on $\mathfrak{g}_1 \otimes F_q$ (resp $\mathfrak{g}'_1 \otimes F_q$) defined in the usual way.

Theorem If $V(q) = \overline{\mathcal{O}}_{\mu}$ then

(a)
$$k(q) = \dim \mathfrak{g}_1 - \sum_i (\mu'_i)^2$$

(b)
$$k'(q) = k(q) - 2\epsilon(\mu)$$

Proof. For $a \in g\ell(n)$ set

$$\overline{a} = \left[\begin{array}{cc} 0 & a \\ a & 0 \end{array} \right].$$

Let $K = \{1, \ldots, n\}$ and calculate $M(\mathfrak{g})$ using the basis $\{\overline{e}_{ij}\}$ of \mathfrak{g}_1 . The rows and columns of $M(\mathfrak{g})$ are indexed by $K \times K$ ordered lexicographically with entry in row (i, j) and column (k, ℓ) given by

$$[\overline{e}_{ij}, \overline{e}_{k\ell}] = \delta_{jk} e_{i\ell} + \delta_{i\ell} e_{kj}.$$

Thus

$$M(\mathbf{g}) = \sum_{i,j,k,\ell} (\delta_{jk} e_{i\ell} + \delta_{i\ell} e_{kj}) \mathbf{e}_{\mathbf{i},\mathbf{k}} \otimes \mathbf{e}_{\mathbf{j},\ell}.$$

If $L = \sum \mathbf{e_{r,s}} \otimes \mathbf{e_{s,r}}$, then L is nonsingular since L^2 is the identity matrix. Let $A = \sum_{i,j \in K} e_{ij} \mathbf{e_{i,j}}$. Then $(\mathbf{e_{i,k}} \otimes \mathbf{e_{j\ell}})L = \mathbf{e_{i,\ell}} \otimes \mathbf{e_{j,k}}$ and hence

$$M(\mathfrak{g})L = A \otimes I_n + I_n \otimes A^t.$$

Since $M(\mathfrak{g})$ and $M(\mathfrak{g})L$ have the same rank, part (a) of the Theorem follows from Lemma 2.4. Part (b) follows from the lemma in the next subsection.

3.12. With $\mathfrak{g}, \mathfrak{g}'$ as in subsection 3.11 we compare the matrices $M(\mathfrak{g})$ and $M(\mathfrak{g}')$. For $1 \leq i \leq n-1$ let $h_i = e_{ii} - e_{i+1,i+1}$ and let h_n be the identity matrix. We calculate $M(\mathfrak{g})$ using the basis

$$\{\overline{e}_{ij}, \overline{h}_k | 1 \le i \ne j \le n, 1 \le k \le n\}$$

of \mathfrak{g}_1 . We order this basis in any way such that the last *n* elements are $\overline{h}_1, \ldots, \overline{h}_n$.

Note that for $1 \leq i \leq n-1$ we have

$$[\overline{h}_i, \overline{e}_{k,k+1}] = (\delta_{i,k+1} - \delta_{i,k-1})e_{k,k+1}$$
(8)

$$[\overline{h_n}, \overline{e}_{k,k+1}] = 2e_{k,k+1}.$$
(9)

The evaluation of $M(\mathfrak{g})$ at J_{μ} has the block-matrix form

$$\begin{bmatrix} * & N(J_{\mu}) \\ \hline & N(J_{\mu})^t & 0 \end{bmatrix}$$

where N is the matrix with entries $[\overline{h}_i, \overline{e}_{k\ell}]$ $(1 \le i \le n, 1 \le k \ne \ell \le n)$. The evaluation of $M(\mathfrak{g}')$ at J_{μ} is obtained by deleting the last row and column.

For $i \neq j$ let C_{ij} be the column of $M(\mathfrak{g})$ corresponding to \overline{e}_{ij} . Also for $1 \leq i \leq n-1$ let C_i be the column corresponding to \overline{h}_i . The evaluation of a column C at J_{μ} is denoted $C(J_{\mu})$.

Lemma (a) If $(\sum_{k,\ell} \lambda_{k,\ell} C_{k,\ell} + \sum_{k=1}^{n} \nu_k C_k)(J_{\mu}) = 0$ then $\sum_{i=1}^{n} \nu_i C_i(J_{\mu}) = 0.$ (b) The linear span of the columns $C_1(J_\mu), \ldots, C_{n-1}(J_\mu)$ contains $C_n(J_\mu)$ if and only if some part of μ is odd.

Proof. (a) This follows since $(\sum_k \nu_k C_k)(J_\mu)$ can have nonzero entries only in rows (i, i + 1) and $(\sum \lambda_{k\ell} C_{k\ell})(J_\mu)$ has zero entries in these rows. (b) Consider the system of equations

$$C_n(J_\mu) = 2\sum_{i=1}^{n-1} x_i C_i(J_\mu)$$
(10)

in the unknowns x_1, \ldots, x_{n-1} . By equations (8) and (9) this system is equivalent to the evaluation of the system of equations

$$e_{k,k+1} = (x_{k+1} - x_{k-1})e_{k,k+1} \tag{11}$$

at J_{μ} . Here we set $x_0 = x_n = 0$. Thus the system (10) is equivalent to the equations

$$1 = x_{k+1} - x_{k-1} \text{ for } 1 \le k \le n - 1, k \ne \mu_1 + \ldots + \mu_i.$$
(12)

If μ_i is even for all *i*, then (12) involves the equations

$$1 = x_n - x_{n-2} = \ldots = x_2 - x_0$$

which are inconsistent.

On the other hand if some μ_i is odd, then $\mu_1 + \ldots + \mu_j$ is odd for some j, so the system (12) is equivalent to a number of systems of equations of the form

$$1 = x_p - x_{p-2} = \dots = x_{q+2} - x_q.$$
(13)

Moreover the sets of variables which occur in two such systems are disjoint, and in each system (13) we have either p < n or q > 0. If p < n (resp. q > 0), we can set $x_q = 0$ (resp. $x_p = 0$) and solve the equations (13) recursively for x_{q+2i} (resp. x_{p-2i}).

Remark. If $V(q) = \overline{\mathcal{O}}_{\mu}$ it follows from Theorem 3.11 and [CM, Corollary 7.2.4] that $k(q) = \dim \overline{\mathcal{O}}_{\mu}$.

3.13. For any classical simple Lie superalgebra \mathfrak{g} considered up to this point, the matrix $M(\mathfrak{g})$ is nonsingular. This fact together with some Clifford

algebra theory can be used to show that $U(\mathfrak{g})$ is prime, [B]. However if $\mathfrak{g} = P(n)$, it is shown in [KK] that $U(\mathfrak{g})$ is not prime, and it follows that $M(\mathfrak{g})$ is singular. Because of this it seems unlikely that $M(\mathfrak{g})$ can be expressed in terms of a Kronecker product. However if \mathcal{O} is a nilpotent orbit in \mathfrak{g}_0 there is a formula for $k(\mathcal{O})$ which is similar to the formula for the corresponding orbit for the Lie superalgebra Q(n).

For $n \ge 2$ the Lie superalgebra P(n) is the subalgebra of $s\ell(n+1, n+1)$ consisting of all matrices of the form

$$\left[\begin{array}{cc} A & B \\ C & -A^t \end{array}\right]$$

where trace (A) = 0, $B^t = B$ and $C^t = -C$.

If $\mathfrak{g} = P(n)$, then $\mathfrak{g}_0 \cong s\ell(n+1)$. As a \mathfrak{g}_0 -module, \mathfrak{g}_1 is the direct sum of two submodules \mathfrak{g}_1^+ where \mathfrak{g}_1^+ (resp. \mathfrak{g}_1^-) consists of all matrices as above with B = 0 (resp. C = 0). Let V be the natural module for $s\ell(n+1)$ with weights $\epsilon_1, \ldots, \epsilon_{n+1}$. Then, as \mathfrak{g}_0 -modules $\mathfrak{g}_1^+ \cong S^2 V$ and $\mathfrak{g}_1^- \cong \wedge^2 V^*$.

Fix μ a partition of n. We assume that the nonzero entries in the Jordan matrix J_{μ} occur immediately below the main diagonal. For $1 \leq i \leq n-1$, let b_i be the entry of J_{μ} in row i + 1, column i, and let $b_0 = b_n = 0$. Denote the orbit of \mathcal{O}_{μ} in \mathfrak{g}_0 by J_{μ} .

Theorem For $\mu \in \mathbf{P}(n)$ we have

$$k(\mathcal{O}_{\mu}) = 2\sum_{i=1}^{n-1} (n-i)b_i = n^2 - \sum_i (\mu'_i)^2.$$

Proof. If we choose a basis for \mathfrak{g}_1 such that elements of \mathfrak{g}_1^+ precede elements of \mathfrak{g}_1^- , then $M(\mathfrak{g})$ has the form

$$\left[\begin{array}{cc} 0 & N \\ N^t & 0 \end{array}\right].$$

Let $\epsilon_1, \ldots, \epsilon_{n+1}$ be the weights of V. We use the weights $-\epsilon_i - \epsilon_j$ of $\wedge^2 V^*$ to index the rows, and the weights $\epsilon_k + \epsilon_\ell$ $(k \leq \ell)$ of $S^2 V$ to index the columns of N. We order the rows of N lexicographically and order the columns so that column (i, j) precedes column (k, ℓ) if and only if $(k, \ell) <_{\ell ex} (i, j)$. Note that the evaluation N_{μ} of N at J_{μ} has the following properties:

(a) The entry in row (i, j) and column (i + 1, j) is nonzero if and only if $b_i = 1$

(b) The entry in row (i, j) and column (i, j + 1) is nonzero if and only if $b_j = 1$.

(c) All other entries in row (i, j) are zero.

We claim that N_{μ} is row equivalent to the matrix \overline{N}_{μ} obtained from N_{μ} by replacing row (i, j) by zero for all j > i whenever $b_i = 0$. We can assume that i > 1, since if $b_1 = 0$ then $J_{\mu} = 0$, and also that $b_j = 1$. Then $b_{i-1} = 1$. Suppose that $b_i = b_{i-q-1} = 0$ but $b_{i-p} \neq 0$ for $p = 1, \ldots, q$. This means that the Jordan block of J_{μ} ending in row i has size q + 1. Since the Jordan blocks of J_{μ} are arranged in order of decreasing size and i < j it follows that the Jordan block of J_{μ} containing row j has size at most q + 1. Hence $b_{j+p} = 0$ for some p with $p \leq q$ and we fix p minimal with this property.

Then the submatrix of N_{μ} formed by rows (i - p - 1 + k, j + p + 1 - k), for $k = 1, \ldots, p + 1$, and columns (i - k, j + k + 1), for $k = 0, \ldots, p - 1$ has the form

000	• • •	0.0 *
$0 \ 0 \ 0$	• • •	0 * *
$0 \ 0 \ 0$	• • •	**0
• • •		• • •
• • •		•••
0 * *	• • •	$0 \ 0 \ 0$
**0	• • •	$0 \ 0 \ 0$
		0 0 0

where each * is nonzero. In addition every nonzero entry in each of the rows listed above occurs in this submatrix. Hence the last row, row (i, j) of N_{μ} , is a linear combination of the preceding rows. The claim follows from this.

Now if $b_i \neq 0$, then for j > i the first entry in row (i, j) of \overline{N}_{μ} occurs in column (i + 1, j). Each such index *i* contributes n - i linearly independent rows to the rank of \overline{N}_{μ} , so we obtain the first formula in the theorem.

To obtain the second formula, note that $b_i = 0$ if $i = \mu_1 + \ldots + \mu_k$ for

some k and that $b_i = 1$ otherwise. Hence

$$k(\mathcal{O}_{\mu}) = 2\sum_{i} (n-i)b_{i}$$

= $n(n-1) - 2\sum_{k} (n - (\mu_{1} + \ldots + \mu_{k})).$

Observe that $n - (\mu_1 + \ldots + \mu_k)$ is the number of boxes in the Young diagram for μ which are not contained in the first k rows. Using the columns instead to count boxes we have

$$k(\mathcal{O}_{\mu}) = n(n-1) - 2\sum_{i\geq 1}\sum_{j\geq 1} (\mu'_i - j)$$

= $n(n-1) - \sum_{i\geq 1} \mu'_i(\mu'_i - 1)$
= $n^2 - \sum_i (\mu'_i)^2.$

4 Parabolic Subalgebras

4.1. Although the connection with Clifford algebras works best for the Lie superalgebras $g\ell(m,n)$, and Q(n) many of our results on induced modules hold more generally. Therefore we adopt an axiomatic approach. Henceforth we assume that

(i) $\mathfrak{g} = \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}(i)$ is a graded Lie superalgebra with \mathfrak{g}_0 reductive.

(ii) $\mathfrak{h}_0 \subseteq \mathfrak{g}(0)$ where \mathfrak{h}_0 is a Cartan subalgebra (CSA) of \mathfrak{g}_0 and \mathfrak{g} is a semisimple \mathfrak{h}_0 -module.

Assume axioms (i) - (ii) and set

$$\mathfrak{m} = \oplus_{i < 0} \mathfrak{g}(i), \quad \mathfrak{l} = \mathfrak{g}(0), \quad \mathfrak{m}^+ = \oplus_{i > 0} \mathfrak{g}(i), \quad \mathfrak{p} = \mathfrak{l} \oplus \mathfrak{m}^+$$

so that

$$\mathfrak{g}=\mathfrak{m}\oplus\mathfrak{l}\oplus\mathfrak{m}^+.$$

Let \mathfrak{h} be the centralizer of \mathfrak{h}_0 in \mathfrak{g} . Axiom (ii) implies the existence of a root space decomposition

$$\mathfrak{g} = \mathfrak{h} \oplus \oplus_{\alpha \in \Delta} \mathfrak{g}^{lpha}$$

where

$$\mathfrak{g}^{\alpha} = \{ x \in \mathfrak{g} | [h, x] = \alpha(h) x \text{ for all } h \in \mathfrak{h}_0 \}$$

and

$$\Delta = \{ \alpha \in \mathfrak{h}_0^* | \alpha \neq 0, \mathfrak{g}^\alpha \neq 0 \}.$$

We also assume that

(iii) $\Delta = \Delta^+ \cup \Delta^-$, a disjoint union, where Δ^{\pm} are subsets of Δ such that $\alpha, \beta \in \Delta^{\pm}$ implies that $\alpha + \beta \in \Delta^{\pm}$ or $\mathfrak{g}^{\alpha+\beta} = 0$, and such that $\mathfrak{g}^{\alpha} \cap \mathfrak{g}_0 \subseteq \mathfrak{p}$ for all $\alpha \in \Delta^+$.

Now let $\Delta(\mathfrak{l})$ be the set of roots of \mathfrak{l} and set $\Delta^{\pm}(\mathfrak{l}) = \Delta^{\pm} \cap \Delta(\mathfrak{l})$. If Γ is a subset of Δ and i = 0, 1 we set $\Gamma_i = \{\alpha \in \Gamma | \mathfrak{g}^{\alpha} \cap \mathfrak{g}_i \neq 0\}$. We refer to the subalgebra

$$\mathfrak{b}=\mathfrak{h}\oplus\oplus_{lpha\in\Delta^+}\mathfrak{g}^lpha$$

as a *Borel subalgebra* of \mathfrak{g} . Note that \mathfrak{b} is determined by Δ^+ in axiom (iii) and that in general there may be several choices for Δ^+ even if Δ_0^+ is specified in advance. By axiom (iii) $\mathfrak{b}_0 \subseteq \mathfrak{p}$. The subalgebra

$$\mathfrak{c} = \mathfrak{h} \oplus \oplus_{\alpha \in \Delta^+(\mathfrak{l})} \mathfrak{g}^{\alpha} = \mathfrak{b} \cap \mathfrak{l}$$

is a Borel subalgebra of \mathfrak{l} . We say that a root $\alpha \in \Delta^+(\mathfrak{l})_0$ (resp. $\alpha \in \Delta^+(\mathfrak{l})$) is indecomposable if we cannot write α in the form $\alpha' + \alpha''$ with $\alpha', \alpha'' \in \Delta^+(\mathfrak{l})_0$ (resp. $\alpha', \alpha'' \in \Delta^+(\mathfrak{l})$). Let S (resp. T) be the set of indecomposable roots of $\Delta^+(\mathfrak{l}_0)$, (resp. $\Delta^+(\mathfrak{l})$).

Let \mathcal{O} be the Richardson orbit induced from a Levi factor of \mathfrak{p}_0 . We say \mathfrak{p} is a *good parabolic* if dim $(\mathfrak{g}/\mathfrak{p})_1 = \ell(\mathcal{O})$, (see Section 2.2 for notation). In Section 5 we show that modules induced from a one dimensional module for a good parabolic have the least possible multiplicity allowed by the Clifford algebra theory.

4.2. We assume axioms (i)-(iii). For $\lambda \in \mathfrak{h}_0^*$ we define the simple highest

weight \mathfrak{l}_0 -module $\widehat{L}_S(\lambda)$ as the unique simple quotient of the Verma module with highest weight λ induced from the Borel subalgebra $\mathfrak{c}(0)$ of \mathfrak{l}_0 , c.f. [Ja2, 5.11]. The module $\widehat{L}_S(\lambda)$ is finite dimensional if and only if $\lambda \in P_S^{++}$ where

$$P_S^{++} = \{ \lambda \in \mathfrak{h}_0^* | (\lambda, \alpha^{\vee}) \in \mathbb{N} \text{ for all } \alpha \in S \}.$$

For $\lambda \in \mathfrak{h}_0^*$ there is a unique graded simple \mathfrak{c} -module V_{λ} such that $\mathfrak{g}^{\alpha}V_{\lambda} = 0$ for all $\alpha \in \Delta^+(\mathfrak{l})$ and $(h - \lambda(h))V_{\lambda} = 0$ for all $h \in \mathfrak{h}_0^*$. We remark that if \mathfrak{c} involves no classical simple Lie superalgebra of type Q, then $\mathfrak{h} = \mathfrak{h}_0$ and $\dim V_{\lambda} = 1$ for all $\lambda \in \mathfrak{h}_0^*$. The induced module $Ind_{\mathfrak{c}}^{\mathfrak{l}} V_{\lambda}$ has a unique simple graded quotient which we denote by $\hat{L}_T(\lambda)$. The conditions for $\hat{L}_T(\lambda)$ to be finite dimensional are rather involved c.f. [K]. However it is easy to show that $\dim \hat{L}_T(\lambda) = 1$ if and only if $\lambda \in \mathfrak{l}^{\perp}$ where

$$\mathfrak{l}^{\perp} = \{ \lambda \in \mathfrak{h}_0^* | \lambda([\mathfrak{l}, \mathfrak{l}] \cap \mathfrak{h}_0) = 0 \}.$$

We can regard $\widehat{L}_S(\lambda)$, (resp. $\widehat{L}_T(\lambda)$) as a $U(\mathfrak{p}_0)$ -module (resp. $U(\mathfrak{p})$ -module) by allowing \mathfrak{m}_0^+ , (resp. \mathfrak{m}^+) to act trivially and form the induced modules

$$M_S(\lambda) = Ind_{\mathbf{p}_0}^{\mathbf{g}_0} \widehat{L}_S(\lambda), \quad M_T(\lambda) = Ind_{\mathbf{p}}^{\mathbf{g}} \widehat{L}_T(\lambda).$$

By Lemma 2.6

$$d(M_T(\lambda)) = \dim(\mathfrak{g}/\mathfrak{p})_0 + d(\widehat{L}_T(\lambda))$$

and

$$e(M_T(\lambda)) = 2^{(\mathfrak{g}/\mathfrak{p})_1} e(\widehat{L}_T(\lambda)).$$

To explain the choice of notation: $M_S(\lambda)$ conforms to the usage in [Ja2] while $\widehat{L}_S(\lambda)$ is denoted $\widehat{L}^S(\lambda)$ in [Ja2]. For $\widehat{L}_T(\lambda)$ and $M_T(\lambda)$ we want something similar which emphasizes the dependence on T rather than S.

4.3. We can obtain a Lie superalgebra satisfying axioms (i) - (iii) as follows. Suppose that \mathfrak{g}_0 is reductive with CSA \mathfrak{h}_0 , V is a \mathbb{Z}_2 -graded \mathfrak{g} -module and $V = \bigoplus_{k=1}^t V(k)$ where V(k) is a \mathbb{Z}_2 -graded, \mathfrak{h}_0 -stable subspace. Set V(k) = 0 unless $1 \leq k \leq t$ and

$$\mathfrak{g}(i) = \{ x \in \mathfrak{g} | xV(k) \subseteq V(k-i) \text{ for all } k \}$$

In all the examples we consider $\mathfrak{g} = \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}(i)$ satisfies axioms (i) - (iii).

For the rest of this subsection suppose that $\mathfrak{g} = g\ell(m, n)$ and that $\mathbf{I} = \mathbf{I}_1 \cup \mathbf{I}_2$ is as in section 3.1. Set $V = span\{e_i | i \in \mathbf{I}\}$, the natural \mathfrak{g} -module. Consider the function $\sigma : \mathbf{I} \longrightarrow \{1, \ldots, t\}$ defined by $e_i \in V(\sigma(i))$ for all i. Then $e_{ij} \in \mathfrak{g}(\sigma(j) - \sigma(i))$. We assume the Borel subalgebra \mathbf{b} of upper triangular matrices in \mathfrak{g}_0 is a subalgebra of \mathfrak{p} . In terms of σ this means that if i < j and either $j \leq m$ or $n+1 \leq i$ we have $\sigma(i) \leq \sigma(j)$. For $1 \leq k \leq t$ set

$$\Lambda_k = \{i \in \mathbf{I} | \sigma(i) = k\}$$

and

$$r_k = |\Lambda_k \cap \mathbf{I_1}|, \quad s_k = |\Lambda_k \cap \mathbf{I_2}|.$$

We can rearrange the sequences $\mathbf{r} = (r_1, \ldots, r_t)$ and $\mathbf{s} = (s_1, \ldots, s_t)$ to obtain partitions $\mu' \in \mathbf{P}(m), \nu' \in \mathbf{P}(n)$. Note that the sequences \mathbf{r}, \mathbf{s} determine the subspaces V(k). Also $\mathfrak{l} = \mathfrak{g}(0) \cong \bigoplus_{i=1}^t g\ell(r_i, s_i)$. It follows that

$$\dim(\mathfrak{g}/\mathfrak{p})_1 = mn - \sum_{i=1}^t r_i s_i.$$

Lemma (a) The Richardson orbit induced from l_0 is $\mathcal{O}_{\mu,\nu}$. (b) \mathfrak{p} is a good parabolic if and only if there is a permutation η of $\{1, \ldots, t\}$ such that $\mu'_i = r_{\eta(i)}$ and $\nu'_i = s_{\eta(i)}$ for $1 \leq i \leq t$.

Proof. (a) follows from [CM, Theorem 7.2.3]. (b) By Lemma 3.4 \mathfrak{p} is good if and only if $\sum \mu'_i \nu'_i = \sum r_i s_i$. We can assume that $\mu'_i = r_i$ for all *i*. Suppose that $r_j > r_{j+1}$ but $s_j < s_{j+1}$ and define $s'_j = s_{j+1}, s'_{j+1} = s_j$ and $s'_i = s_i$ for $i \neq j, j+1$. Then $\sum r_i s'_i > \sum r_i s_i$. The result follows from this observation.

We define $\epsilon_i \in \mathfrak{h}_0^*$ so that $\epsilon_i(x)$ is the i^{th} diagonal entry of x. We take $\Delta^+ = \{\epsilon_i - \epsilon_j | i < j\}$. For this choice of Δ we have dim $\widehat{L}_T(\lambda) < \infty$ if and only if dim $\widehat{L}_S(\lambda) < \infty$. Note that \mathfrak{p} need not contain the distinguished Borel subalgebra of \mathfrak{g} as the following examples show.

Example. Let (m, n) = (4, 3) and define σ by

$$\sigma(1) = \sigma(2) = \sigma(5) = 1$$

$$\sigma(3) = \sigma(6) = \sigma(7) = 2$$

$$\sigma(4) = 3$$

Then

$$\Lambda_1 = \{1, 2, 5\}, \Lambda_2 = \{3, 6, 7\}, \Lambda_3 = \{4\}$$

so $\mathbf{r} = (2, 1, 1), \mathbf{s} = (1, 2, 0).$ Also $S = \{\epsilon_1 - \epsilon_2, \epsilon_6 - \epsilon_7\}$ and
 $\mathfrak{l} \cong g\ell(2, 1) \oplus g\ell(1, 2) \oplus g\ell(1, 0).$

In this case p is not a good parabolic.

If we arrange instead that

$$\Lambda_1 = \{1, 2, 5, 6\}, \Lambda_2 = \{3, 7\}, \Lambda_3 = \{4\}$$

then

$$\mathfrak{l} \cong g\ell(2,2) \oplus g\ell(1,1) \oplus g\ell(1,0).$$

In this case p is a good parabolic.

4.4. Now let V, ψ, \mathfrak{g} and \mathfrak{g}' be as in section 3.11 and set $\overline{\mathfrak{g}} = \mathfrak{g}'/\mathfrak{z}$, the simple Lie superalgebra of type Q(n-1). We show how to associate a good parabolic in \mathfrak{g} and $\overline{\mathfrak{g}}$ to most nilpotent orbits. Suppose that $V = \bigoplus_{k=1}^{t} V(k)$ where V(k) is a \mathbb{Z}_2 -graded subspace of V stable under \mathfrak{h}_0 and ψ . Set $r_k = \dim V(k)_0$ and rearrange the sequence $\mathbf{r} = (r_1, \ldots, r_t)$ to obtain a partition $\mu' \in \mathbf{P}(n)$. The grading on \mathfrak{g} defined in section 4.3 induces a grading on \mathfrak{g}' and $\overline{\mathfrak{g}}$.

Let $\mathfrak{l}_{\mathbf{r}}$ be the block diagonal subalgebra of $g\ell(n)$ with diagonal entries of size r_1, \ldots, r_t and set $\mathfrak{l}'_{\mathbf{r}} = \{x \in \mathfrak{l}_{\mathbf{r}} | trace(x) = 0\}$. By [CM, Theorem 7.2.3] the Richardson orbit in $g\ell(n)$ (resp. $s\ell(n)$) induced from $\mathfrak{l}_{\mathbf{r}}$ (resp $\mathfrak{l}'_{\mathbf{r}}$) is \mathcal{O}_{μ} . Also $\mathfrak{g}(0)$ consists of all matrices of the form

$$\left[\begin{array}{rrr}a&b\\b&a\end{array}\right]$$

with $a, b \in \mathfrak{l}_{\mathbf{r}}$, while $\overline{\mathfrak{g}}(0)$ consists of the images mod \mathfrak{z} of matrices of this form with $a, b \in \mathfrak{l}'_{\mathbf{r}}$. Set $\mathfrak{p} = \bigoplus_{i \ge 0} \mathfrak{g}(i)$ and $\overline{\mathfrak{p}} = \bigoplus_{i \ge 0} \overline{\mathfrak{g}}(i)$. Then $\dim(\mathfrak{g}/\mathfrak{p})_1 = n^2 - \sum_i (\mu'_i)^2$. Thus from Theorem 3.12 we get the following result.

Lemma (a) \mathfrak{p} is a good parabolic in \mathfrak{g} . (b) If some part of μ is odd then $\overline{\mathfrak{p}}$ is a good parabolic in $\overline{\mathfrak{g}}$.

5 Induced Modules and Primitive Ideals

5.1. The connection between the Clifford algebras C_q and modules of low multiplicity is based on the following result.

Lemma Let N be a nonzero finitely generated graded $grU(\mathfrak{g})$ -module such that $q = ann_{S(\mathfrak{g}_0)}N$ is prime and N is torsion free as a $S(\mathfrak{g}_0)/q$ -module. If \mathcal{V} is the closed set in \mathfrak{g}_0^* defined by q then $d(N) = d(S(\mathfrak{g}_0)/q)$ and $e(N) \geq 2^{\ell(\mathcal{V})}e(\mathcal{V})$. Furthermore if $e(N) = 2^{\ell(\mathcal{V})}e(\mathcal{V})$ then C_q is split.

Proof. Clearly $d(N) \leq d(S(\mathfrak{g}_0)/q)$. Let $\mathcal{C} = \mathcal{C}(q)$ so that $N_{\mathcal{C}}$ is a $(grU(\mathfrak{g})/q)_{\mathcal{C}}$ -module. There is a factor module of $N_{\mathcal{C}}$ which is a simple module over $C_q = (grU(\mathfrak{g})/\pi(q))_{\mathcal{C}}$. By [GW, Theorem 9.17 (a)] this factor has the form $\overline{N}_{\mathcal{C}}$ for some $grU(\mathfrak{g})/\pi(q)$ factor module \overline{N} of N. Hence by the remarks in Section 2.5 and Lemma 2.1

$$d(N) \ge d(\overline{N}) = d(S(\mathfrak{g}_0)/q)$$

and

$$e(N) \ge e(\overline{N}) = 2^{\ell(\mathcal{V})} e(\mathcal{V}).$$

The last statement follows from Lemma 2.1.

5.2. To apply Lemma 5.1 let M be a finitely generated $U(\mathfrak{g})$ -module. We equip M with a good filtration and consider an affiliated series

$$0 = N_0 \subset N_1 \subset \ldots \subset N_k = N$$

for the graded module N = grM. Let p_1, \ldots, p_k be the affiliated primes of this series and $q_i = \pi^{-1}(p_i)$. By [GW, Prop. 2.13] each factor N_i/N_{i+1} is torsion-free as a $grU(\mathfrak{g})/p_i$ -module and hence also as a $S(\mathfrak{g}_0)/q_i$ -module. Thus

$$e(M) = e(N) = \sum e(N_i/N_{i+1}) \ge \sum 2^{\ell(q_i)} e(S(\mathfrak{g}_0)/q_i)$$

where both sums are taken over all indices i such that $d(N_i/N_{i+1}) = d(N)$.

By [GW, Proposition 2.14], any prime ideal of which is minimal over ann N is equal to one of the p_i and it follows easily that $\sqrt{ann_{S(\mathbf{g}_0)}N} =$ $q_1 \cap q_2 \ldots \cap q_k$. The closed subset of $SpecS(\mathfrak{g}_0)$ defined by $ann_{S(\mathfrak{g}_0)}N$ is called the *associated variety* of M. This definition is independent of the choice of good filtration [Ja2, 17.2]. These considerations motivate the study of modules M whose associated variety has a unique component \mathcal{V} with dimension equal to d(M) and such that $e(M) = 2^{\ell(\mathcal{V})}e(\mathcal{V})$.

For primitive factors $U(\mathfrak{g})/P$ the Goldie rank, $rank(U(\mathfrak{g})/P)$ is a more important invariant than $e(U(\mathfrak{g})/P)$ so we should try to find primitives P such that $rank(U(\mathfrak{g})/P) \leq 2^{\ell(q)}$. where $q = \sqrt{grP} \cap S(\mathfrak{g}_0)$.

5.3. For the remainder of the paper we assume that conditions (i) - (iii) of Section 4.1 hold.

Theorem Suppose that dim $\widehat{L}_T(\lambda) < \infty$ and that dim $(\mathfrak{g}/\mathfrak{p})_1 = c$. Then $M_T(\lambda)$ has a filtration by \mathfrak{g}_0 -submodules

$$0 = M_0 \subset M_1 \subset \ldots \subset M_k = M_T(\lambda)$$

such that for $i = 1, \ldots, k$

$$M_i/M_{i-1} \cong M_S(\lambda_i)$$

for certain $\lambda_i \in P_S^{++}$ and

$$\sum_{i=1}^{k} \dim \widehat{L}_{S}(\lambda_{i}) = 2^{c} \dim \widehat{L}_{T}(\lambda).$$
(14)

Proof. To simplify notation set $M = M_T(\lambda)$. We extend the grading on \mathfrak{g} defined in section 4.1 to $U(\mathfrak{g})$ and $\wedge \mathfrak{m}_1$. Note that \mathfrak{m}_1 is an \mathfrak{l}_0 -module. Antisymmetrization gives an injective map of \mathfrak{l}_0 -modules

$$\wedge \mathfrak{m}_1 \longrightarrow U(\mathfrak{m})$$

and we identify $\wedge \mathfrak{m}_1$ with its image. Then

$$U(\mathfrak{m}) = U(\mathfrak{m}_0) \otimes \wedge \mathfrak{m}_1.$$

It is easy to see that the extended grading satisfies

$$[\mathfrak{m}_0^+(j), (\wedge \mathfrak{m}_1)(-i)] \subseteq \oplus_{r,s,t} U(\mathfrak{m}_0)(-r) \otimes (\wedge \mathfrak{m}_1)(-s) \otimes U(\mathfrak{p})(t)$$
(15)

for all i, j > 0, where the sum is over all $r, s, t \ge 0$ such that j - i = t - r - s. Furthermore since $[\mathfrak{m}_0^+(j), \mathfrak{m}_1(-k)] \subseteq \mathfrak{g}(j-k)$ we can restrict the sum on the right to terms with t < j. In particular, each summand satisfies s < i.

Now for $i \geq 0$, set $\wedge_i = \bigoplus_{j \leq i} \wedge \mathfrak{m}_1(-j)$, $L'_i = \wedge_i \otimes \widehat{L}_T(\lambda) \subseteq M$, and define $M'_i = U(\mathfrak{g}_0)L'_i$. This process terminates when $L'_N = (\wedge \mathfrak{m}_1) \otimes \widehat{L}_T(\lambda)$ and $M'_N = M$ for some N. Note that each L'_i is an \mathfrak{l}_0 -module. Since $\widehat{L}_T(\lambda)$ is a $U(\mathfrak{p})$ -module with $\mathfrak{m}_0^+ \widehat{L}_T(\lambda) = 0$ it follows from equation (15) that $\mathfrak{m}_0^+ L'_i \subseteq M'_{i-1}$.

We refine the series $0 = L'_0 \subset L'_1 \subset \ldots \subset L'_N = \wedge \mathfrak{m}_1 \otimes \widehat{L}_T(\lambda)$ to a composition series

$$0 = L_0 \subset L_1 \subset \ldots \subset L_k = \wedge \mathfrak{m}_1 \otimes \widehat{L}_T(\lambda)$$

of $\wedge \mathfrak{m}_1 \otimes \widehat{L}_T(\lambda)$ as an \mathfrak{l}_0 -module and define $M_i = U(\mathfrak{g}_0)L_i$. Since each L_i/L_{i-1} is finite dimensional it follows that $L_i/L_{i-1} \cong \widehat{L}_S(\lambda_i)$ for $\lambda_i \in P_S^{++}$.

Also for each *i* we have $L'_{j-1} \subseteq L_{i-1} \subset L_i \subseteq L'_j$ for some *j* and hence $\mathfrak{m}_0^+ L_i \subseteq \mathfrak{m}_0^+ L'_j \subseteq M'_{i-1} \subseteq M_{i-1}$. Thus

$$\overline{L}_i = (L_i + M_{i-1})/M_{i-1}$$

is a $U(\mathfrak{p}_0)$ -module and $M_i/M_{i-1} = U(\mathfrak{g}_0)\overline{L}_i$. Hence M_i/M_{i-1} is a homomorphic image of $Ind_{\mathfrak{p}_0}^{\mathfrak{g}_0}\overline{L}_i$ and \overline{L}_i is a homomorphic image of L_i/L_{i-1} . It follows that

$$[M_i/M_{i-1}] \le [Ind_{\mathbf{p}_0}^{\mathbf{g}_0} L_i/L_{i-1}].$$

$$(16)$$

Therefore

$$[M] = \sum_{i=1}^{k} [M_i/M_{i-1}] \le \sum_{i=1}^{k} [Ind_{\mathbf{p}_0}^{\mathbf{g}_0} L_i/L_{i-1}] = [M]$$

where the last equality is obtained by comparing characters using the PBW theorem. Thus equality holds in (16) and it follows that $M_i/M_{i-1} \cong M_S(\lambda_i)$.

Remark. From the proof we see that as an l_0 -module

$$\oplus_i \widehat{L}_S(\lambda_i) \cong \wedge \mathfrak{m}_1 \otimes \widehat{L}_T(\lambda).$$

With this additional information the theorem generalizes [M1, Theorem 3.2].

5.4. The next result is an analog of [Ja2,15.5(a)].

Corollary If $\widehat{L}_T(\lambda)$ is finite dimensional then $M_T(\lambda)$ is a homogeneous $U(\mathfrak{g})$ -module.

Proof. Let N be a nonzero submodule of $M_T(\lambda)$ and choose *i* minimal such that $N \cap M_i \neq 0$. Then $N \cap M_i$ is isomorphic to a nonzero submodule of $M_S(\lambda_i)$ which is a homogeneous $U(\mathfrak{g}_0)$ -module by [Ja2, Satz 15.5(a)]. Hence

$$d(M_S(\lambda_i)) = d(N \cap M_i) \leq d(N) \leq d(M_T(\lambda)).$$

The result follows since $d(M_S(\lambda_i)) = d(M_T(\lambda)) = \dim(\mathfrak{g}/\mathfrak{p})_0$.

5.5. The following result is an analog of [Ja2, 17.16].

Lemma The associated variety $V(grann_{U(\mathfrak{g}_0)}M_T(\lambda))$ is the closure of the Richardson orbit induced from a Levi factor of \mathfrak{p}_0 .

Proof. Consider the series $M_0 \subset M_1 \subset \ldots \subset M_k = M_T(\lambda)$ of Theorem 5.3 and set $I_S(\lambda_i) = ann_{U(\mathfrak{g}_0)}M_S(\lambda_i)$. Then

$$I_S(\lambda_1) \dots I_S(\lambda_k) \subseteq ann_{U(\mathfrak{g}_0)} M_T(\lambda) \subseteq I_S(\lambda_i)$$

so that

$$\prod grI_S(\lambda_i) \subseteq grann_{U(\mathfrak{g}_0)}M_T(\lambda) \subseteq grI_S(\lambda_i)$$

for all *i*. On the other hand by [Ja2, 17.16] $V(grann_{U(\mathfrak{g}_0)}M_S(\lambda_i)) = G\mathfrak{m}_0$ for all *i* so the result follows.

5.6. For the proof of Theorem 5.7 we need a good filtration on $M_T(\lambda)$ with special properties.

Lemma If $M = M_T(\lambda)$ and $q = S(\mathfrak{g}_0)\mathfrak{p}_0$, there is a good filtration on M such that $ann_{S(\mathfrak{g}_0)}(grM) = q$ and grM is a torsion free $S(\mathfrak{g}_0)/q$ -module.

Proof. Let $U_n(\mathfrak{m}) = U_n(\mathfrak{g}) \cap U(\mathfrak{m})$ and

$$M_n = U_n(\mathfrak{m}) \otimes L_T(\lambda)$$

Since $\mathfrak{p}\widehat{L}_T(\lambda) \subseteq \widehat{L}_T(\lambda)$, an easy induction shows that $\mathfrak{g}_0 M_n \subseteq M_{n+2}$ and $\mathfrak{g}_1 M_n \subseteq M_{n+1}$, so $\{M_n\}$ is a filtration of M as a $U(\mathfrak{g})$ -module.

Similarly we have $\mathfrak{p}_0 M_n \subseteq M_n$ which implies $S(\mathfrak{g}_0)\mathfrak{p}_0 \subseteq ann_{S(\mathfrak{g}_0)}grM$. On the other hand $grM \simeq grU(\mathfrak{m}) \otimes \widehat{L}_T(\lambda)$ is a free $S(\mathfrak{m}_0)$ -module. The result follows from this.

5.7. Part (a) of the next result is an analog of [Ja2, Satz 15.5b)].

Theorem Suppose that \mathfrak{p} is a good parabolic in \mathfrak{g} and that dim $\widehat{L}_T(\lambda) = 1$. Set $M = M_T(\lambda)$ and $q = S(\mathfrak{g}_0)\mathfrak{p}_0$. Then (a) M is a critical $U(\mathfrak{g})$ -module with $e(M) = 2^{\ell(q)}$ (b) $ann_{U(\mathfrak{g})}M$ is a primitive ideal.

Proof. (a) Consider a good filtration on M as in Lemma 5.6. If M' is a nonzero submodule of M then N' = grM' is a nonzero submodule of N = grM, and we have d(N') = d(M') and e(N') = e(M'). Let q' be the prime ideal of $S(\mathfrak{g}_0)$ defining the Richardson orbit induced from \mathfrak{l}_0 and $q = S(\mathfrak{g}_0)\mathfrak{p}_0$. Since \mathfrak{p} is a good parabolic

$$\dim(\mathfrak{g}/\mathfrak{p})_1 = \ell(q).$$

By Lemma 5.6 N' is torsionfree, so by Lemma 5.1 we have

$$\dim(\mathfrak{g}/\mathfrak{p})_0 \le d(N') \le d(M) = \dim(\mathfrak{g}/\mathfrak{p})_0$$

and

$$2^{\dim(\mathfrak{g/p})_1} < e(N') < e(M) = 2^{\dim(\mathfrak{g/p})_1}.$$

Thus equality holds in both cases and this proves the result.

(b) Note that M has finite length, so the arguments in [Ja2, 8.14-8.15] show that socM is simple and $ann_{U(\mathfrak{g})}M = ann_{U(\mathfrak{g})}socM$.

5.8. If M, N are $U(\mathfrak{g}_0)$ -modules we set as in [Ja2]

$$\mathcal{L}(M, N) = \{ \phi \in Hom_{\mathbb{C}}(M, N) | \dim U(\mathfrak{g}_0)\phi < \infty \}.$$

Then $\mathcal{L}(M, N)$ is a $U(\mathfrak{g}_0)$ -bimodule, and if M, N are actually $U(\mathfrak{g})$ -modules then $\mathcal{L}(M, N)$ is a $U(\mathfrak{g})$ -bimodule.

For X a $U(\mathfrak{g}_0)$ -bimodule we write RannX for the annihilator of X as a right $U(\mathfrak{g}_0)$ -module. If \wedge is a coset of the integral weight lattice of \mathfrak{g}_0 in \mathfrak{h}_0^* the set \wedge^{++} is defined as in [Ja2, 2.5].

Lemma If $\lambda \in \wedge^{++}$, $\widehat{L}_T(\mu)$ is finite dimensional and $M = Ind_{\mathfrak{p}}^{\mathfrak{g}} \widehat{L}_T(\mu)$ then $Rann_{U(\mathfrak{g}_0)}\mathcal{L}(M(\lambda), M)$ is a primitive ideal of $U(\mathfrak{g}_0)$.

Proof. There is a surjective map of $U(\mathfrak{g}_0)$ -modules

$$M' = U(\mathfrak{g}) \otimes_{U(\mathfrak{p}_0)} \widehat{L}_T(\mu) \longrightarrow M.$$

Since the finite dimensional module $\hat{L}_T(\mu)$ is semisimple as a \mathfrak{l}_0 -module, we can write

$$\widehat{L}_T(\mu) \cong \oplus \widehat{L}_S(\mu_i)$$

with $\mu_i \in \wedge \cap P_S^{++}$. Thus as a \mathfrak{g}_0 -module

$$M' \cong \bigoplus_{i} U(\mathfrak{g}) \otimes_{U(\mathfrak{g}_{0})} U(\mathfrak{g}_{0}) \otimes_{U(\mathfrak{g}_{0})} \widehat{L}_{S}(\mu_{i})$$
$$= \bigoplus_{i} U(\mathfrak{g}) \otimes_{U(\mathfrak{g}_{0})} M_{S}(\mu_{i})$$
$$\cong \bigoplus_{i} E \otimes_{U(\mathfrak{g}_{0})} M_{S}(\mu_{i})$$

for some finite dimensional $U(\mathfrak{g}_0)$ -module E. Since $\lambda \in \wedge^{++}$ the functor $\mathcal{L}(M(\lambda), _)$ is exact on the category \mathcal{O} , see [Ja2, Lemma 4.8 and 6.9 (9)]. Hence $X' = \mathcal{L}(M(\lambda), M')$ maps onto $X = \mathcal{L}(M(\lambda), M)$ and $RannX \supseteq RannX'$. Similarly since $M'' = M_S(\mu) \subseteq M_T(\mu)$ we have $X'' = \mathcal{L}(M(\lambda), M'') \subseteq \mathcal{L}(M(\lambda), M)$ and so $RannX'' \supseteq RannX$. Finally [Ja2, 6.8 (2') and Lemma 15.7] imply that RannX'' = RannX' is a primitive ideal in $U(\mathfrak{g}_0)$.

5.9. By Corollary 5.4 and Lemma 5.8 the hypotheses of [Ja2, Satz 12.3] are satisfied. We apply this below.

Theorem If dim $\widehat{L}_T(\lambda) < \infty$, $M = M_T(\lambda)$ and $c = \dim(\mathfrak{g}/\mathfrak{p})_1$, then $\mathcal{L}(M, M)$ is prime Noetherian with Goldie rank $2^c \dim \widehat{L}_T(\lambda)$.

Proof. By [Ja2, Satz 12.3 (a), (c)] $\mathcal{L}(M, M)$ is prime Noetherian and

rank
$$\mathcal{L}(M, M) = \sum_{L} [M : L]$$
 rank $\mathcal{L}(L, L)$

where the sum runs over composition factors L of M as a $U(\mathfrak{g}_0)$ -module such that d(L) = d(M), and [M : L] is the multiplicity of L in M. Now if

$$0 = M_0 \subset M_1 \subset \ldots \subset M_k = M$$

is the series given in Theorem 5.3 then

$$[M:L] = \sum_{i=1}^{k} [M_S(\lambda_i):L]$$

Using [Ja2, 15.8], then [Ja2, 15.21 (2)] and finally equation (14) we obtain

$$\operatorname{rank} \mathcal{L}(M, M) = \sum_{i=1}^{k} \operatorname{rank} \mathcal{L}(M_{S}(\lambda_{i}), M_{S}(\lambda_{i}))$$
$$= \sum_{i=1}^{k} \dim \widehat{L}_{S}(\lambda_{i}) = 2^{c} \dim \widehat{L}_{T}(\lambda).$$

5.10. In the final result of this subsection we assume that $\mathfrak{g} = \bigoplus_{i \in \mathbf{z}} \mathfrak{g}(i)$ is a graded Lie superalgebra as in section 4.1 and set $\mathfrak{p} = \bigoplus_{i \geq 0} \mathfrak{g}(i), \mathfrak{l} = \mathfrak{g}(0)$. Suppose that $\lambda \in \mathfrak{l}^{\perp}$ and set $M = M_T(\lambda), q = S(\mathfrak{g}_0)\mathfrak{p}_0$.

Corollary. Suppose that \mathfrak{p} is a good parabolic in \mathfrak{g} . Then $e(M) = 2^{\ell(q)}$, ann_{U(\mathfrak{g})}M is a prime ideal of $U(\mathfrak{g})$ and $\mathcal{L}(M, M)$ is a primitive ring with Goldie rank $2^{\ell(q)}$. Furthermore the Clifford algebra C_q is split. **Proof.** Since dim $\widehat{L}_T(\lambda) = 1$, the statements about $\mathcal{L}(M, M)$ follow from Theorem 5.9, while the claims about M and $ann_{U(\mathfrak{g})}M$ hold by Theorem 5.7. Since $e(M) = 2^{\ell(q)}$, C_q is split by Lemma 2.1.

Observe that $U(\mathfrak{g})/ann_{U(\mathfrak{g})}M$ embeds in $\mathcal{L}(M, M)$. It follows from a version of the additivity principle [GW, Corollary 7.26] that $\overline{U} = U(\mathfrak{g})/ann_{U(\mathfrak{g})}M$

has Goldie rank at most $2^{\ell(q)}$. However the Goldie rank of \overline{U} can be strictly less than $2^{\ell(q)}$. For example suppose that $\mathfrak{g} = osp(1,2)$ and let M be a Verma module. The associated variety of M in \mathfrak{g}_0 is the nilpotent cone \mathcal{N} and we have $k(\mathcal{N}) = \ell(\mathcal{N}) = 1$, by Theorem 3.9. By the Corollary the Goldie rank of $\mathcal{L}(M, M)$ is 2, but for an appropriate choice of M, \overline{U} is isomorphic to the first Weyl algebra, which has Goldie rank 1, see [P].

6 Nilpotent Orbits in Lie superalgebras

6.1. Suppose that \mathfrak{g} is a Lie superalgebra such that \mathfrak{g}_0 is reductive, and that there is a non-degenerate even invariant bilinear form B on \mathfrak{g} . We use B to identify \mathfrak{g}_0^* with \mathfrak{g}_0 . For $x \in \mathfrak{g}_0$, let \mathfrak{g}^x be the centralizer of x in \mathfrak{g} .

Lemma We have

$$k(m_x) = dim\mathfrak{g}_1 - dim\mathfrak{g}_1^x.$$

Proof. If $u \in \mathfrak{g}_1$, then $u \in \mathfrak{g}^x$ if and only if 0 = B([x, u], w) = B(x, [u, w]) for all $w \in \mathfrak{g}_1$. This holds if and only if u is in the radical of the \mathbb{C} -valued bilinear form on \mathfrak{g}_1 whose matrix is obtained by reducing $M(\mathfrak{g}) \mod m_x$. The result follows since $k(m_x)$ is the rank of this bilinear form.

Remarks. If ad x is nilpotent, then the value of $k(m_x)$ is given by the formulas in Section 3. When $\mathfrak{g} = g\ell(m, n)$, and $x \in \mathfrak{g}_0$, we can compute $dim\mathfrak{g}_1^x$ directly as follows. If

$$x = \begin{bmatrix} J_{\mu} & 0\\ 0 & J_{\nu} \end{bmatrix}, y = \begin{bmatrix} 0 & C\\ D & 0 \end{bmatrix},$$

we have $y \in \mathfrak{g}_1^x$ if and only if $J_{\mu}C = CJ_{\nu}$ and $DJ_{\mu} = J_{\nu}D$. Let $U = \{C \in M_{m,n} | J_{\mu}C = CJ_{\nu}\}$. Then U is the space of highest weight vectors in $Hom(\bigoplus_{i\geq 1}L(\mu_i), \bigoplus_{i\geq 1}L(\nu_i))$. Hence as in the proof of Lemma 2.4, we have $\dim U = \sum_i \mu'_i \nu'_i$. This easily gives $\dim \mathfrak{g}_1^x = 2\sum_i \mu'_i \nu'_i$.

6.2. Consider the action of an algebraic group K on its Lie algebra \mathfrak{k} by the adjoint representation $Ad: K \longrightarrow GL(\mathfrak{k})$. It follows from [H,Theorem 10.4] that the tangent space to the orbit at $x \in \mathfrak{k}$ is given by $T_x(K \cdot x) = \mathfrak{k}/\mathfrak{k}^x$.

We prove a parallel result for certain Lie superalgabras. Before we can state it however, we need to review some notions concerning the functor of points and Lie supergroups, see [DG],[Ja3] and [Ma].

The category of supercommutative \mathbb{C} -algebras will be denoted **Alg** and the category of sets by **Set**. Whenever we construct a functor X from **Alg** to **Set**, we do so by specifying the value of X on an object R of **Alg** in a way which is functorial in R. Hence there is no need to say anything about the effect of X on morphisms. We call X(R) the set of R-points of X. We say that X is a subfunctor of Y if $X(R) \subseteq Y(R)$ for all supercommutative R. An affine superscheme X is a representable functor from **Alg** to **Set**. Thus there is a supercommutative \mathbb{C} -algebra $\mathcal{O}(X)$ such that $X(R) = Mor_{Alg}(\mathcal{O}(X), R)$ for any supercommutative algebra R.

6.3. Suppose that $V = V_0 \oplus V_1$ is a \mathbb{Z}_2 -graded vector space and let V^* be the dual vector space. To specify V as a representable functor we need to define $\mathcal{O}(V)$. This is done by setting

$$\mathcal{O}(V) = S(V_0^*) \otimes \wedge (V_1^*),$$

the tensor product of the symmetric algebra on the vector space V_0^* and the exterior algebra on the vector space V_1^* . It is easy to see that for any supercommutative algebra R

$$V(R) = V_0 \otimes R_0 + V_1 \otimes R_1.$$

It should be clear from the context whether V is to be thought of as a \mathbb{Z}_2 graded vector space or as a functor. We say that an affine superscheme X is a *closed subscheme* of V if $\mathcal{O}(X)$ is a \mathbb{Z}_2 -graded factor algebra of $\mathcal{O}(V)$. If \mathfrak{g} is a Lie superalgebra, then for any supercommutative algebra R, $\mathfrak{g}(R)$ becomes a Lie algebra when we set

$$[u \otimes r, v \otimes s] = [u, v]rs$$

for all $u \otimes r \in \mathfrak{g}_i \otimes R_i, v \otimes s \in \mathfrak{g}_j \otimes R_j, (i, j = 0, 1).$

6.4. Let *H* be a supercommutative Hopf superalgebra with coproduct Δ . For $h \in H$, write

$$\Delta(h) = \sum h_1 \otimes h_2.$$

Then for any $R \in Ob$ Alg, $Hom_{\mathbb{C}}(H, R)$ is an algebra under the convolution product

$$(\phi \cdot \omega)(h) = \sum \phi(h_1)\omega(h_2) \tag{17}$$

for

$$\phi, \omega \in Hom_{\mathbb{C}}(H, R).$$

Note that the identity of $Hom_{\mathbb{C}}(H, R)$ is the composite of the counit $H \longrightarrow \mathbb{C}$ followed by the inclusion $\mathbb{C} \longrightarrow R$. Also $Mor_{Alg}(H, R)$ is a subgroup of the group of units of $Hom_{\mathbb{C}}(H, R)$. The inverse of $\phi \in Mor_{Alg}(H, R)$ is $\phi^{-1} = \phi \circ \sigma$ where σ is the antipode of H.

6.5. Let V be a \mathbb{Z}_2 -graded vector space and K = GL(V). By choice of a basis we identify K with GL(m, n). For $R \in Ob$ Alg the R-points of K are matrices over R. We use the set $\mathbf{I} = \mathbf{I_1} \cup \mathbf{I_2}$ as in section 3.1 to index the rows and columns of these matrices, as well as elements of $\mathfrak{k} = g\ell(m, n)$. We think of K as the group scheme represented by the Hopf superalgebra $H = \mathcal{O}(K)$ which we describe below.

Treating \mathfrak{k} as a \mathbb{Z}_2 -graded vector space, the construction of the previous subsection yields an algebra $\mathcal{O}(\mathfrak{k})$. It is often convenient to arrange the generators $x_{ij}, (i, j \in \mathbf{I})$ of $\mathcal{O}(\mathfrak{k})$ in standard matrix format [Ma, page158]. This means that we arrange them in the form

$$x = (x_{ij}) = \left[\begin{array}{cc} x_1 & x_2 \\ x_3 & x_4 \end{array} \right]$$

where x_1 is the matrix of indeterminates $(x_{ij})_{i,j\in\mathbf{I}_1}$ and the other submatrices are defined similarly. All entries in the matrices x_1, x_4 are even while those in x_2, x_3 are odd. As an algebra $\mathcal{O}(\mathfrak{k})$ is the tensor product of the polynomial algebra generated by the even entries of x with the exterior algebra on the vector space spanned by the odd entries of x.

We can make $\mathcal{O}(\mathfrak{k})$ into a bialgebra by defining the coproduct Δ and counit ϵ on the generators x_{ij} , $(i, j \in \mathbf{I})$ by

$$\Delta(x_{ij}) = \sum_{\ell \in \mathbf{I}} x_{i\ell} \otimes x_{\ell j}$$

$$\epsilon(x_{ij}) = \delta_{ij}.$$

This implies that the product defined by equation (17) is just matrix multiplication.

Note that $d = (\det x_1)(\det x_4)$ is a polynomial in the central variables x_{ij}, x_{kl} with $i, j \in \mathbf{I}_1, k, l \in \mathbf{I}_2$. Inverting d we obtain the Hopf superalgebra $H = \mathcal{O}(K)$. The coproduct and counit for H are uniquely determined by their counterparts for $\mathcal{O}(\mathcal{M}_{m,n})$. The antipode σ is defined on generators x_{ij} symbolically by

$$\sigma\left(\left[\begin{array}{cc} x_1 & x_2 \\ x_3 & x_4 \end{array}\right]\right) = \left[\begin{array}{cc} y_1 & -x_1^{-1}x_2y_4 \\ -x_4^{-1}x_3y_1 & y_4 \end{array}\right]$$

where

$$y_1 = (x_1 - x_2 x_4^{-1} x_3)^{-1}, \quad y_4 = (x_4 - x_3 x_1^{-1} x_2)^{-1}.$$

Thus for example if $i, j \in \mathbf{I}_1$, then $\sigma(x_{ij})$ is the entry in row *i* and column *j* of y_1 .

We say that G is a closed subgroup of GL(V) if G is a subfunctor of $Mor_{Alg}(H, \)$ of the form $G = Mor_{Alg}(H/I, \)$ for some Hopf ideal I of H. If this is the case we set $\mathcal{O}(G) = H/I$. We say that the functor G is a (linear) Lie supergroup if it is isomorphic to a closed subgroup of GL(V) for some V. A Lie supergroup G acts on an affine superscheme X if $\mathcal{O}(X)$ is an $\mathcal{O}(G)$ -comodule algebra, [Mo,4.1.2]. This means that there is a natural transformation of functors $G \times X \longrightarrow X$ satisfying the usual axioms for group actions, see [DG, page 160].

6.6. We define orbits for actions of Lie supergroups and study their tangent spaces. Suppose that G is a Lie supergroup which acts on an affine superscheme X. We write X_{red} for the \mathbb{C} -points of X, that is

$$X_{red} = Mor_{\mathbf{Alg}}(\mathcal{O}(X), \mathbb{C}).$$

If $x \in X_{red}$, then using the inclusion $\mathbb{C} \longrightarrow R$ we can regard x as an element of X(R) for any supercommutative algebra R. It is therefore meaningful to define a subfunctor $G \cdot x$ of X by setting

$$(G \cdot x)(R) = \{g \cdot x | g \in G(R)\},\$$

compare [DG, page 243]. The orbit map $g \to g \cdot x$ gives a natural transformation of functors $\mu : G \to G \cdot x \subseteq X$. The orbit closure $\overline{G \cdot x}$, can be described as follows, cf [S1]. By Yoneda's lemma, μ induces an algebra map $\mu^* : \mathcal{O}(X) \longrightarrow \mathcal{O}(G)$, and $\overline{G \cdot x}$ is the closed subfunctor of X defined by the ideal $Ker\mu^*$. 6.7. For a supercommutative algebra R, the algebra of dual numbers over R is the algebra $R[\varepsilon]$ obtained from R by adjoining an even central indeterminate ε such that $\varepsilon^2 = 0$. Suppose that X is a subfunctor of a \mathbb{Z}_2 graded vector space V. We define the tangent space, $T_x(X)$ to X at $x \in X_{red}$. As a first attempt we consider the subfunctor of V given by

$$t_x(X)(R) = \{ y \in Hom_{\mathbb{C}}(\mathcal{O}(V), R) | x + y\varepsilon \in X(R[\varepsilon]) \}.$$

That is $t_x(X)(R)$ is the fiber over x under the map $X(R[\varepsilon]) \longrightarrow X(R)$. However it is not clear that $t_x(X)$ is a subspace of V, compare the discussion in [EH, VI.1.3] on the tangent space to a functor. So we define $T_x(X)$ to be the smallest \mathbb{Z}_2 -graded subspace of V such that $t_x(X)(R) \subseteq T_x(X)(R)$ for all supercommutative R. For $X \subseteq K = GL(V)$, $t_x(X)$ and $T_x(X)$ are defined in similar ways except that V is replaced by $\mathfrak{k} = g\ell(V)$. This definition works well in the cases of interest to us which are as follows.

Case 1. Suppose that X is a closed subscheme of V and let \mathbf{m}_x be the maximal ideal of $\mathcal{O}(X)$ corresponding to x. Using a Taylor expansion centered at x, see [Le,II.2], we can see that $t_x(X) = T_x(X)$ is naturally isomorphic to $(\mathbf{m}_x/\mathbf{m}_x^2)^*$. Note also that if $y \in Hom_{\mathbb{C}}(\mathcal{O}(V), R)$, then $y \in T_x(X)(R)$ if and only if y is an R-valued point derivation at x (compare [H, page 38]), that is

$$y(fg) = x(f)y(g) + y(f)x(g)$$

for all $f, g \in \mathcal{O}(X)$.

Case 2. If G is a closed subgroup of K, the tangent space to the identity $1 \in G$ is

$$T_1(G)(R) = \{ y \in Hom_{\mathbb{C}}(\mathcal{O}(\mathfrak{k}), R) | 1 + y\varepsilon \in G(R[\varepsilon]) \}.$$

An easy computation, [A, Chapter 8, (6.19)], shows that $T_1(G)(R)$ is a Lie subalgebra of $\mathfrak{k}(R)$ for any supercommutative algebra R. Thus $T_1(G)$ is a Lie superalgebra which we denote by Lie(G).

Case 3. For G, K as in Case 2, set $\mathfrak{g} = Lie(G)$ and $\mathfrak{k} = Lie(K)$. Then G(R) acts by conjugation on $\mathfrak{g}(R)$. Since the action is functorial in R, we can say that G acts on \mathfrak{g} . Consider the orbit of a \mathbb{C} -point of \mathfrak{g} , that is of an element $x \in \mathfrak{g}_0$. We have

$$t_x(G \cdot x)(R) = \{ y \in Hom_{\mathbb{C}}(\mathcal{O}(\mathfrak{k}), R) | x + y\varepsilon \in (G \cdot x)(R[\varepsilon]) \}.$$

To compute this, suppose $g_0 \in G(R)$ and $g_1 \in Hom_{\mathbb{C}}(\mathcal{O}(\mathfrak{k}), R)$ and set $g = g_0 + g_1 \varepsilon$. Then $g^{-1} = g_0^{-1} - g_0^{-1} g_1 g_0^{-1} \varepsilon$. Since $G(R) \subseteq G(R[\varepsilon])$ we have $g \in (G(R[\varepsilon]))$ if and only if $gg_0^{-1} = 1 + g_1 g_0^{-1} \varepsilon \in G(R[\varepsilon])$ and this is equivalent to $z = g_1 g_0^{-1} \in \mathfrak{g}(R)$.

We have $gxg^{-1} = x + y\varepsilon$ if and only if $g_0xg_0^{-1} = x$ and $gxg^{-1} = x + [z, x]\varepsilon$. The calculations take place in the algebra $Hom_{\mathbb{C}}(\mathcal{O}(\mathfrak{k}), R[\varepsilon])$. It follows that $x + y\varepsilon \in (G \cdot x)(R[\varepsilon])$ if and only if $y = [z, x] \in [\mathfrak{g}(R), x] = [\mathfrak{g}, x](R)$. Therefore $t_x(G \cdot x) = [\mathfrak{g}, x]$, which is a subspace of \mathfrak{g} . We have proved the following result.

Theorem We have

$$T_x(G \cdot x) = [\mathfrak{g}, x]$$

and the map $z \longrightarrow [z, x]$ gives a natural isomorphism of functors

 $\mathfrak{g}/\mathfrak{g}^x \longrightarrow T_x(G \cdot x).$

Remark. We do not know whether, in the situation of the Theorem, we have $T_x(G \cdot x) = T_x(\overline{G \cdot x})$.

We define the *superdimension* of a \mathbb{Z}_2 -graded vector space $U = U_0 \oplus U_1$ to be

$$\underline{\dim} U = (\dim U_0, \dim U_1).$$

In the non-super case, if X is an irreducible variety, we have $\dim T_x(X) \ge \dim X$ with equality on a dense subset of X, [H,Theorem 5.2]. For an orbit $X = G \cdot x$ as above, it only makes sense to consider the tangent space at a \mathbb{C} -point y of X. In this case, clearly $G \cdot x = G \cdot y$ so $T_y(G \cdot x) = T_y(G \cdot y)$ has the same dimension as $T_x(G \cdot x)$. Hence it is reasonable to define $\dim G \cdot x$ to be $\dim T_x(G \cdot x)$.

6.8. Since there are Lie algebras which are not the Lie algebra of any algebraic group, see [McR,14.7.4], the question now arises whether Theorem 6.7 applies to classical simple Lie superalgebras. This is the case at least in the following examples.

Example 1. If G = GL(V), then clearly $Lie(G) = g\ell(V)$.

Example 2. Let $Ber \in \mathcal{O}(GL(V))$ be the Berezinian, or superdeterminant, [Ma, Section 3.3]. This is a grouplike element of $\mathcal{O}(GL(V))$.

We define SL(V) to be the group scheme represented by the Hopf superalgebra $\mathcal{O}(GL(V))/(Ber - 1)$. It is well known that if G = SL(V), then $Lie(G) = s\ell(V)$. This is easy to see using our definition of $T_1(G)$.

Example 3. Let K = GL(V), and $\mathfrak{k} = g\ell(V)$ and suppose that (,) is a homogeneous bilinear form on V. The Lie superalgebra \mathfrak{g} preserving this form is defined, see [Sch page 129], by setting

$$\mathfrak{g}_a = \{ x \in \mathfrak{k}_a | (xu, v) + (-1)^{a\overline{u}}(u, xv) = 0 \text{ for all } u, v \in V, \text{with } deg \ u = \overline{u} \}.$$

We extend (,) to a bilinear form $(,)_R$ on V(R) by the rule

$$(u \otimes r, v \otimes s)_R = (u, v)rs$$

for all $u \otimes r \in V_i \otimes R_i$, $v \otimes s \in V_j \otimes R_j$, (i, j = 0, 1). It is easy to show that if R_1 is sufficiently large, then

$$\mathfrak{g}(R) = \{g \in \mathfrak{k}(R) | (gu, v)_R + (u, gv)_R = 0 \text{ for all } u, v \in V(R) \}.$$

That is $\mathfrak{g}(R)$ is the Lie subalgebra of $\mathfrak{k}(R)$ preserving the form $(,)_R$. On the other hand the Lie supergroup G preserving (,) is the functor defined by

$$G(R) = \{ g \in K(R) | (gu, gv)_R = (u, v)_R \text{ for all } u, v \in V(R) \}.$$

A simple computation shows that $Lie(G) = \mathfrak{g}$.

Theorem Let $\mathfrak{g} = s\ell(m,n), (m \neq n), g\ell(m,n), \text{ or } osp(m,n)$ and let G be the Lie supergroup with $Lie(G) = \mathfrak{g}$ defined above. Then if $x \in \mathfrak{g}_0$ we have

$$\underline{\dim} \ G \cdot x = (\dim \ G_0 \cdot x, k(m_x)).$$

Proof. In these cases there is a non-degenerate even invariant bilinear form on \mathfrak{g} . Hence by Lemma 6.1 we have $k(m_x) = \dim \mathfrak{g}_1 - \dim \mathfrak{g}_1^x$. But by Theorem 6.7 we have $\dim T_x(G \cdot x)_1 = \dim \mathfrak{g}_1/\mathfrak{g}_1^x$. This proves the statement about $\dim T_x(G \cdot x)_1$, and the claim about $\dim T_x(G \cdot x)_0$ follows similarly.

We remark that the values of $\dim G_0 \cdot x$ for nilpotent orbits in classical Lie algebras are given in [CM, Corollary 6.1.4].

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