Associated Varieties for Classical Simple Lie Superalgebras

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Abstract

Let $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ be a finite dimensional Lie superalgebra over an algebraically closed field K of characteristic zero. We consider a filtration on the enveloping algebra $U(\mathfrak{g})$ such that the associated graded ring is isomorphic to $U(\tilde{\mathfrak{g}})$ where $\tilde{\mathfrak{g}} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ but \mathfrak{g}_0 is central in $\tilde{\mathfrak{g}}$. This filtration was used by A. D. Bell to show that if a certain determinant $d(\mathfrak{g})$ is nonzero, then $U(\mathfrak{g})$ is prime. We show in this case that $d(\mathfrak{g})$ defines the non-Azumaya locus of $U(\tilde{\mathfrak{g}})$ provided dim \mathfrak{g}_1 is even.

When \mathfrak{g} is classical simple we study the associated graded ideal gr P of a primitive ideal P in $U(\mathfrak{g})$. We show that the radical $\sqrt{gr(P)}$ of gr P is prime. This is an analog of a result of Borho-Brylinski and Joseph concerning the irreducibility of the associated variety.

1. Introduction.

1.1. If \mathfrak{g} is a finite dimensional Lie algebra, we can filter $U(\mathfrak{g})$ in such a way that the associated graded ring is isomorphic to the symmetric algebra $S(\mathfrak{g})$. This allows methods from algebraic geometry to be brought into play in the study of primitive ideals of $U(\mathfrak{g})$. A key result is that if \mathfrak{g} is semisimple and Qa primitive ideal of $U(\mathfrak{g})$, then the radical $\sqrt{gr(Q)}$ of gr Q is prime. It follows that the associated variety $V(\sqrt{gr Q})$ is the closure of a single nilpotent orbit. We refer to Borho's survey, [B] for more details on the significance of this result.

Now suppose that \mathfrak{g} is a finite dimensional Lie superalgebra with $\mathfrak{g}_1 \neq 0$. The usual filtration on $U(\mathfrak{g})$ produces an associated graded ring which contains a nonzero nilpotent ideal. If we wish to use methods from algebraic geometry to study $U(\mathfrak{g})$ there are two different responses that could be made to this situation. We could apply the theory of supermanifolds developed by Manin in [Ma]. This has been done for example by Penkov [Pe] to obtain versions of the Borel-Weil-Bott and Beilinson-Bernstein theorems.

In this paper we use a second filtration on $U(\mathfrak{g})$. For this filtration the symmetric algebra $S(\mathfrak{g}_0)$ on \mathfrak{g}_0 is a central subalgebra of $gr U(\mathfrak{g})$.

If a certain determinant $d(\mathfrak{g})$ is non-zero, then Bell shows in [Be] that $gr U(\mathfrak{g})$ and hence $U(\mathfrak{g})$ are prime rings. He also shows that $d(\mathfrak{g}) \neq 0$ for all classical simple Lie superalgebras except P(n).

We show that if $d(\mathfrak{g}) \neq 0$ and $\dim \mathfrak{g}_1$ is even then $S(\mathfrak{g}_0)$ is precisely the center of $gr U(\mathfrak{g})$. From a geometric point of view the best we could hope for next is that $gr U(\mathfrak{g})$ would be Azumaya, but this is not the case. Indeed we show that $V(d(\mathfrak{g}))$ is the non-Azumaya locus of $gr U(\mathfrak{g})$. Furthermore if \mathfrak{g} is basic classical simple we use a theorem of Chevalley and an unpublished observation of A.I. Ooms to give an explicit formula for $d(\mathfrak{g})$.

Evidence that the second filtration can be used to study geometric properties of prime and primitive ideals is provided by two further results. It follows from work of E. S. Letzter [L2] that the graded spectrum of $gr U(\mathfrak{g})$ is homeomorphic to $Spec S(\mathfrak{g}_0)$. We give a new proof of this fact which hopefully sheds more light on the situation. We also show that for any primitive ideal P of $U(\mathfrak{g}), \sqrt{gr(P)}$ is prime. This result can be seen as an analog of the irreducibility of the associated variety.

I am grateful to Ooms for allowing me to include his result and also to L. LeBruyn for his comments on a preliminary version of this paper.

1.2. The Filtration on $U(\mathfrak{g})$.

Let $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ be a Lie superalgebra and $U = U(\mathfrak{g})$. From now on we discuss only the filtration $\{U_n\}$ on U which is given as follows. Informally it is defined by giving elements of \mathfrak{g}_1 degree 1, and elements of \mathfrak{g}_0 degree 2. More precisely we define the filtration $\{U_n\}$ on U by setting

$$U_0 = K, \quad U_1 = K \oplus \mathfrak{g}_1, \quad U_2 = U_1^2 + \mathfrak{g}_0.$$

Next if $\mathbf{i} = (i_1, i_2, \dots, i_\ell)$ is a sequence with $i_j = 1$ or 2 for $1 \leq j \leq \ell$ we define

$$U_{\mathbf{i}} = U_{i_1} U_{i_2} \dots U_{i_d}$$

and

$$w(\mathbf{i}) = |\{j|i_j = 1\}| + 2|\{j|i_j = 2\}|.$$

Finally for all $n \in \mathbb{N}$ we set

$$U_n = \sum_{w(\mathbf{i})=n} U_{\mathbf{i}}.$$

To describe the associated graded ring for this filtration, we define a new Lie superalgebra $\mathfrak{g}^{(t)}$ for all $t \in K$. As a vector space

$$\mathfrak{g}^{(t)}=\mathfrak{g}=\mathfrak{g}_0\oplus\mathfrak{g}_1.$$

The bracket $[\;,\;]_t$ on $\mathfrak{g}^{(t)}$ is defined by setting

$$\begin{split} & [x,y]_t = [x,y] & \text{if } x,y \in \mathfrak{g}_1 \\ & [x,y]_t = t^2[x,y] & \text{if } x \in \mathfrak{g}_1, y \in \mathfrak{g}_0 \\ & [x,y]_t = t^2[x,y] & \text{if } x,y \in \mathfrak{g}_0 \end{split}$$

Note that if $t \neq 0$ then $\mathfrak{g}^{(t)} \cong \mathfrak{g}$. Set $\tilde{g} = \mathfrak{g}^{(0)}$. In this way \mathfrak{g} may be viewed as a deformation of \tilde{g} .

Theorem. If $\{U_n\}$ is the second filtration on $U = U(\mathfrak{g})$ then the identity map on \mathfrak{g} induces an isomorphism of algebras

$$U(\tilde{g}) \longrightarrow grU.$$

Proof. This follows easily from the PBW Theorem. See also [AL].

Remark 1. If we regard t as an indeterminate then the algebra generated by t and \mathfrak{g} with defining relations those of $\mathfrak{g}^{(t)}$ is the Rees ring of the filtration $\{U_n\}$. This ring is also called the homogenized enveloping algebra

$$H(\mathfrak{g}) = \bigoplus_{n \ge 0} U_n t^n \subseteq U[t].$$

2. If \mathfrak{g} is simple then by [Sch, Lemma 2, page 93] $\mathfrak{g}_0 = [\mathfrak{g}_1, \mathfrak{g}_1]$. It follows that U is generated by \mathfrak{g}_1 and the definition of the filtration $\{U_n\}$ simplifies to $U_n = U_1^n$ for $n \ge 1$.

Now suppose that v_1, \ldots, v_n is a basis of \mathfrak{g}_1 , over K and set $d(\mathfrak{g}) = det([v_i, v_j])$, where the determinant is computed in the symmetric algebra $S(\mathfrak{g}_0)$. By [Be, Theorem 1.5], $U(\mathfrak{g})$ is prime if $d(\mathfrak{g})$ is non-zero. Note that $d(\mathfrak{g}) = d(\tilde{\mathfrak{g}})$.

2. The Case where \mathfrak{g}_0 is Central in \mathfrak{g} .

2.1. In this section we assume that \mathfrak{g} is a Lie superalgebra such that \mathfrak{g}_0 is central in \mathfrak{g} . Set $R = U(\mathfrak{g}_0)$ and $S = U(\mathfrak{g})$.

If $M = \mathfrak{g}_1 \otimes_K R$, the restriction of the Lie bracket to $\mathfrak{g}_1 \times \mathfrak{g}_1$ extends to a symmetric R-bilinear map from $M \times M$ to R. Let $C_R(M)$ be the Clifford algebra of the quadratic map associated to this bilinear form as in [Hn, Chapter 5]. The universal property of $C_R(M)$ gives a map $C_R(M) \to U(\mathfrak{g})$. Using [Hn, (5.3)] and the PBW Theorem for $U(\mathfrak{g})$, it follows that this map is an isomorphism.

Let \mathcal{C} be the set of nonzero elements of R, F = Fract(R) and $V = M \otimes_R F$. Then $S_{\mathcal{C}} = C_R(M) \otimes_R F = C_F(V)$, the Clifford algebra of the *F*-bilinear form on *V* obtained by extending the Lie bracket.

Now suppose that $d(\mathfrak{g}) \neq 0$, so that S is prime. If dim \mathfrak{g}_1 is odd then $d(\mathfrak{g})$ is not a square in F and $Z(S_{\mathcal{C}} \cong F(\sqrt{d(\mathfrak{g})}))$ by [Lam, V.2.4]. If dim \mathfrak{g}_1 is even then $Z(S_{\mathcal{C}}) = F$ [Lam, V.2.5]. In particular we deduce the following result.

Corollary. If dim \mathfrak{g}_1 , is even and $d(\mathfrak{g}) \neq 0$, then Z(S) = R.

Proof. This follows since $Z(S) = Z(S_{\mathcal{C}}) \cap S = R$.

Remark. The same argument shows that if M is any ideal of R such that $d \notin M$, then the center of S/SM equals R/M.

2.2. Let A be an algebra over a commutative ring C and set

$$A^e = A \otimes_C A^{op}.$$

We make A into a left A^e -module via

$$(a \otimes a')b = aba'$$

for $a, a', b \in A$.

If A is a projective left A^{e} -module, we say A is separable over C. If A is separable over its center we say that A is Azumaya, see [DM-I] for background. If A is Azumaya and P a prime ideal of Z(A) then by [DM-I, Corollary 1.7, page 44], A_{P} is Azumaya. Recall that the Zariski topology on SpecA is defined by taking closed sets to be of the form

$$\mathcal{V}(I) = \{ P \in SpecA | I \subseteq P \}$$

where I is an ideal of A. It follows that the set

$$X = \{P \in SpecZ(A) | A_P \text{ is not Azumaya} \}$$

is Zariski-closed in SpecZ(A). We call X the non-Azumaya locus of A.

We say that a ring C is a Jacobson ring if every prime ideal is an intersection of primitive ideals. Observe that if C is commutative and Jacobson and X, Y are closed subsets of SpecC, then X = Y provided $X \cap MaxC =$ $Y \cap MaxC$. The next result may be used to identify the non-Azumaya locus in a fairly general context.

Lemma. Suppose that A is a finitely generated algebra over its center C which is a Jacobson ring. Suppose also that I is an ideal of C such that if M is a maximal ideal of C, then A/MA is separable over C/M if and only if $I \not\subseteq M$. Then $\mathcal{V}(I)$ is the non-Azumaya locus of A.

Proof. We must show that if M is a maximal ideal of C, then A_M is Azumaya if and only if $M \notin \mathcal{V}(I)$. By [DM-I, Theorem 7.1, page 72] applied to

the finitely generated C_M -algebra A_M , A_M is separable over C_M if and only if $(A/MA) \cong (A_M/MA_M)$ is separable over C/M, which by hypothesis is equivalent to $M \notin \mathcal{V}(I)$.

2.3. Theorem. If dim \mathfrak{g}_1 is even and $d(\mathfrak{g}) \neq 0$, then $V(d(\mathfrak{g}))$ is the non-Azumaya locus of S.

Proof. By Corollary 2.1 $R = U(\mathfrak{g}_0)$ is the center of $S = U(\mathfrak{g})$. Let M be a maximal ideal of R. We need to show that $\overline{S} = S/MS$ is separable over $R/M = \overline{R}$ if and only if $d(\mathfrak{g}) \notin M$. Now R/M is a field so \overline{S} is separable if and only if it is separable in the classical sense ([DM-I, Cor. 2.4, p. 49]). Also \overline{S} has generators x_1, \ldots, x_n over R/M and relations

$$x_i x_j + x_j x_i = b_{ij}$$

for some $n \times n$ matrix B such that det $B = \overline{d(\mathfrak{g})}$

Hence for any field extension F of R/M, $\overline{S} \otimes_{R/M} F$ is the Clifford algebra of a symmetric bilinear form with matrix B. Such an algebra is semisimple if and only if det $B \neq 0$, that is $d(\mathfrak{g}) \notin M$.

Remark. If $\mathfrak{g}_0 = Kx$, $\mathfrak{g}_1 = Ky$ with [y, y] = x central in \mathfrak{g} then $U(\mathfrak{g}) = K[y]$, so the non-Azumaya locus is empty. On the other hand $V(d(\mathfrak{g})) = \{(y)\}$. Thus Theorem 2.3 may fail if dim \mathfrak{g}_1 is odd.

We remark that if \mathfrak{k} is classical simple and $\mathfrak{g} = \mathfrak{k}$, then dim \mathfrak{g}_1 is even if $\mathfrak{g} \neq Q(n)$, *n* odd or P(n), *n* even and $d(\mathfrak{g}) \neq 0$ if $\mathfrak{g} \neq P(n)$.

2.4. Recall that if A is an Azumaya algebra with center C, then extension and contraction give a one-one correspondence between the ideals of A and those of C [McR, Proposition 13.7.9]. While Theorem 2.3 shows that S is not Azumaya if $\mathfrak{g}_1 \neq 0$, we still get a one-one correspondence between the prime ideals of R and the graded prime ideals of S. In fact since \mathfrak{g} is nilpotent, a result of Letzter, [L2] states that there is a homeomorphism

$$SpecU(\mathfrak{g}_0) \longrightarrow GrSpecU(\mathfrak{g}).$$

where $GrSpecU(\mathfrak{g})$ denotes the space of \mathbb{Z}_2 -graded prime ideals. When \mathfrak{g}_0 is central in \mathfrak{g} we give a more transparent proof of this result, which yields additional information.

Theorem. There is a homeomorphism

$$\phi: GrSpecS \longrightarrow SpecR$$

given by $\phi(P) = P \cap R$. The inverse of ϕ is given by $\psi(Q) = \sqrt{QS}$. Furthermore if $\phi(P) = Q$ we have GK(S/P) = GK(R/Q).

Proof. We proceed in a number of steps.

- 1. If $P \in GrSpecS$, then $Q = P \cap R$ is prime since R is a central subring of S consisting of homogeneous elements. In addition GK(S/P) =GK(R/Q) by [KL, Proposition 5.5].
- 2. Suppose that $Q \in SpecR$ and let \mathcal{C} be the set of regular elements in R/Q. Since S_R is free $\overline{S} = S/SQ$ is \mathcal{C} torsion-free.

Let $T = \overline{S}_{\mathcal{C}}$, the localization of \overline{S} at \mathcal{C} . Let F = Fract(R/Q), a central subfield of T. Denote the exterior algebra on \mathfrak{g}_1 by $\Lambda \mathfrak{g}_1$. The restriction of the Lie bracket to \mathfrak{g}_1 extends to a symmetric F-bilinear form on $\mathfrak{g}_1 \otimes_K F$. Since $\overline{S} = \Lambda \mathfrak{g}_1 \otimes_K (R/Q)$ as an R/Q-module, it is easy to see that T is the Clifford algebra of this form over F. Observe that since \mathcal{C} consists of even elements the \mathbb{Z}_2 -grading on T extends that on \overline{S} . The nilradical N of T is generated by the radical of the bilinear form on $\mathfrak{g}_1 \otimes_K F$, and T/N is the Clifford algebra of a nonsingular bilinear form. Thus T/N is a central simple graded algebra over F by [Lam, Theorem V.2.1]. In particular N is a graded ideal of T and is either maximal or the intersection of two maximal ideals.

Set $\overline{P} = N \cap \overline{S}$. If N is a maximal ideal of T then by [GW, Theorem 9.20], \overline{P} is prime and graded prime. If N is not maximal then $N = N_1 \cap N_2$ for maximal ideals N_1, N_2 and if $\overline{P}_i = N_i \cap \overline{S}$ then \overline{P}_i is prime for i = 1, 2 and $\overline{P} = \overline{P}_1 \cap \overline{P}_2$ is graded prime. Thus in all cases, the inverse image P of \overline{P} under the map $S \longrightarrow \overline{S}$ is graded prime. Since obviously $\overline{P} = P/SQ$ is nilpotent, we have $P = \sqrt{SQ}$. In addition, S/P is C-torsionfree by [GW, Theorem 9.17]. Thus S/Pis an S/P - R/Q bimodule which is finitely generated and faithful on both sides, so GK(S/P) = GK(R/Q) by [KL, Lemma 5.3].

3. We have now shown that there exist maps ϕ, ψ as in the statement of the theorem. To see that that they are inverse bijections suppose that

 $Q \in SpecR$ and $P \in SpecS$. Since

$$Q \subseteq \sqrt{QS} \cap R = Q_1.$$

and

$$GK(R/Q_1) = GK(S/\sqrt{QS}) = GK(R/Q),$$

we have $Q = Q_1 = \phi \psi(Q)$ by [KL, Proposition 3.15]. Similarly since

$$\psi\phi(P) = \sqrt{(P \cap R)S} \subseteq P$$

we get $\psi \phi(P) = P$. To conclude the proof we observe that

$$\phi^{-1}\mathcal{V}(I) = \mathcal{V}(\sqrt{IS})$$

and

$$\psi^{-1}\mathcal{V}(J) = \mathcal{V}(J \cap R)$$

for all ideals I of R and J of S.

2.5. We can extract further information from the proof of Theorem 2.4. Suppose that v_1, \ldots, v_n is a basis of \mathfrak{g}_1 , over K and consider the matrix $(x_{ij}) = ([v_i, v_j])$. Suppose Q is a prime ideal of R and set

 $k(Q) = \{ \max m | \text{some } m \times m \text{ minor of } (x_{ij}) \text{ is nonzero mod } Q \}$

and $\ell(Q) = n - k(Q)$. Then retaining the notation of the proof, we see that k(Q) is the rank of the bilinear form on $\mathfrak{g}_1 \otimes_K F$. Hence we have

Corollary. With the notation as above, suppose $P = \sqrt{QS}$.

(a) T/N is a central simple graded algebra of dimension $2^{k(Q)}$ over F. (b) P/SQ has index of nilpotence $\ell(Q) + 1$.

Remark. By (b) the function ℓ measures the degree by which the correspondence $\psi : SpecR \longrightarrow GrSpecS$ is not given by extension of scalars.

2.6. For a positive integer m, set

$$X_m = \{Q \in Spec(R) | \ell(Q) \ge m\}.$$

Under the hypotheses of Theorem 2.3 X_1 is the non-Azumaya locus of S. In general we have

Lemma. The function $Q \longrightarrow \ell(Q)$ is upper semicontinuous on Spec R, that is the sets X_m are closed (c.f [Ha, page 125]).

Proof. It is immediate from the definitions that if $Q \subseteq Q'$ then $k(Q) \ge k(Q')$ so $\ell(Q) \le \ell(Q')$. The result follows from this.

2.7 We need another result in the next section. When referring to graded–(semi) prime ideals of S we mean the \mathbb{Z}_2 -grading.

Lemma. If I is a graded-semiprime ideal of S such that $I \cap R$ is prime then I is graded-prime.

Proof. Write $I = \bigcap_{i=1}^{r} P_i$ where the P_i are graded-prime ideals of S and set $Q = I \cap R$. Then $\bigcap_i (P_i \cap R) \subseteq Q$, so $P_i \cap R \subseteq Q$ for some i, since Qis assumed prime. By Theorem 2.4 $P_i = \sqrt{(P_i \cap R)S}$ which is contained in \sqrt{QS} . Thus there is an integer m with $P_i^m \subseteq QS = (I \cap R)S \subseteq I$. Since I is graded-semiprime this forces $P_i \subseteq I$, so $I = P_i$ is graded-prime.

3. The Classical Simple Case.

3.1. Throughout this section we assume that \mathfrak{k} is a classical simple Lie superalgebra and set $U = U(\mathfrak{k}_0)$, $V = U(\mathfrak{k})$. After giving these algebras the filtrations introduced in section 1.2, we form the associated graded algebras R = grU and S = grV. We denote the set of even (resp. odd) roots by Δ_0 (resp Δ_1). Let $\Delta = \Delta_0 \cup \Delta_1$.

Lemma. If $P \in SpecV$ and Q_1, Q_2 and both minimal over $P \cap U$, then $\sqrt{grQ_1} = \sqrt{grQ_2}$.

Proof. By [GW, Lemma 7.15] there exists a $U/Q_1 - U/Q_2$ bimodule C which is torsion-free and finitely generated on both sides. Furthermore the proof of [GW, Lemma 7.15] shows that we can choose C to be a subfactor C = B/Aof V as a U - U-bimodule.

If $x \in \mathfrak{k}_0$ and $b \in B$ then $xb, bx \in B$. Therefore B, A and C are \mathfrak{k}_0 -modules

and the adjoint action of \mathfrak{k}_0 on C is locally finite. Hence in the terminology of [B, Lemma 4.9] C is a Harish-Chandra bimodule link from U/Q_1 to U/Q_2 and we obtain the result.

3.2 Corollary. Suppose that $P \in Spec V$, $Q \in Spec U$ and Q is minimal over $P \cap U$. Then

- (a) GK(V/P) = GK(U/Q)
- (b) $\sqrt{grQ} = \sqrt{grP} \cap R.$

Proof. Let $Q = Q_1, \ldots, Q_r$ be the primes of U minimal over $P \cap U$.

(a) Since V/P is a finitely generated $U/(P \cap U)$ -module we have $GK(V/P) = GK(U/(P \cap U))$ by [KL, Proposition 5.5]. Also since GK-dimension is exact for U-modules, we have

$$GK(U/(P \cap U)) = GK(U/Q_i),$$

for some i, by [KL, Proposition 5.7].

Now $\{(U_m + Q_i)/Q_i\}$ defines a good filtration on U/Q_i such that the associated graded module satisfies

$$gr(U/Q_i) \cong grU/grQ_i.$$

It follows from [KL, Propositions 6.6 and 5.7] that

$$GK(U/Q_i) = GK(grU/grQ_i)$$

= $GK(grU/\sqrt{grQ_i}).$

Since $\sqrt{grQ_i}$ is independent of *i* by Lemma 3.1 this proves the result.

(b) There is a finite product of the ideals Q_i which is contained in $P \cap U$. Hence there is a finite product of the ideals grQ_i which is contained in $grP \cap R$ and so in $\sqrt{grP} \cap R$. Since $\sqrt{grQ_i} = \sqrt{grQ}$ for all *i* by Lemma 3.1 this implies $(grQ)^N \subseteq \sqrt{grP} \cap R$ for some N, so $\sqrt{grQ} \subseteq \sqrt{grP} \cap R$. The other inclusion follows easily using the fact that $P \cap U \subseteq Q$.

3.3. Theorem. If $P \in Gr Prim V$, then $\sqrt{grP} \in Gr Spec R$.

Proof Choose $Q \in SpecU$ minimal over $P \cap U$. Then Q is primitive by [L, Theorem 1.4] and then \sqrt{grQ} is prime by [BB] or [J]. Since $\sqrt{grQ} = \sqrt{grP} \cap R$ by Corollary 3.2 it follows from Lemma 2.7 that \sqrt{grP} is graded-prime.

3.4. Let $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ be a finite dimensional Lie superalgebra, and suppose that x_1, \ldots, x_n is a basis for \mathfrak{g}_1 . We consider the algebraic subgroup.

$$G = \{ x \in Aut(\mathfrak{g}) | x(\mathfrak{g}_i) = \mathfrak{g}_i \quad \text{for } i = 0, 1 \}.$$

The following result and its corollary are due to A.I. Ooms, see [O] for a related argument.

Proposition. For all $x \in G$ we have

$$x(d(\mathfrak{g})) = (detx|\mathfrak{g}_1)^2 d(\mathfrak{g}).$$

Proof. Consider the matrix $B = ([x_i, x_j])$ with entries in $S(\mathfrak{g}_0)$. Define a matrix A by

$$x(x_i) = \sum_{r=1}^n a_{ri} x_r \quad \text{for all} \quad i = 1, \dots, m.$$

Since $x|_{\mathfrak{g}_0}$ is an automorphism of \mathfrak{g}_0 which extends to an automorphism of $S(\mathfrak{g}_0)$ we have the following

$$\begin{aligned} x(d(\mathfrak{g})) &= x(\det([x_i, x_j])) \\ &= \det([x(x_i), x(x_j)]) \\ &= \det(\sum_{r,s} a_{ri}[x_r, x_s]a_{sj}) \\ &= \det(A^t B A) = (\det A)^2 \det B \\ &= (\det x|\mathfrak{g}_1)^2 d(\mathfrak{g}). \end{aligned}$$

Corollary. Denote by $\mathcal{D}_0(\mathfrak{g})$ the Lie algebra of even derivations of \mathfrak{g} . For all $D \in \mathcal{D}_0(\mathfrak{g})$ we have

$$D(d(\mathfrak{g})) = 2tr(D|\mathfrak{g}_1)d(\mathfrak{g}).$$

In particular $d(\mathfrak{g}) \in S(\mathfrak{g}_0)$ is a semi-invariant under the action of $ad \mathfrak{g}_0$.

Proof. This follows since the Lie algebra of the algebraic group G is precisely $\mathcal{D}_0(\mathfrak{g}).$

3.5. Now assume that there exists a nondegenerate even supersymmetric invariant bilinear form (,) on \mathfrak{k} . The second assumption holds provided that \mathfrak{k} does not belong to any of the series P(n) or Q(n), in the notation of [Kac]. In this case it was shown that $d(\mathfrak{k})$ is nonzero by Bell, and we give a new interpretation of his specialization argument, see [Be, 3.4].

By Corollary 3.4 $d(\mathfrak{k})$ is a semi-invariant in $S(\mathfrak{k}_0)$. Therefore, since \mathfrak{k}_0 is reductive $d(\mathfrak{k})$ must belong to the invariant ring $S(\mathfrak{k}_0)^{\mathfrak{k}_0}$. Now the inclusion $\mathfrak{h} \subseteq \mathfrak{k}_0$ induces a restriction homomorphism $S(\mathfrak{k}_0^*) \longrightarrow S(\mathfrak{h}_0^*)$. Identifying \mathfrak{k}_0 with \mathfrak{k}_0^* and \mathfrak{h} with \mathfrak{h}^* by means of the form (,) we obtain a homomorphism

$$\theta: S(\mathfrak{k}_0) \longrightarrow S(\mathfrak{h}).$$

Let $h_{\alpha} \in \mathfrak{h}$ be the element that corresponds to $\alpha \in \mathfrak{h}^*$ under the above identification.

By Chevalley's Theorem [H, Theorem 23.1] θ induces an isomorphism from $S(\mathfrak{k}_0)^{\mathfrak{k}_0}$ to $S(\mathfrak{h})^W$.

Theorem. Up to a scalar multiple we have

$$d(\mathfrak{k}) = \theta^{-1} \left(\Pi_{\alpha \in \Delta_1} h_{\alpha}^{\dim} \mathfrak{k}^{\alpha} \right).$$

Proof. For ease of notation we assume that dim $\mathfrak{k}^{\alpha} = 1$ for all roots α . This assumption holds unless \mathfrak{k} is isomorphic to a simple algebra of type A(1,1). The exceptional case can be handled easily using an argument similar to the one we give below. Our assumption allows us to choose a basis e_{α} for \mathfrak{g}^{α} for all $\alpha \in \Delta_1$ such that $[e_\alpha, e_{-\alpha}] = h_\alpha$. Thus if $\{e_\alpha | \alpha \in \Delta_1\} = \{x_1, \ldots, x_n\}$ then the basis elements x_i of \mathfrak{k}_1 satisfy either

$$[x_i, x_j] \in \mathfrak{h} \tag{1}$$

or

$$[x_i, x_j] \in \mathfrak{n}_0^+ \cup \mathfrak{n}_0^- \tag{2}$$

Now θ vanishes on terms of the form (4). As for terms of the form (3) we can arrange that x_1, \ldots, x_n correspond to pairs of roots $\pm \alpha_1, \ldots, \pm \alpha_m$, so that $\theta([x_i, x_j])$ is a block diagonal matrix with diagonal blocks of the form

$$\left[\begin{array}{cc} 0 & h_{\alpha} \\ h_{\alpha} & 0 \end{array}\right]$$

where $\alpha \in \Delta_1^+$. Clearly this implies the result.

Remarks.

- 1. Bell's argument consists of specializing the terms in (4) to zero to show that $d(\mathfrak{k}) \neq 0$.
- 2. If $\mathfrak{k} = Q(n)$, Bell uses a different specialization argument to show that $d(\mathfrak{k})$ is nonzero and so $U(\mathfrak{k})$ is prime.
- 3. If $\mathfrak{k} = P(n)$, it is shown in [KiK] that $U(\mathfrak{k})$ is not prime, and that $U(\mathfrak{k})$ has a unique minimal prime ideal.

4. Examples.

4.1 Suppose first that \mathfrak{k} is a classical simple Lie superalgebra of type I, and set $\mathfrak{g} = \mathfrak{k}$. This means that $\mathfrak{k}_1 = \mathfrak{k}_1^+ \oplus \mathfrak{k}_1^-$ is a direct sum of two simple \mathfrak{k}_0 -modules. We note that \mathfrak{k} has type I if and only if $\mathfrak{k} = A(m, n), C(n)$ or P(n). Setting $\mathfrak{g}_1^{\pm} = \mathfrak{k}_1^{\pm}$ we see that $\mathfrak{g}_1^+, \mathfrak{g}_1^-$ are maximal isotropic subspaces of \mathfrak{g}_1 with respect to the bilinear form defined by the Lie bracket. Hence \mathfrak{g}_1 is a hyperbolic space. Similarly using the notation of 2.4 and 2.5 the image of $\mathfrak{g}_1 \otimes_K F$ in T/N is a hyperbolic space, and thus $T/N \cong M_{2^{k(Q)-1}}(F)$.

4.2 Let $\mathfrak{k} = s\ell(2,1)$, Then $\Delta_1 = \{\pm\beta,\pm\gamma\}$ $\Delta_0 = \{\pm\alpha\}$ as in [M]. Set $e_{\alpha} = e_{12}, e_{\beta} = e_{23}, e_{\gamma} = e_{13}$ and for $\eta = \alpha, \beta, \gamma$ let $e_{-\eta}$ be the transpose of e_{η} and $h_{\eta} = [e_{\eta}, e_{-\eta}]$. Then $grU(\mathfrak{k}) = U(\mathfrak{g})$ is the Clifford algebra of the bilinear form over $U(\mathfrak{g}_0)$ with matrix

$$\begin{bmatrix} 0 & 0 & h_{\beta} & e_{-\alpha} \\ 0 & 0 & e_{\alpha} & h_{\gamma} \\ h_{\beta} & e_{\alpha} & 0 & 0 \\ e_{-\alpha} & h_{\gamma} & 0 & 0 \end{bmatrix}$$

Thus $d(\mathfrak{g}) = -(h_{\beta}h_{\gamma} - e_{\alpha}e_{-\alpha})^2 = -\frac{1}{4}(z^2 - h_{\alpha}^2 - 4e_{\alpha}e_{-\alpha})^2$ where z is central in \mathfrak{k}_0 . If I is a minimal primitive ideal of $U(\mathfrak{k}_0)$ and $Q = \sqrt{gr(I)}$ then Q contains z and $h_{\alpha}^2 + 4e_{\alpha}e_{-\alpha}$, so $d(\mathfrak{g}) \in Q$.

4.3 Let $\mathfrak{k} = osp(1, 2r)$. The root system of \mathfrak{k} is described in [K, 2.5.4]. We have $\Delta_1 = \{\pm \epsilon_1, \ldots, \pm \epsilon_r\}$. Let $x_{\pm i}$ be a basis for the root space $\mathfrak{k}^{\pm \epsilon_i}$. Then $[x_i, x_j]$ has weight $\epsilon_i + \epsilon_j$, $[x_{-i}, x_{-j}]$ has weight $-\epsilon_i - \epsilon_j$ and $[x_{-i}, x_{-j}]$ has weight $-\epsilon_i - \epsilon_j$ if $i \neq j$. Also each of these products is nonzero so forms a basis for the corresponding root space. On the other hand the products $[x_i, x_{-i}]$ with $1 \leq i \leq r$ form a basis for the Cartan subalgebra of \mathfrak{g}_0 . In other words the upper triangular enries of the symmetric matrix $([x_{\pm i}, x_{\pm j}])$ form a basis for \mathfrak{g}_0 . It follows that the algebra $grU(\mathfrak{k}) = U(\mathfrak{g})$ is a generic Clifford algebra as defined, and studied in [LB, Chapter 2].

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