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Rings of Differential Operators and Zero Divisors

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Throughout k will denote an algebraically closed field of characteristic zero. If A is a (commutative) k-algebra, we denote by $\mathscr{D}(A)$ the ring of all k-linear differential operators on A.

If A is a domain it is well known and easy to see that $\mathscr{D}(A)$ is also a domain. In most studies of $\mathscr{D}(A)$, see, for example, [8], it is assumed that A is a domain. Here we are mainly concerned with the case where A has zero divisors, and we study annihilator conditions in $\mathscr{D}(A)$. For example, in Section 2 we prove

THEOREM A. *Let A be a finitely generated k-algebra. The following conditions are equivalent.*

(1) *$\mathscr{D}(A)$ has a semisimple artinian (classical) quotient ring.*

(2) *$\mathscr{D}(A)$ has an artinian quotient ring.*

(3) *$\mathscr{D}(A)$ has the maximum condition on left annihilators.*

(4) *A has an artinian quotient ring.*

In Theorem B we obtain a description of the prime radical N of $\mathscr{D}(A)$ for A a finitely generated k-algebra. The statement of Theorem B requires the introduction of some notation so we postpone it until Section 3.1. However, as an amusing consequence we show that $N^{n+1}=0$, where n is the Krull dimension of A (Corollary 3.7). We also show that $\mathscr{D}(A)$ is semiprime if and only if A has an artinian quotient ring (Corollary 3.8).

In Section 4 we study an analogue of Nakai's conjecture for algebras which are not necessarily domains. We prove that if $\mathscr{D}(A)$ is generated by operators of order at most one then A is reduced and we conjecture that in fact A is a direct sum of domains.

Many of the results in this paper are motivated by an interesting example of Muhasky [5]. Let $A=k[x, y]/(x^2, xy)$. An explicit calculation of

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$\mathscr{D}(A)$ is given in [5, Example 7.2] and this calculation is used to show that $\mathscr{D}(A)$ is right but not left Noetherian.

In a forthcoming paper we shall study $\mathscr{D}(A)$ for algebras of Krull dimension at most one in greater detail, extending the main results of [5] and [8], where it is assumed that A is reduced or is a domain, respectively.

1. Background Results

1.1. Let A be a commutative k-algebra and M, N A-modules. We denote by $\mathscr{D}^n_A(M, N)$ the space of k-linear operators from M to N of order $\leqslant n$ and $\mathscr{D}_A(M, N) = \bigcup_{n=0}^{\infty} \mathscr{D}^n_A(M, N)$ as defined, for example, in [8, Sect. 1.2]. Write $\mathscr{D}_A(M) = \mathscr{D}_A(M, M)$.

1.2. If I and J are subsets of M we define

$$\Delta_M(I, J) = \{d \in \mathscr{D}_A(M) \mid d(I) \subseteq J\}.$$

We shall drop the subscript A in the notation $\mathscr{D}_A(M, N)$, $\Delta_A(I, J)$, etc., whenever no confusion is likely to result.

LEMMA. (a) *If I (resp. J) is a $\mathscr{D}(M)$-submodule of M then $\Delta_M(I, J)$ is a right (resp. left) ideal of $\mathscr{D}(M)$.*

(b) *If $J \subseteq I$ are $\mathscr{D}(M)$-submodules of M, there is a ring homomorphism ϕ: $\mathscr{D}(M) \to \mathscr{D}(I/J)$ defined by $\phi(d)(m+J) = d(m) + J$ with* $\operatorname{Ker} \phi = \Delta_M(I, J)$.

Proof. Straightforward.

We caution that $\Delta(I, J)$ should not be confused with $\mathscr{D}(I, J)$ when both are defined. For example, $\mathscr{D}(I, 0)$ is always zero whereas $\Delta(I, 0)$ may not be. We note that $\Delta(A, I) = \mathscr{D}(A, I)$ for any ideal I of A. Although we mainly work with $\Delta(I, J)$, $\mathscr{D}(I, J)$ is often useful, since it is defined for any A-modules I and J. Also as noted in [8, Sect. 1.3]. $\mathscr{D}(A, -)$ is a left exact functor from A-modules to right $\mathscr{D}(A)$-modules. A similar notational problem is discussed in [8, Sect. 2.7].

1.3. If I is a right ideal in a ring R, the idealiser of I in R is the ring $\mathbb{I}_R(I) = \{r \in R \mid rI \subseteq I\}$.

LEMMA. *Let $R = k[x_\Lambda]$ be a polynomial ring in indeterminates $\{x_\lambda\}_{\lambda \in \Lambda}$, I an ideal of R, and $A = R/I$. Then*

(a) *There is a k-algebra ismorphism*

$$\Delta_R(I, I)/\Delta_R(R, I) \cong \mathscr{D}(A)$$

under which an operator $d \in \Delta_R(I, I)$ maps $a + I \in A$ to $d(a) + I$.

(b) *We have* $\Delta_R(I, I) = \mathbb{I}_{\mathscr{D}(R)}(I\mathscr{D}(R))$ *and* $\Delta_R(R, I) = I\mathscr{D}(R)$.

Proof. (a) This is [5, Lemma 1.4].

(b) This follows in much the same way as (a); see also [8, Proposition 1.6].

1.4. If S is a multiplicatively closed subset of A and M an A-module we denote by $S(0)$ the kernel of the localisation map $M \to M_S$. Thus $S(0) = \{m \in M \mid sm = 0 \text{ some } s \in S\}$.

LEMMA. *$S(0)$ is a $\mathscr{D}(M)$-submodule of M.*

Proof. Set $N = S(0)$. If $d \in \mathscr{D}(M)$ is an operator of order r we show by induction on r that $d(N) \subseteq N$. This is clear if $r = 0$. Let $n \in N$ and suppose $sn = 0$ for $s \in S$. Then $-sd(n) = [d, s](n) \in N$ by induction. Hence $s_1 sd(n) = 0$ for some $s_1 \in S$ and $d(n) \in N$.

1.5. For $d \in \mathscr{D}(M)$ we define $\Phi(d) \in \operatorname{Hom}_k(M_S, M_S)$ by $\Phi(d)(m/s) = \sum_{p=0}^{\infty} (-1)^p [d, s]_p (m)/s^{p+1}$ for $m \in M$, $s \in S$, where $[d, s]_p$ is defined inductively by $[d, s]_0 = d$ and $[d, s]_p = [[d, s]_{p-1}, s]$. It is known that Φ gives a well defined ring homomorphism $\Phi\colon \mathscr{D}(M) \to \mathscr{D}(M_S)$. The image of Φ is contained in $\mathscr{D}(M/S(0))$. For $m \in M$ we have $\Phi(d)(m + S(0)) = d(m) + S(0)$. Hence by 1.2, $\operatorname{Ker} \Phi = \Delta_M(M, S(0))$. Also by [5, Lemma 1.8] we have $\operatorname{Ker} \Phi = \{d \in \mathscr{D}(M) \mid sd = 0 \text{ some } s \in S\}$. Finally, note that if d has order n and $sd = 0$ for some $s \in S$ then by induction on n, $ds^{n+1} = 0$. Thus $\operatorname{Ker} \Phi = \{d \in \mathscr{D}(M) \mid ds = 0 \text{ for some } s \in S\}$.

1.6. Suppose A, M, S are as above and set $\bar{M} = M/S(0)$.

LEMMA. *If $c \in A$ and $cS(0) = 0$ then $c\mathscr{D}(\bar{M}) \subseteq \operatorname{Im} \Phi$.*

Suppose $d \in \mathscr{D}(\bar{M})$ and let $\tilde{d}\colon M \to M$ be any k-linear map which lifts d. It suffices to show that $c\tilde{d} \in \mathscr{D}(M)$ since then clearly $cd = \Phi(c\tilde{d}) \in \operatorname{Im} \Phi$. We use induction on the order of d. If d has order 0, then for all $a \in A$ and $m \in M$ we have $d(am) - ad(m) = 0$. Therefore $\tilde{d}(am) - a\tilde{d}(m) \in S(0)$. Hence $c\tilde{d}(am) - ac\tilde{d}(m) = 0$ and $c\tilde{d} \in \operatorname{Hom}_A(M, M) \in \mathscr{D}(M)$. For the inductive step note that for $a \in A$, the map $[\tilde{d}, a]$ lifts $[d, a]$. Hence $[c\tilde{d}, a] = c[\tilde{d}, a] \in \mathscr{D}(M)$ by induction. It follows that $c\tilde{d} \in \mathscr{D}(M)$.

Remarks. (1) If M is a finitely generated module over a Noetherian ring A, we can find $c \in S$ such that $cS(0) = 0$, and the above applies.

(2) If $d \in \mathscr{D}(\bar{M})$ has order n and c is as in the lemma then by induction on n, $dc^{n+1} \in \operatorname{Im} \Phi$.

1.7. EXAMPLE. We give an example to show that $\operatorname{Im} \Phi$ may be strictly contained in $\mathscr{D}(\bar{M})$.

Let $A=k[x, y]/(xy)$. Then A is isomorphic to the subring $k(1, 1)+(xk[x], yk[y])$ of $k[x]\oplus k[y]$. It is easily seen that $\mathscr{D}(A)\cong k(1, 1)+(x\mathscr{D}(k[x]), y\mathscr{D}(k[y]))$. Let $P=(y)$, $S=A-P$, then $S(0)=P$ and $A/S(0)\cong k[x]$. However, the image of $\mathscr{D}(A)$ under the localisation map is $k+x\mathscr{D}(k[x])\subsetneq\mathscr{D}(k[x])$.

1.8. We need a slight generalization of [5, Proposition 1.14].

LEMMA. *Let M, N be A-modules such that* $\operatorname{Hom}_A(M, N)=0$ *then* $\mathscr{D}(M, N)=0$.

Proof. An easy induction on n shows $\mathscr{D}^n(M, N)=0$ for all n.

COROLLARY. *Let $A=A_1\oplus\cdots\oplus A_t$ and suppose I is an ideal of A. Write $I=I_1\oplus\cdots\oplus I_t$, where $I_i=I\cap A_i$.*
Then $\mathscr{D}(I)\cong\mathscr{D}(I_1)\oplus\cdots\oplus\mathscr{D}(I_t)$

Proof. We may assume $t=2$. Since $e_1=(1, 0)$, $e_2=(0, 1)$ belong to A, I is a direct sum as above. If say $f\in\operatorname{Hom}_A(I_1, I_2)$ then for $r\in I_1$, $f(r)=f(re_1)=f(r)e_1=0$ so $f=0$. Hence $\mathscr{D}(I_1, I_2)=0$ by the lemma. For $d\in D(I)$, let d_i be the restriction of d to I_i. It is easily seen that the map $d\rightarrow(d_1, d_2)$ is an isomorphism of $\mathscr{D}(I)$ onto $\mathscr{D}(I_1)\oplus\mathscr{D}(I_2)$.

2. ARTINIAN QUOTIENT RINGS

2.1. In this section we prove Theorem A. The implications $(1)\Rightarrow(2)$ and $(2)\Rightarrow(3)$ are trivial so it suffices to prove $(3)\Rightarrow(4)$ and $(4)\Rightarrow(1)$. The next result is used in both parts of the proof.

LEMMA. *Suppose $A=\bar{A}\oplus N$, where N is a nilpotent ideal and $\bar{A}$ a subalgebra of A. If L, M are A-modules, then $\mathscr{D}_A(L, M)=\mathscr{D}_{\bar{A}}(L, M)$.*

Proof. Clearly $\mathscr{D}_A(L, M)\subseteq\mathscr{D}_{\bar{A}}(L, M)$. If $d\in\mathscr{D}^n_{\bar{A}}(L, M)$ and $N^p=0$, we show that $d\in\mathscr{D}^{n+2p-2}_A(L, M)$. If S and T are subsets of $\operatorname{Hom}_k(L, M)$ and A, respectively, we write $[S, T]_0=S$ and for $i\geqslant 0$, $[S, T]_{i+1}=\{[\partial, t]\mid\partial\in[S, T]_i,\quad t\in T\}$. Since $[d, A]=[d, \bar{A}]+[d, N]$ and $[[d, N], \bar{A}]=[[d, \bar{A}], N]$ we have $[d, A]_i=\sum_{j+k=i}[[d, \bar{A}]_j, N]_k\subseteq\sum_{l+j+m=i}N^l[d, \bar{A}]_jN^m$. In the last sum all terms are zero unless $l, m\leqslant p-1$ and $j\leqslant n-1$. Hence $[d, A]_{n+2p-2}=0$ as required.

2.2. *Proof of Theorem* A. $(3)\Rightarrow(4)$. We actually prove the contrapositive. Suppose $A\cong R/I$, where I is an ideal of $R=k[x_1, ..., x_n]$, and A does not have an artinian quotient ring. This is equivalent to the assumption that there exist prime ideals P, Q of R belonging to I such that

$P \subsetneq Q$. By [1, Proposition 7.17], Q/I is an annihilator ideal of R/I. Thus if $J/I = \operatorname{ann}_{R/I} Q/I$, we have $Q/I = \operatorname{ann}_{R/I} J/I$.

Let $Q^{(t)}$ be the tth symbolic power of Q, and $L_t = l - \operatorname{ann}_{\mathscr{D}(A)}(((Q^{(t)} + I)/I)\,\mathscr{D}(A))$. Since $Q^{(t)} \supseteq Q^{(t+1)}$ we have $L_t \subseteq L_{t+1}$. We show below that for fixed t, there exists $\partial \in \mathscr{D}(R)$ such that

(1) $\partial(P) \subseteq Q$,

(2) $\partial(Q^{(t)}) \not\subseteq Q$,

(3) $\partial(Q^{(t+1)}) \subseteq Q$.

Hence by (2) there exists $y \in Q^{(t)}$ such that $\partial(y) \notin Q$. By the first paragraph of the proof we can find $x \in J$ such that $x\partial(y) \notin I$. Now by (1), $x\partial(I) \subseteq x\partial(P) \subseteq JQ \subseteq I$, so by 1.3, $x\partial$ induces a differential operator (also denoted $x\partial$) on A. Since $x\partial(Q^{(t)}) \not\subseteq I$ we have $x\partial \notin L_t$. However, by (3), $x\partial(Q^{(t+1)}) \subseteq xQ \subseteq I$. This will show that $L_t \subsetneq L_{t+1}$ and give an ascending chain of left annihilators in $\mathscr{D}(A)$ as required.

Let $\bar{R} = R/Q^{(t+1)}$ and use the overbar to denote images of elements and ideals of R in $\bar{R}$. Suppose we can find $\partial_1 \in \mathscr{D}(\bar{R})$ such that

(1)′ $\partial_1(\bar{P}) \subseteq \bar{Q}$ and (2)′ $\partial_1\overline{(Q^{(t)})} \not\subseteq \bar{Q}$.

Then by 1.3 there exists $\partial \in \mathscr{D}(R)$ with $\partial(Q^{(t+1)}) \subseteq Q^{(t+1)}$ and $\partial(r) + Q^{(t+1)} = \partial_1(r + Q^{(t+1)})$ for $r \in R$. Then ∂ will satisfy (1)–(3), so it suffices to find ∂_1 satisfying (1)′ and (2)′.

Let $M = Q_Q$, $S = \bar{R}_{\bar{Q}} = R_Q/M^{t+1}$, and $\bar{M} = M/M^{t+1} = \bar{Q}_{\bar{Q}}$. Then S is a complete local artinian ring with maximal ideal $\bar{M}$. Hence by Cohen's theorem [4, 28.J], there exists a subfield K of S with $S = K \oplus \bar{M}$.

Suppose that $\bar{M}^t = M^t/M^{t+1} \subseteq (P_Q + M^{t+1})/M^{t+1} = \bar{P}_Q$. Then $M^t \subseteq P_Q + M^{t+1}$, and so

$$\left(\frac{M^t + P_Q}{P_Q}\right) M = \frac{M^{t+1} + P_Q}{P_Q} = \frac{M^t + P_Q}{P_Q}.$$

By Nakayama's lemma this would imply $M^t \subseteq P_Q$, but this is impossible since P_Q is a prime ideal of R_Q strictly contained in M. Therefore $\bar{M}^t \not\subseteq \bar{P}_Q$ and these are K-subspaces of S.

By Lemma 2.1, $\operatorname{Hom}_K(S, K)$ is a K-subspace of $\mathscr{D}(S)$. Hence by vector space duality we can find $\partial_2 \in \mathscr{D}(S)$ such that $\partial_2(\bar{P}_Q) = 0$, $\partial_2(\bar{M}^t) = K$, and ∂_2 is K-linear. In particular there exists $r \in \bar{M}^t$ such that $\underline{\partial_2(r)} \notin \bar{M}$. Now there exist $c_1, c_2 \in \bar{R} - \bar{Q}$ such that $c_1 r \in \bar{R} \cap \bar{M}^t = \overline{Q^{(t)}}$ and $\partial_1 = c_2\partial_2 \in \mathscr{D}(\bar{R})$. Write $c_1 \in \bar{R} \subseteq S$ in the form $c_1 = c_3 + m$ with $c_3 \in K$, $m \in \bar{M}$. Then $c_1 r = c_3 r + mr$ and $mr \in \bar{M}^{t+1} = 0$. Hence $c_3 r \in \overline{Q^{(t)}}$ and since ∂_1 is K-linear, $\partial_1(c_3 r) = c_2 c_3 \partial_2(r) \notin \bar{M}$. Therefore $\partial_1(\overline{Q^{(t)}}) \not\subseteq \bar{Q}$ and $\partial_1(\bar{P}) \subseteq c_2\partial_2(\bar{P}_Q) = 0$. Hence we have found ∂_1 satisfying (1)′ and (2)′ and this completes the proof.

2.3. LEMMA. *Suppose* $A=\bar{A}\oplus N$, *where* N *is a nilpotent ideal and* $\bar{A}$ *a subalgebra of* A. *If* V *is an* $\bar{A}$-*module direct summand of* A *there exists an idempotent* $e\in\mathscr{D}(A)$ *such that* $e\mathscr{D}(A)e\cong\mathscr{D}_{\bar{A}}(V)$.

Proof. Suppose $A=V\oplus W$ as $\bar{A}$-modules and let e be the projection of A onto V relative to this decomposition. By Lemma 2.1 with $L=M=A$, $e\in\mathscr{D}_{\bar{A}}(A)=\mathscr{D}(A)=\mathscr{D}$. Clearly elements of $e\mathscr{D}e$ act as k-linear maps on V. For $a\in\bar{A}$ and $d\in\mathscr{D}$ we have $[ede,a]=edea-aede=edae-eade=e[d,a]e$. Hence elements of $e\mathscr{D}e$ act as differential operators on the $\bar{A}$-module V and we obtain a ring homomorphism ϕ: $e\mathscr{D}e\to\mathscr{D}_{\bar{A}}(V)$. If $d\in\mathscr{D}$ and $\phi(ede)=0$ then since $e(W)=0$, we obtain $ede=0$ in $e\mathscr{D}e$ so ϕ is injective. For $d\in\mathscr{D}_{\bar{A}}(V)$ we extend d to a k-linear map d' on A by defining $d'(W)=0$. For $a\in\bar{A}$ we have $[d',a]=[d,a]'$. It follows that $d'\in\mathscr{D}_{\bar{A}}(A)=\mathscr{D}(A)$. Since $\phi(ed'e)=d$ we have shown that $e\mathscr{D}e\cong\mathscr{D}_{\bar{A}}(V)$.

2.4. LEMMA. *Suppose* $A=\bar{A}\oplus N$, *where* N *is a nilpotent ideal and* $\bar{A}$ *a subalgebra of* A, *and that* N *is free of rank* $n-1$ *as an* $\bar{A}$-*module. Then* $\mathscr{D}(A)\cong\mathrm{Mat}_n(\mathscr{D}(\bar{A}))$ *the ring of* $n\times n$ *matrices over* $\mathscr{D}(\bar{A})$.

Proof. Let $v_1=1$ and let $v_2,\ldots,v_n$ be a basis for N as an $\bar{A}$-module. Let e_{ij} be the $\bar{A}$-linear map defined by $e_{ij}(v_k)=\delta_{jk}v_i$, where δ_{ik} is the Kronecker delta. Then $e_{ij}\in\mathscr{D}(A)=\mathscr{D}$ by Lemma 2.1, $e_{ij}e_{kl}=\delta_{jk}e_{il}$, and $1=e_{11}+e_{22}+\cdots+e_{nn}$. Also by the proof of Lemma 2.3, $e_{11}\mathscr{D}e_{11}\cong\mathscr{D}(\bar{A})$. Hence by [6, Lemma 6.1.5], $\mathscr{D}(A)\cong\mathrm{Mat}_n(\mathscr{D}(\bar{A}))$.

2.5. *Proof of Theorem* A. (4) $\Rightarrow$ (1). Assume that A has an artinian quotient ring, and let $P_1,\ldots,P_t$ be the minimal primes of A. If $S=A-\bigcup_{i=1}^{t}P_i$, then S is the set of non-zero divisors of A and $A_S\cong A_1\oplus\cdots\oplus A_t$, where $A_i=A_{P_i}$. Since $\mathscr{D}(A_S)=A_S\otimes_A\mathscr{D}(A)$ is a localisation of $\mathscr{D}(A)$ at a set of regular elements it suffices to show that $\mathscr{D}(A_S)$ has a semisimple artinian quotient ring. By [5, Proposition 1.14], $\mathscr{D}(A_S)=\mathscr{D}(A_1)\oplus\cdots\oplus\mathscr{D}(A_t)$.

Let M be the maximal ideal of the local artinian k-algebra A_i. By Cohen's theorem there exists a subfield L of A_i with $A_i=L\oplus M$. If $n=\dim_L A_i$ then by Lemma 2.4, $\mathscr{D}(A_i)\cong\mathrm{Mat}_n(\mathscr{D}(L))$. Since L is a finitely generated field extension of k, $\mathscr{D}(L)$ is a Noetherian domain by [5, Proposition 2.6], for example. Hence $\mathscr{D}(L)$ has a simple artinian quotient ring Q and $\mathrm{Mat}_n(Q)$ is the simple artinian quotient ring of $\mathscr{D}(A_i)$. It follows that $\mathscr{D}(A_S)$ has a semisimple artinian quotient ring.

2.6. COROLLARY. $\mathscr{D}(A)$ *has a simple artinian quotient ring if and only if* A *has a local artinian quotient ring.*

Proof. This is immediate from the proof of 2.5.

2.7. We denote by $Q(R)$ the quotient ring of R if it exists. In the next section we shall require the following generalization of 2.5.

LEMMA. *If A has an artinian quotient ring and I is an ideal of A, then $\mathscr{D}(I)$ has a semisimple artinian quotient ring. In fact there is an idempotent $e \in Q(\mathscr{D}(A))$ such that $\mathscr{D}(I)$ has quotient ring $eQ(\mathscr{D}(A))\,e$.*

Proof. It is enough to prove the last statement. Let S be the set of non-zero divisors in A and $A_S = A_1 \oplus \cdots \oplus A_t$ as in 2.5. We have $I_S = I_1 \oplus \cdots \oplus I_t$, where $I_i = I_S \cap A_i$ and $\mathscr{D}(I_S) \cong \mathscr{D}(I_1) \oplus \cdots \oplus \mathscr{D}(I_t)$ by Corollary 1.8.

As in 2.5 we have $A_i = L \oplus M$, where M is the maximal ideal of A_i and L a subfield. By Lemmas 2.3 and 2.1 there exists an idempotent $e_i \in \mathscr{D}(A_i)$ such that $e_i \mathscr{D}(A_i)\, e_i \cong \mathscr{D}_L(I_i) = \mathscr{D}_A(I_i) = \mathscr{D}(I_i)$. Hence $e_i Q(\mathscr{D}(A_i))\, e_i \cong Q(\mathscr{D}(I_i))$ by [7, Theorem 3]. If $e = e_1 + \cdots + e_t \in \mathscr{D}(A_S) \subseteq Q(\mathscr{D}(A))$, it follows that $Q(\mathscr{D}(I)) \cong eQ(\mathscr{D}(A))\, e$.

2.8. It is convenient also to have the following description of $\mathscr{D}(I)$ which is implicit in the above. For simplicity we assume that I is an ideal in a local artinian ring A and that $A = L \oplus N$, where L is a subfield and N the nilpotent radical of A. Let $v_1, ..., v_r$ be a basis for I over L and extend to a basis $v_1, ..., v_s$ of A. For each i, let e_i denote the L-linear map from A to L defined by $e_i(v_i) = 1$, $e_i(v_j) = 0$, $j \neq i$. Under composition of maps $v_j\, \mathscr{D}_L(L)\, e_i$ acts as differential operators from $v_i L$ to $v_j L$.

LEMMA. *With the above notation*

$$v_j \mathscr{D}_L(L)\, e_i = \mathscr{D}_L(v_i L, v_j L)$$

and

$$\mathscr{D}_A(I) = \mathscr{D}_L(I) = \sum_{1 \leqslant i, j \leqslant r} v_j \mathscr{D}_L(L)\, e_i.$$

3. THE PRIME RADICAL

3.1. In this section we describe the prime radical of $\mathscr{D}(A)$, where A is a finitely generated k-algebra. We first establish some notation. Let $0 = \bigcap_{\lambda \in \Lambda} K_\lambda$ be an irredundant primary decomposition of 0 in A, where K_λ is P_λ-primary. Suppose A has Krull dimension n. For $0 \leqslant i \leqslant n$ we set $\Lambda_i = \{\lambda \in \Lambda \mid \operatorname{rank}(P_\lambda) \leqslant i\}$, $I_i = \bigcap_{\lambda \in \Lambda_i} K_\lambda$, and $S_i = A - \bigcup_{\lambda \in \Lambda_i} P_\lambda$. Then by [1, Proposition 4.9], $S_i(0) = I_i$ and in particular I_i is independent of the chosen primary decomposition. It is convenient to set $I_{-1} = A$. By Lemma 1.4 each ideal in the chain $A = I_{-1} \supseteq I_0 \supseteq I_1 \supseteq \cdots \supseteq I_n = 0$ is a $\mathscr{D}(A)$-submodule of A. For $0 \leqslant i \leqslant n$, set $J_i = \Delta_A(I_{i-1}, I_i)$. Then by Lemma 1.2, J_i is an ideal of $\mathscr{D}(A)$ and by construction $J_n \cdots J_1 J_0 = 0$. Hence if we set $N = J_0 \cap J_1 \cap \cdots \cap J_n$, then $N^{n+1} = 0$.

THEOREM B. *With the above notation N is the prime radical of $\mathscr{D}(A)$.*

Since N is nilpotent it will suffice to show that each J_i is a semiprime ideal. We do this by showing that each factor ring $\mathscr{D}(A)/J_i$ has a semisimple artinian quotient ring.

3.2. From now on fix i and set $J = J_i$, $I = I_{i-1}$. Let $\Omega = \Lambda_i - \Lambda_{i-1}$ and for each $\lambda \in \Omega$ set $I_\lambda = I \cap K_\lambda$ and $S_\lambda = S_{i-1} \cap (A - P_\lambda)$. By [1, Proposition 4.9], $I_\lambda = S_\lambda(0)$, so by Lemma 1.4, I_λ is a $\mathscr{D}(A)$-submodule of A. Hence by Lemma 1.2, $J_\lambda = \Delta_A(I, I_\lambda)$ is an ideal of $\mathscr{D}(A)$. Since $\bigcap_{\lambda \in \Omega} I_\lambda = I_i$ we have $\bigcap_{\lambda \in \Omega} J_\lambda = J$. We prove

THEOREM. (a) *Each factor ring $\mathscr{D}(A)/J_\lambda$ has a simple artinian quotient ring Q_λ.*

(b) *The ideals J_λ/J, $\lambda \in \Omega$, are the minimal primes of $\mathscr{D}(A)/J$.*

(c) *$\mathscr{D}(A)/J$ is an order in the semisimple artinian ring $\bigoplus_{\lambda \in \Omega} Q_\lambda$.*

3.3. LEMMA. *Let $S = \{s + K_\lambda \mid s \in S_\lambda\}$. Then S is precisely the set of non-zero divisors in A/K_λ.*

Proof. Since $S_\lambda \subseteq A - P_\lambda$, elements of S are non-zero divisors in A/K_λ. Conversely suppose $s + K_\lambda$ is a non-zero divisor in A/K_λ, then $s \notin P_\lambda$. Number the maximal elements of the set $\{P_\mu \mid \mu \in \Lambda_{i-1}\}$ as $P_1, ..., P_m, P_{m+1}, ..., P_n$, where $s \in P_j$ if and only if $1 \leqslant j \leqslant m$. Let $B = P_{m+1} \cap \cdots \cap P_n \cap K_\lambda$. If $B \subseteq P_1 \cup \cdots \cup P_m$ then by [3, Theorem 81], $B \subseteq P_j$ for some j with $1 \leqslant j \leqslant m$. Since P_j is prime it follows that $P_l \subseteq P_j$ for some l with $m+1 \leqslant l \leqslant n$ or $K_\lambda \subseteq P_j$. The first case is impossible by the incomparability of P_l, P_j and the second case gives $P_\lambda \subseteq P_j$, which contradicts the facts that $\operatorname{rank}(P_\lambda) = i$, $\operatorname{rank}(P_j) \leqslant i - 1$. Hence we can find $x \in B$ with $x \notin P_1, ..., P_m$. It then follows that $s + x + K_\lambda = s + K_\lambda$ and $s + x \in S_\lambda$, which proves the lemma.

3.4. LEMMA. *Let I and K be ideals of the finitely generated algebra A and suppose A/K has a local artinian quotient ring. Let S be the set of non-zero divisors in A/K. Given $d \in \mathscr{D}((I+K)/K)$ there exist $s \in S$ and $d' \in \mathscr{D}(A/I \cap K)$ such that for all $a \in I$, $d'(a + (I \cap K)) + K = sd(a + K)$.*

Proof. Write A as a homomorphic image of a polynomial algebra $\tilde{A}$

and let $\tilde{I}$, $\tilde{K}$ be the inverse images of I, K, respectively. Since $\tilde{A}/\tilde{K} \cong A/K$, $\tilde{A}/\tilde{I} \cap \tilde{K} \cong A/I \cap K$, and $I+K/K \cong \tilde{I}+\tilde{K}/\tilde{K}$ as A-modules we may replace A by $\tilde{A}$ in proving the lemma, to assume that A is a polynomial algebra.

Since $S^{-1}(A/K)$ is a local artinian ring it contains a copy of its residue field L by Cohen's theorem. We can choose $v_1, ..., v_r \in I$ such that $v_1 + K, ..., v_r + K$ form a basis for $S^{-1}(I+K/K)$ as a vector space over L. If $e_1, ..., e_r$ are as in 2.8 we have

$$\mathscr{D}(S^{-1}(I+K/K)) = \sum_{i \leqslant i,\, j \leqslant r} (v_j + K)\, \mathscr{D}_L(L)\, e_i.$$

Write $d \in \mathscr{D}(I+K/K)$ in the form $d = \sum_j (v_j + K)\, \delta_j$, where $\delta_j \in \sum_i \mathscr{D}_L(L)\, e_i \subseteq \mathscr{D}(S^{-1}(A/K))$. There exists $s \in S$ such that $s\delta_j \in \mathscr{D}(A/K)$ for all j. Therefore by Lemma 1.3, there exists $\delta_j' \in \mathscr{D}(A)$ such that $\delta_j'(K) \subseteq K$ and $\delta_j'(a) + K = s\delta_j(a+K)$ for all $a \in A$. Let $d_1 = \sum v_j \delta_j' \in I\mathscr{D}(A)$. Then $d_1(K) \subseteq K$ and

$$d_1(a) + K = sd(a+K) \qquad \text{for all} \quad a \in I.$$

Also since $d_1(A) \subseteq I$, we have $d_1(I \cap K) \subseteq I \cap K$. Hence d_1 induces a differential operator $d' \in \mathscr{D}(A/I \cap K)$ such that $d'(a + (I \cap K)) = d_1(a) + (I \cap K)$ for all $a \in A$. In particular for $a \in I$ we have

$$d'(a + (I \cap K)) + K = sd(a+K) \qquad \text{as required.}$$

3.5. We can now prove part (a) of Theorem 3.2. By Lemma 1.2 we can regard $\mathscr{D}(A)/J_\lambda$ as a subring of $\mathscr{D}(I/I_\lambda)$. Since $I/I_\lambda \cong I + K_\lambda/K_\lambda$ as A-modules we have $\mathscr{D}(I/I_\lambda) \cong \mathscr{D}(I + K_\lambda/K_\lambda)$. Now $(I + K_\lambda)/K_\lambda$ is an ideal in the primary ring A/K_λ so $\mathscr{D}(I/I_\lambda)$ has a simple artinian quotient ring Q_λ by Corollary 2.6 and Lemma 2.7.

If $s \in S_\lambda$, then $s + J_\lambda$ is a non-zero divisor in $\mathscr{D}(A)/J_\lambda$ and $\mathscr{D}(I/I_\lambda)$ since s acts as a non-zero divisor on the module I/I_λ. We show that given $d \in \mathscr{D}(I/I_\lambda)$, there exists $c \in S_\lambda$ such that $(c + J_\lambda)\, d \in \mathscr{D}(A)/J_\lambda$. If d is an operator of order n then we shall also have $d(c^{n+1} + J_\lambda) \in \mathscr{D}(A)/J_\lambda$. It follows from this that Q_λ is the simple artinian quotient ring of $\mathscr{D}(A)/J_\lambda$.

Define $d_1 \in \mathscr{D}(I + K_\lambda/K_\lambda)$ by

$$d_1(a + K_\lambda) = d'(a) + K_\lambda \qquad \text{for} \quad a \in I, \tag{1}$$

where $d'(a)$ is any element of I such that $d'(a) + I_\lambda = d(a + I_\lambda)$.

By Lemmas 3.3 and 3.4, there exist $s \in S_\lambda$ and $d_2 \in \mathscr{D}(A/I_\lambda)$ such that

$$d_2(a + I_\lambda) + K_\lambda = (s + K_\lambda)\, d_1(a + K_\lambda) \qquad \text{for} \quad a \in I. \tag{2}$$

Now consider the localisation map Φ: $\mathscr{D}(A) \to \mathscr{D}(A/I_\lambda)$. By Lemma 1.6

there exists $t \in S_\lambda$ such that $(t+I_\lambda)\,\mathscr{D}(A/I_\lambda) \subseteq \operatorname{Im} \Phi$. Hence we can find $d_3 \in \mathscr{D}(A)$ such that

$$d_3(a) + I_\lambda = (t + I_\lambda)\, d_2(a + I_\lambda) \qquad \text{for} \quad a \in A, \tag{3}$$

Combining Eq. (1)–(3) we have, since $I_\lambda \subseteq K_\lambda$,

$$\begin{aligned} d_3(a) + K_\lambda &= (t + K_\lambda)(d_2(a + I_\lambda) + K_\lambda) \\ &= (ts + K_\lambda)\, d_1(a + K_\lambda) \\ &= (tsd'(a) + K_\lambda) \qquad \text{for} \quad a \in I. \end{aligned}$$

Hence $d_3(a) - tsd'(a) \in K_\lambda$. However, $d_3(I) \subseteq I$, since I is a $\mathscr{D}(A)$-submodule of A, and $d'(a) \in I$, so $d_3(a) - tsd'(a) \in I \cap K_\lambda = I_\lambda$. Therefore

$$\begin{aligned} d_3(a) + I_\lambda &= tsd'(a) + I_\lambda \\ &= tsd(a + I_\lambda) \qquad \text{for all} \quad a \in I. \end{aligned}$$

It follows that $d_3 + J_\lambda = (c + J_\lambda)\, d$ with $c = ts \in S_\lambda$ as claimed.

3.6. The proof of Theorem 3.2 is now easy to complete. For each $\lambda \in \Omega$, I/I_λ is isomorphic to an ideal of A/K_λ. Hence $K_\lambda \subseteq \operatorname{ann}_A(I/I_\lambda) \subseteq P_\lambda$. If $\bigcap_{\mu \neq \lambda} K_\mu \subseteq P_\lambda$ then $K_\mu \subseteq P_\lambda$ for some $\mu \neq \lambda$ since P_λ is prime and so $P_\mu \subseteq P_\lambda$, which is impossible. Hence for each $\lambda \in \Omega$ we can choose $c_\lambda \in \bigcap_{\mu \neq \lambda} K_\mu$, $c_\lambda \notin P_\lambda$. In particular, it follows that $c_\lambda \in J_\mu = \Delta_A(I, I_\mu)$ for $\mu \neq \lambda$ and $c_\lambda \notin J_\lambda$. Hence the ideals $\{J_\lambda \mid \lambda \in \Omega\}$ are incomparable. Since these ideals are prime by part (a) of the theorem and $\bigcap J_\lambda = J$, part (b) follows.

Also we have an embedding

$$\mathscr{D}(A)/J \subseteq \bigoplus_{\lambda \in \Omega} \mathscr{D}(A)/J_\lambda = R \subseteq \bigoplus_{\lambda \in \Omega} Q_\lambda = Q.$$

An element of Q will be written (q_λ), where q_λ is the component in Q_λ for all λ. To show that $\mathscr{D}(A)/J$ is an order in Q it will suffice in view of part (a) to show that if $d = (d_\lambda + J_\lambda) \in R$, with $d_\lambda \in \mathscr{D}(A)$, there exists a non-zero divisor $c \in R$ such that $cd \in \mathscr{D}(A)/J$. If d has order n we will also have $dc^{n+1} \in \mathscr{D}(A)/J$.

Since $c_\lambda \notin P_\lambda$, $c_\lambda + J_\lambda$ is a non-zero divisor in $\mathscr{D}(A)/J_\lambda$. Hence $c = (c_\lambda + J_\lambda)$ is a non-zero divisor in R. Set $\delta = \sum c_\mu d_\mu \in \mathscr{D}(A)$. Since $c_\mu \in J_\lambda$ for $\mu \neq \lambda$ we have $\delta = c_\lambda d_\lambda \bmod J_\lambda$. Therefore δ maps to $(c_\lambda d_\lambda + J_\lambda) = (c_\lambda + J_\lambda)(d_\lambda + J_\lambda) = cd$. Hence $\delta + J = cd$ as required.

3.7. An immediate consequence of Theorem B is the following.

COROLLARY. *If A is a finitely generated k-algebra of Krull dimension n, and N is the prime radical of $\mathscr{D}(A)$, then $N^{n+1}=0$.*

This is perhaps surprising since clearly we cannot bound the index of nilpotence of the nilradical of A in terms of any function of n.

It is easy to construct examples where the bound in the corollary is achieved. For example, let $B=k[x_0, x_1, ..., x_n]$ and for $0 \leqslant i \leqslant n$, $P_i = x_0B + \cdots + x_iB$, $K_i = P_i^{i+1}$, $K = K_0 \cap K_1 \cap \cdots \cap K_n$, and $A = B/K$. Then A has Krull dimension n and K_i is P_i-primary ideal. In the notation of 3.1 we have $I_i = (K_0 \cap \cdots \cap K_i)/K$. Let $x = x_0 + K$. Then $xI_i \subseteq I_{i+1}$ for $-1 \leqslant i < n$. Therefore $x \in \bigcap J_i = N$ and $x^n \neq 0$. Hence $N^n \neq 0$.

3.8. COROLLARY. *If A is a finitely generated k-algebra then $\mathscr{D}(A)$ is semiprime (resp. prime) if and only if A has an artinian (resp. local artinian) quotient ring.*

Proof. The sufficiency of the conditions follows from 2.5 and 2.6. Conversely if $\mathscr{D}(A)$ is semiprime we need to show that $I_0 = 0$ (in the notation of 3.1). It is easily seen that I_0 is a nilpotent ideal of A. If $I_0 \neq 0$ suppose that $I_0^l \neq 0$ but $I_0^{l+1} = 0$ for some integer $l \geqslant 1$. Then we have $I_0^l \cdot A \subseteq I_0$ and $I_0^l \cdot I_0 = 0$. This gives $I_0^l \subseteq N$, which contradicts $\mathscr{D}(A)$ semiprime. If in addition $\mathscr{D}(A)$ is prime then by the proof of 2.5, the artinian quotient ring of A is local.

4. AN ANALOGUE OF NAKAI'S CONJECTURE

4.1. If A is a finitely generated k-algebra, we can ask for conditions under which $\mathscr{D}(A)$ is generated by $\mathscr{D}^1(A)$. If A is a domain then Nakai's conjecture asserts that this is equivalent to A being the coordinate ring of a non-singular variety. In general we show A must be reduced. It seems likely that A must in fact be a direct sum of domains.

LEMMA. *If S is a multiplicatively closed subset of A and $\mathscr{D}(A)$ is generated by $\mathscr{D}^1(A)$, then $\mathscr{D}(A_S)$ is generated by $\mathscr{D}^1(A_S)$.*

Proof. This follows easily from the fact that $\mathscr{D}(A_S) \cong A_S \otimes_A \mathscr{D}(A)$.

4.2. THEOREM. *If A is a finitely generated k-algebra such that $\mathscr{D}(A)$ is generated by $\mathscr{D}^1(A)$ then A is reduced.*

Proof. If we filter $\mathscr{D}(A)$ by the order of the differential operators, then the associated graded ring $\operatorname{gr} \mathscr{D}(A)$ is generated by A and the image of $\operatorname{der}(A)$. Since $\operatorname{der}(A)$ is a finitely generated A-module, it follows that $\operatorname{gr} \mathscr{D}(A)$ is a finitely generated commutative A-algebra and hence

Noetherian. Therefore $\mathscr{D}(A)$ is left Noetherian. It follows from Theorem A that A has an artinian quotient ring.

Now let N be the nilradical of A and S the set of non-zero divisors of A. If $N \neq 0$, then $N_S \neq 0$ and N_S is the nilradical of A_S. By Lemma 4.1, $\mathscr{D}(A_S)$ is generated by A_S and $\operatorname{der}(A_S)$. Now $A_S \cong A_1 \oplus \cdots \oplus A_t$, where each A_i is a local artinian ring. Since $\mathscr{D}(A_S) \cong \mathscr{D}(A_1) \oplus \cdots \oplus \mathscr{D}(A_t)$, each $\mathscr{D}(A_i)$ is generated by A_i and $\operatorname{der}(A_i)$. If M is the maximal ideal of A_i, then any derivation of A_i preserves M by [2, 4.1]. By Cohen's theorem, there exists a subfield K of A_i such that $A_i = K \oplus M$. By Lemma 2.1 any K-linear endomorphism of A_i is a differential operator. It follows that $M = 0$, and each A_i is a field, but this contradicts the assumption that $N_S \neq 0$.

For the case where $A = k[x_1, ..., x_n]/(f)$ is a factor algebra of a polynomial algebra by a principal ideal, the above result has been proved by D. P. Patil and B. Singh; see [9, Note Added in Proof]. It was their result which inspired Theorem 4.2.

4.3. Lemma. *Suppose A is reduced with minimal primes $P_1, ..., P_n$. If $\mathscr{D}(A)$ is generated by $\mathscr{D}^1(A)$, and each A/P_i is the coordinate ring of a non-singular variety, then $A \cong A/P_1 \oplus \cdots \oplus A/P_n$.*

Proof. Set $A_i = A/P_i$. Since $P_1 \cap \cdots \cap P_n = 0$ we can identify A with a subalgebra of $A_1 \oplus \cdots \oplus A_n$. Suppose that

$$A \cap (A_1, 0, ..., 0) \subsetneqq (A_1, 0, ..., 0),$$

that is, $P_1 + (P_2 \cap \cdots \cap P_n) \neq A$. Let M be a maximal ideal of A containing $P_1 + (P_2 \cap \cdots \cap P_n)$. By replacing A with A_M we can assume that A is local. If $S_i = A - P_i$, then $P_i = S_i(0)$ and thus $\mathscr{D}(A)$ may be identified with a subalgebra of $\mathscr{D}(A_1) \oplus \cdots \oplus \mathscr{D}(A_n)$. Let $I = P_1 + (P_2 \cap \cdots \cap P_n)$. Since each P_i is invariant under every derivation of A by [2, 4.1] so also is $P_1 + I^m$ for all $m \geqslant 1$. Therefore if $\mathscr{D}(A)$ is generated by A and $\operatorname{der}(A)$, $P_1 + I^m$ is a $\mathscr{D}(A)$-submodule of A. If $x \in P_2 \cap \cdots \cap P_n$, $x \neq 0$, then by Lemma 1.6, $((x + P_1)\, \mathscr{D}(A_1), 0, ..., 0) \subseteq \mathscr{D}(A)$. We can choose m such that $x \notin P_1 + I^m$. Since $P_1 + I^m$ is a non-zero ideal in the regular local ring A_1, there exists $\partial \in \mathscr{D}(A_1)$ such that $\partial(P_1 + I^m)$ contains a unit of A_1. Thus $(x + P_1)\, \partial(P_1 + I^m) \nsubseteq P_1 + I^m$. This contradicts the fact that $P_1 + I^m$ is a $\mathscr{D}(A)$-submodule. It follows that $(A_1, 0, ..., 0) \subseteq A$ and similarly $(0, ..., A_i, ..., 0) \subseteq A$ for all i. Hence $A = A_1 \oplus \cdots \oplus A_n$.

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