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Injective modules for group algebras of locally finite groups

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1. INTRODUCTION

Two recent results relate the existence of injective modules for group algebras which are 'small' in some sense to the structure of the group.

(1) The trivial kG-module is injective if and only if G is a locally finite group with no elements of order $p = \operatorname{char} k$ (9).

(2) If G is a countable group, then every irreducible kG-module is injective if and only if G is a locally finite p' group which is abelian-by-finite (9) and (11).

In this paper we investigate several situations in which kG has injective modules with countable k-dimension. The main result can be seen as a generalization of (2).

THEOREM 4·8. *If G is a locally finite group and k is a field of characteristic $p \geqslant 0$, then the injective hull of every irreducible kG-module has countable k-dimension if and only if G is abelian-by-finite and has no infinite p subgroup.*

Here if $p = 0$, we take the last condition as being vacuously satisfied.

However, we begin by studying the centralizer $C_G(V)$ of an arbitrary injective kG-module V. This is always a locally finite p' group (compare (1) above) and furthermore, if W is any kG-module with $C_G(W)$ a locally finite p' subgroup, then W is kG-injective if and only if it is injective when viewed as a natural $k[G/C_G(W)]$-module.

In Section 3 we study injective modules which satisfy the minimum condition on annihilators of subsets of the ring. It is convenient to introduce some notation at this point.

If S is a subset of a ring R and V is a right R module we denote by

$$\operatorname{Ann}_V S = \{v \in V \mid vS = 0\}, \quad \text{the annihilator of } S \text{ in } V.$$

We abbreviate the condition that V satisfies the minimum condition on annihilators of subsets of R by writing *V satisfies min-ann.* We denote by $l(S)$, $r(S)$ respectively the left and right annihilator ideals of S in R. We assume throughout that all rings have 1's and all modules are unital. If W is a right R module we denote the injective hull of W by $E(W)$.

Faith (7) shows that an injective module V is Σ-injective if and only if it satisfies min-ann. We find the latter condition easier to work with and in Lemma 3·2 (adapted from an argument due to D. S. Passman), we show that if R is a k-algebra and V is an injective R-module with countable k-dimension, then V satisfies min-ann.

Suppose now that V is a right R-module satisfying min-ann. and set $R_0 = \operatorname{Ann}_R V$. Then R/R_0 has the maximum condition (max.) on right annihilator ideals (Lemma 3·3, see also (8), corollary 7). If in addition, we suppose that V is irreducible and R is locally artinian, then R/R_0 is a primitive ring and every non-zero right ideal contains a non-zero idempotent (Lemma 3·4). This forces R/R_0 to be simple artinian and so $R/R_0 \cong M_n(E)$ where $E = \mathrm{End}_R V$ and $n = \dim_E V$. In particular V is finite dimensional as a (left) module over its endomorphism ring.

However, we can improve this result by lifting idempotents in locally artinian rings to show that if R is a locally artinian ring and the injective hull of an irreducible R module V satisfies min-ann., then again $R/R_0 \cong M_n(E)$, where E, R and n are as in the previous paragraph (Theorem 3·7).

In Section 4 we specialize to the case where $R = kG$, the group algebra of a locally finite group has an injective module V satisfying min-ann. If $R_0 = \mathrm{Ann}_R V$ and $\bar{R} = R/R_0$, then using the arguments of section 3 we see that $S = \bar{R}/J(\bar{R})$ is semisimple artinian, where $J(\bar{R})$ denotes the Jacobson radical of $\bar{R}$, so $S \cong \oplus M_{n_i}(k_i)$ for certain division algebras k_i. If $\mathrm{char}\, k > 0$ or if k contains all roots of 1, we see that all the division algebras which occur are fields (this follows from Lemma 4·4).

Now there is a natural homomorphism from $G/C_G(V)$ to $\Pi GL(n_i, k_i)$ and from the assumption that V satisfies min-ann. we can deduce easily that G has no infinite p subgroups. It follows from a theorem of Brauer and Feit (3) and (12), theorem 1.L.4, that the image of $G/C_G(V)$ in each summand $GL(n_i, k_i)$ is abelian-by-finite and hence the image of $G/C_G(V)$ in $\Pi GL(n_i, k_i)$ is also abelian-by-finite. Also the kernel of this homomorphism is a finite p subgroup and in fact we show that $G/C_G(V)$ is abelian-by-finite. Assembling these results we obtain a characterisation of Σ-injective modules over group algebras of locally finite groups (Theorem 4·5).

However, we do not need the full force of this characterization in order to prove Theorem 4·8. In fact it follows from Theorem 3·7 that if G is a locally finite group such that the injective hull of every irreducible kG-module has countable k-dimension, then G is a *restricted* group, that is every irreducible kG-module is finite dimensional over its endomorphism ring, and we show that a restricted locally finite group with no infinite p subgroups is abelian-by-finite, thus proving the harder part of Theorem 4·8. Without the assumption that G has no infinite p subgroups, Hartley (unpublished) has shown that it G is a locally finite restricted group in char p, then $G/O_p(G)$ is abelian-by-finite.

Finally, in Section 5 we apply some of the above techniques to study the injective hull of the regular module.

2. The centralizer of an injective module

We assemble some well-known results on induction and restriction.

Lemma 2·1. *Let H be a subgroup of a group G and k any field.*

(i) *If V is an injective (right) kH-module, then the coinduced module* $\mathrm{Hom}_{kH}(kG, V)$ *is an injective kG-module containing a copy of $V \otimes_{kH} kG$. Moreover if the index $|G{:}H|$ is finite then $V \otimes_{kH} kG \cong \mathrm{Hom}_{kH}(kG, V)$.*

(ii) *If V is an injective kG-module, then the restriction V_H is an injective kH-module.*

Proof. We make $\mathrm{Hom}_{kH}(kG, V)$ into a right kG-module by defining

$$(fr)(s) = f(rs) \quad \text{for} \quad f \in \mathrm{Hom}_{kH}(kG, V) \quad r, s \in kG.$$

It is easily seen that $\mathrm{Hom}_{kH}(kG,\ )$ is a functor from right kH-modules to right kG-modules. If $\theta: A \to B$ is a map between kH-modules we define

$$\theta^G: \mathrm{Hom}_{kH}(kG, A) \to \mathrm{Hom}_{kH}(kG, B)$$

by

$$(\theta^G(f))(x) = \theta(f(x))$$

for

$$f \in \mathrm{Hom}_{kH}(kG, A), \quad x \in kG.$$

The embedding of $V \otimes_{kH} kG$ into $\mathrm{Hom}_{kH}(kG, V)$ may be described as follows. If $\{s_i | i \in I\}$ is a right transversal to H in G, then $\{s_i^{-1} | i \in I\}$ is a left transversal and for

$$v = \Sigma v_i \otimes s_i \in V \otimes_{kH} kG \quad \text{and} \quad a = \Sigma s_j^{-1} a_j \in kG$$

we define $f_v \in \mathrm{Hom}_{kH}(kG, V)$ by $f_v(a) = \Sigma v_1 a_1$.

For further details see (4), p. 866, or (6), lemma 57·7.

(ii) To any diagram of kH-modules,

$$\begin{tikzcd} O \arrow[r, dash] & M \arrow[r] \arrow[d, "\theta"] & N \\ & V_H & \end{tikzcd}$$

there corresponds a diagram of kG-modules

$$\begin{tikzcd} 0 \arrow[r, dash] & \mathrm{Hom}_{kH}(kG, M) \arrow[r] \arrow[d, "\theta^G"] & \mathrm{Hom}_{kH}(kG, N) \\ & \mathrm{Hom}_{kH}(kG, V_H) & \end{tikzcd}$$

Also since V is an injective kG-module and $V \cong \mathrm{Hom}_{kG}(kG, V) \leqslant \mathrm{Hom}_{kH}(kG, V)$, V is isomorphic to a direct summand of $\mathrm{Hom}_{kH}(kG, V)$ and combining θ^G with the projection map gives a kG-map from $\mathrm{Hom}_{kH}(kG, M)$ to V. Then since V is an injective kG-module we can extend θ^G to a kG map $\overline{\theta^G}$, say from $\mathrm{Hom}_{kH}(kG, N)$ to V, and finally restricting $\overline{\theta^G}$ to the kH-submodule $\mathrm{Hom}_{kH}(kH, N) \cong N$ gives a kH-map extending the original map θ.

LEMMA 2·2. *Let G be any group and k a field of chracteristic $p \geqslant 0$. If V is an injective kG-module, then $C = C_G(V)$ is a locally finite p' group.*

Proof. The restriction V_C is an injective kC-module and is trivial as a kC-module. Therefore the 1-dimensional trivial module k is a direct summand of V_C and so is injective. Thus C is a locally finite p' group by (9), theorem 1.

LEMMA 2·3. *If V is an injective kG-module then V is injective as a $k[G/C_G(V)]$-module.*

Proof. A routine calculation which we leave to the reader.

LEMMA 2·4. *Suppose H is a normal locally finite p' subgroup of a group G. If V is an injective $k[G/H]$-module then V is an injective kG-module when H is allowed to act trivially.*

Proof. Suppose I is a right ideal of kG and $\phi: I \to V$ is a kG map. Let $\omega_G H$ denote the (two-sided) augmentation ideal of H in kG.

If r is an element of $I \cap \omega_G H$ we can write $r = \Sigma a_i(g_i - 1)$ where $a_i \in kG$ and $g_1, \ldots, g_n$ are finitely many elements of H.

Now kH is von Neumann regular by (15), §24, and hence there is an idempotent $e \in kH$ such that

$$kH(g_1 - 1) + \ldots + kH(g_n - 1) = kHe.$$

Since $(g_i - 1) \in kHe$ we have

$$(g_i - 1) = (g_i - 1)e \quad \text{for} \quad i = 1, \ldots, n.$$

Therefore $r = \Sigma a_i(g_i - 1) = \Sigma a_i(g_i - 1)e = re$ and so $\phi(r) = \phi(r)e = 0$, as $\phi(r) \in V$, e is in the augmentation ideal of kH and H acts trivially on V.

Hence we can extend ϕ to a map from $I + \omega_G H$ to V by setting $\phi(\omega_G H) = 0$ and this gives a $k[G/H]$-map

$$\bar{\phi}: \frac{I + \omega_G H}{\omega_G H} \to V.$$

Since V is injective as a $k[G/H]$-module $\bar{\phi}$ extends to $k[G/H]$ and then composition with the natural map $kG \to k[G/H]$ gives a map $kG \to V$ which extends ϕ.

3. Modules satisfying min-ann.

Definition 3·1 (Faith (7)). *An injective R-module V is Σ-injective if and only if the following equivalent conditions hold:*

(1) *Any direct sum of copies of V is injective.*

(2) *Any countable direct sum of copies of V is injective.*

(3) *R satisfies max. on annihilators of subsets of V.*

(4) *V satisfies min-ann.*

We shall work mainly with the last condition.

Lemma 3·2. *If R is a k-algebra and V an injective R-module with $\dim_k V$ countable, then V satisfies min-ann.*

Proof. Suppose $V_1 > V_2 > \ldots$ is a strictly descending chain of annihilators where $V_i = \mathrm{Ann}_V S_i$, say for certain subsets S_i of R.

If we set $I_i = \mathrm{Ann}_R V_i = \{r \in R \mid V_i r = 0\}$, then $V_i = \mathrm{Ann}_V I_i$ and so we may work now with I_i in place of S_i. Note that I_i is a right ideal and we have a strictly ascending chain $I_1 < I_2 < \ldots$.

For each integer n choose an element $v_n \in V_n \backslash V_{n+1}$.

Then $v_n I_{n+1} \neq 0$, so $\exists\, a_{n+1} \in I_{n+1}$ such that $v_n a_{n+1} \neq 0$. It follows that

$$a_{n+1} \notin I_n = \mathrm{Ann}_R V_n.$$

Set $W = \bigcap_n V_n$ a k-subspace of V and let $T = k^{\mathbb{N}}$ be the k-space of countably infinite sequences $(t_1, t_2, \ldots)$ of elements t_i in k. We aim to construct a 1–1 k-linear transformation $\phi: T \to V/W$. Since $\dim_k V$ is countable and $\dim_k T$ is uncountable this will give a contradiction.

Set $I = \cup I_i$ a right ideal in R. If $t = (t_1, t_2, \ldots) \in T$ we define $f_t: I \to V$ by

$$f_t(r) = \sum_{n=1}^{\infty} v_n t_n r.$$

If $r \in I$, then $r \in I_j$ for some j and hence $v_n t_n r = 0$ for all $n > j$ and the above sum is finite for all $r \in I$.

Now f_t is an R module homomorphism and so extends to an R map $f_t^*: R \to V$, since V is injective. Suppose that $g: R \to V$ also extends f_t. Then

$$g(1)r = g(r) = f_t^*(r) = f_t^*(1)r \quad \text{for all} \quad r \in I.$$

Therefore $(f_t^*(1) - g(1)) \in \operatorname{Ann}_V I_i$ for all i, and so $f_t^*(1) - g(1) \in W$.

Hence we have a well-defined map $\phi: T \to V/W$ given by

$$\phi(t) = f_t^*(1) + W/W.$$

Now if $a, b \in k$ and $s, t \in T$ then $f_{as+bt} = af_s + bf_t$ and so $af_s^* + bf_t^*$ extends f_{as+bt} and $\phi(as+bt) = a\phi(s) + b\phi(t)$.

Therefore ϕ is a k-linear transformation.

Now suppose $t = (t_1, t_2, \ldots) \in \ker \phi$. Then $f_t^*(1) \in W$ so

$$f_t(r) = f_t^*(1)r = 0 \quad \text{for all} \quad r \in I.$$

If $t \neq 0$ choose j minimal with $t_j \neq 0$. Then

$$\begin{aligned} 0 = f_t(a_{j+1}) &= \Sigma v_n t_n a_{j+1} \\ &= t_j v_j a_{j+1}. \end{aligned}$$

Hence $v_j a_{j+1} = 0$ contradicting the choice of a_{j+1}. Therefore ϕ is 1–1 and the lemma is proved.

LEMMA 3·3. *If R is a ring, V a right R module satisfying min-ann. and $R_0 = \operatorname{Ann}_R V$, then $\bar{R} = R/R_0$ has max. on right annihilator ideals. In particular $\bar{R}_R$ has max. on direct summands.* (Cf. (8), cor. 11.)

Proof. Suppose $I_1 < I_2 < \ldots$ is a strictly ascending chain of right annihilator ideals in $\bar{R}$. By replacing I_n by $r(l(I_n))$, we may assume that $I_n = r(J_n)$ where $J_1 > J_2$ and J_n is the left annihilator ideal of I_n.

Set $W_n = VJ_n$, then clearly $W_n \subseteq \operatorname{Ann}_V I_n$. If $j \in J_n \backslash J_{n+1}$, then since V is a faithful $\bar{R}$-module $\exists v \in V$ such that $vjI_{n+1} \neq 0$. Hence $W_n \not\subseteq \operatorname{Ann}_V I_{n+1}$ and it follows that $\operatorname{Ann}_V I_{n+1} < \operatorname{Ann}_V I_n$ for all n contradicting the assumption that V satisfies min-ann.

Since $\bar{R}$ has a 1 any direct summand of $\bar{R}_R$ has the form $e\bar{R}$ for some idempotent $e \in \bar{R}$ and $e\bar{R} = r(\bar{R}(1-e))$. Therefore $\bar{R}_R$ has max. on direct summands.

LEMMA 3·4. *Let R be a locally artinian semisimple ring. If I is a non-zero right ideal in R, then I contains a non-zero idempotent.*

Proof. Since R is semisimple, I is not nil so there is an element $a \in R$ which is not nilpotent. If S is an artinian subring of R containing a, then $I \cap S$ is an ideal of S which is not nil. Hence $I \cap S \not\subseteq J(S)$. It follows from (1), theorem 2·4 A, that $I \cap S$ contains a non-zero idempotent.

LEMMA 3·5. *Let M be an R-module such that:*

(i) *M has max. on direct summands.*

(ii) *Every non-zero submodule of M contains a non-zero direct summand.*

Then M is a finite direct sum of irreducible R-modules.

Proof. Let N be a non-zero submodule of M and choose a direct summand K of M maximal subject to being contained in M. Suppose that $K \oplus L = M$.

If the intersection $L \cap N$ is non-zero it contains a non-zero direct summand K_1 of M, where $K_1 \oplus L_1 = M$ say.

But then since $K_1 \subset L$ we have

$$K_1 + (L \cap L_1) = L \cap (K_1 + L_1) = L$$

by the modular law and hence $(K + K_1) \oplus (L \cap L_1) = M$, contradicting the maximality of K. Therefore $L \cap N = 0$ and $K = N$ is a direct summand of M.

Hence the lattice of submodules of M is complemented and the result follows from (13), § 3·3, proposition 2.

We are now in a position to show that if R is a locally artinian ring and V an irreducible R-module satisfying min-ann. then V is finite dimensional as a vector space over its endomorphism ring. First, however we prove a lemma on idempotent lifting which will give a stronger result.

LEMMA 3·6. *Let I be a two-sided ideal in a ring R. Let $\bar{R} = R/I$ and write $\bar{x}$ for the image of x in the factor ring $\bar{R}$. Suppose that either* (i) *I is a nil ideal, or* (ii) *R is locally artinian.*

Then we may lift ascending (and descending) chains of principal one-sided ideals generated by idempotents over I. More precisely, if $\bar{e}_1 \bar{R} < \bar{e}_2 \bar{R} < \dots$ is a strictly ascending chain of ideals in $\bar{R}$ where $\bar{e}_i$ is an idempotent, then there exist idempotents f_i in R such that $\bar{f}_i = \bar{e}_i$ and $f_1 R < f_2 R < \dots$.

Similarly for the other chains of ideals.

Proof. Note that $\bar{e}_1 \bar{R} < \bar{e}_2 \bar{R}$ if and only if $\bar{R}(1-\bar{e}_1) > \bar{R}(1-\bar{e}_2)$. Hence by symmetry it suffices to prove the result for ascending chains of right ideals.

(i) Let I be a nil ideal. Suppose first that $\bar{e}$ is an idempotent in $\bar{R}$. Then $e^2 - e \in I$, and so $(e^2-e)^k = 0$ for some integer k.

Now, by the Binomial Theorem,

$$\begin{aligned} 1 = \{e + (1-e)\}^{2k} &= \sum_{r=0}^{2k} \binom{2k}{r} e^{2k-r}(1-e)^r \\ &= \sum_{r=0}^{k-1} \binom{2k}{r} e^{2k-r}(1-e)^r + \binom{2k}{k} e^k(1-e)^k + \sum_{r=k+1}^{2k} \binom{2k}{k} e^{2k-r}(1-e)^r. \end{aligned}$$

Notice that the middle term is zero by choice of k.

Set
$$\lambda(e) = \sum_{r=0}^{k-1} \binom{2k}{r} e^{k-r}(1-e)^r,$$

$$\mu(e) = \sum_{r=k+1}^{2k} \binom{2k}{r} e^{2k-r}(1-e)^{r-k}$$

and
$$\sigma(e) = e^k \lambda(e), \quad \tau(e) = (1-e)^k \mu(e).$$

Then λ, μ, σ and τ are all polynomials in e,

$$1 = \sigma(e) + \tau(e)$$

and
$$\sigma(e)\tau(e) = \tau(e)\sigma(e) = e^k(1-e)^k\lambda(e)\mu(e) = 0.$$

Hence
$$\sigma(e) = \sigma(e)\{\sigma(e)+\tau(e)\} = \sigma(e)^2.$$

Now
$$\sigma(e) = e^{2k} + \binom{2k}{1}e^{2k-1}(1-e) + \dots$$
$$= e^{2k} = e \text{ modulo } I.$$

Similarly $\tau(e)$ is an idempotent which is congruent to $(1-e)$ modulo I. (The above argument was shown to me by T. K. Carne of Trinity College, Cambridge.)

Now suppose that $\bar{e}_1 \bar{R} < \bar{e}_2\bar{R} < \dots$ is an ascending chain of ideals in $\bar{R}$, with $\bar{e}$ idempotents.

Then, clearly, $\bar{R}(1-\bar{e}_i)$ is the left annihilator of $\bar{e}_i\bar{R}$ in $\bar{R}$ and so $\bar{R}(1-\bar{e}_1) > \bar{R}(1-\bar{e}_2)$ is a descending chain with $1-\bar{e}_i$ idempotents. We lift $1-\bar{e}_1$ to an idempotent f_1 in R as above.

Then $\bar{R}(1-\bar{e}_1) > \bar{R}(1-\bar{e}_2)$ gives $(1-\bar{e}_2) = (1-\bar{e}_2)(1-\bar{e}_1) = (1-\bar{e}_2)\bar{f}_1$. So replacing $1-e_2$ by $(1-e_2)f_1$ we may assume that $(1-e_2) \in Rf_1$.

Next we lift $1-e_2$ to an idempotent $f_2 \in R$. Then f_2 is a polynomial in $1-e_2$ and hence $f_2 \in Rf_1$ and so $Rf_2 < Rf_1$.

Therefore $(1-f_1)R < (1-f_2)R$ and $1-f_i$ is an idempotent in R which is congruent to e_i modulo I, for $i = 1, 2$.

Repeating this process gives the result when I is a nil ideal.

(ii) Now let I be any ideal in a locally artinian ring R, and $\bar{e}_1, \bar{e}_2$ idempotents in $\bar{R}$ such that $\bar{e}_1\bar{R} < \bar{e}_2\bar{R}$. Then e_1 and e_2 are contained in some artinian subring S of R and $I \cap S$ is an ideal in S.

Therefore in order to lift e_1 and e_2 we may suppose that $R = S$, i.e. that R is artinian.

First suppose that $I \cap J(R) = 0$.

Then
$$I = \frac{I}{I \cap J(R)} \cong \frac{I+J(R)}{J(R)} \quad \text{and} \quad \frac{I+J(R)}{J(R)} \oplus \frac{K}{J(R)} = \frac{R}{J(R)},$$

for some ideal K of R, since $R/J(R)$ is semisimple artinian.

Therefore $I+K = R$, and $I \cap K \leqslant I \cap J(R) = 0$. Hence $I \oplus K = R$. Now for $i = 1, 2$, $e_i = x_i + f_i$ where $x_i \in I$ and $f_i \in K$ and so $\bar{e}_i = \bar{f}_i$.

Also
$$\bar{x}_i + \bar{f}_i = \bar{e}_i = \bar{e}_i^2 = (\bar{x}_i + \bar{f}_i)^2 = \bar{x}_i + \bar{f}_i,$$

so
$$f_i - f_i^2 \in (x_i^2 - x_i) + I = I.$$

Hence $f_i - f_i^2 = 0$, since $I \cap K = 0$, and f_i is an idempotent which is congruent to e_i modulo I.

We claim that $f_1 R \leqslant f_2 R$, i.e. that $f_1 = f_2 f_1$. This is certainly true modulo I since $\bar{f}_1 - \bar{f}_2\bar{f}_1 = \bar{e}_1 - \bar{e}_2\bar{e}_1 = 0$.

Therefore $f_1 - f_2 f_1 \in I \cap K = 0$.

The inclusion is strict since $f_1 R = f_2 R$ gives $f_2 = f_1 f_2$ and $\bar{e}_2 = \bar{f}_2 = \bar{f}_1\bar{f}_2 = \bar{e}_1\bar{e}_2$, and so $\bar{e}_1\bar{R} = \bar{e}_2\bar{R}$.

In general, if $I \cap J(R) \neq 0$ we can lift $\bar{e}_1, \bar{e}_2$ to idempotents f_1, f_2 in $R/I \cap J(R)$ such that $f_1(R/I \cap J(R)) < f_2(R/I \cap J(R))$ but if R is artinian, $I \cap J(R)$ is nilpotent and part (i) allows us to lift f_1 and f_2 to idempotents in R.

Theorem 3·7. *Let R be a locally artinian ring and $W \subset V$ right R modules such that W is irreducible and V satisfies min-ann.*

Let $R_0 = \mathrm{Ann}_R W$ and $E = \mathrm{End}_R W$. Then W has finite dimension n, say as a (left) module over E and $\bar{R} = R/R_0 \cong M_n(E)$, the ring of $n \times n$ matrices over E.

In particular this occurs if V is the injective hull of W and is Σ-injective (by 3·1).

Proof. Let $R_1 = \mathrm{Ann}_R V$. Then by Lemma 3·3 R/R_1 has max. on right ideals generated by an idempotent.

Now R_0/R_1 is an ideal in the locally artinian ring R/R_1 and Lemma 3·6 enables to conclude that $\bar{R}$ also has max. on right ideals generated by an idempotent.

However, $\bar{R}$ is a locally artinian, primitive ring and hence by Lemma 3·4 any non-zero right ideal contains a non-zero idempotent. Lemma 3·5 applied to the right regular module then shows that $\bar{R}$ is artinian and hence simple artinian.

Finally W is a faithful irreducible module for $\bar{R}$ and so $\bar{R} \cong M_n(E)$, where

$$E = \mathrm{End}_R W \quad \text{and} \quad n = \dim_E W$$

by the Jacobson density theorem (6), (26·8), (12), 3·1, proposition 3.

We remark that if $R = kG$ is the group algebra of a locally finite group, and if V is any Σ-injective kG-module then $kG/\mathrm{Ann}_{kG}(V)$ is artinian (Theorem 4·5). We have been unable to decide whether any locally artinian ring with a faithful Σ-injective module is actually artinian.

4. Group algebras of locally finite groups

The first result of this section is really part of the folklore.

Lemma 4·1. *Suppose G is a finite p group and k a field of characteristic p. Then the injective hull of the trivial kG-module is isomorphic to the regular module.*

Proof. kG is a quasi-Frobenius algebra and hence is self-injective by (6), theorem 58·6. Also since 0 and 1 are the only idempotents kG is indecomposable and so contains a unique minimal submodule which must be the trivial module. It is readily seen that any element of kG which is invariant under all elements of G must be a scalar multiple of $\hat{G} = \sum_{g \in G} g$.

Lemma 4·2. *Let G be a locally finite group and k a field of characteristic $p > 0$. If kG has a non-zero injective module V satisfying min-ann. then G has no infinite p subgroups.*

Proof. If the result is false then G has a strictly ascending chain $G_1 < G_2 < \ldots$ of finite p subgroups.

Let ωkG_n denote the augmentation ideal of kG_n and set

$$V_n = \mathrm{Ann}_V(\omega kG_n).$$

Then clearly $$V_{n+1} \leqslant V_n.$$

Consider the restriction of V to the finite subgroup G_{n+1}. $V_{G_{n+1}}$ is a module for the finite dimensional algebra kG_{n+1} and so contains a non-zero finitely generated submodule.

Hence $V_{G_{n+1}}$ must contain an irreducible submodule which is the trivial module k since G_{n+1} is a p group.

Now $V_{G_{n+1}}$ is injective by Lemma 2·1 and so contains a copy of the injective hull of k which is isomorphic to kG_{n+1} by the previous lemma.

We identify kG_{n+1} with this submodule of $V_{G_{n+1}}$.

If
$$\hat{G}_n = \sum_{g \in G_n} g \quad \text{then} \quad \hat{G}_n\,\omega(kG_n) = 0,$$
but
$$\hat{G}_n(g-1) \neq 0 \quad \text{for any} \quad g \in G_{n+1} \backslash G_n.$$

This shows that $V_1 > V_2 > \ldots$ is a strictly descending chain of annihilators and this is a contradiction. Hence G has no infinite p subgroups.

LEMMA 4·3. *Let H be a finite group, k a field of characteristic $p > 0$. Then $kH/J(kH)$ is isomorphic to a direct sum of matrix rings over commutative fields.*

Proof. Let $\mathbb{F}_p$ denote the prime subfield of k. We have $\mathbb{F}_p H/J(\mathbb{F}_p H) \cong \bigoplus_i M_{n_i}(k_i)$ where the k_i are commutative fields by Wedderburn's theorem on finite division rings.

Now $k \otimes_{\mathbb{F}_p} J(\mathbb{F}_p H)$ is a nilpotent ideal in kH and so
$$k \otimes_{\mathbb{F}_p} J(\mathbb{F}_p H) \leqslant J(kH).$$

On the other hand, $\mathbb{F}_p H/J(\mathbb{F}_p H)$ is a separable $\mathbb{F}_p H$ algebra by (2), §7, no. 5, and hence $J(kH) \leqslant k \otimes_{\mathbb{F}_p} J(F_p H)$.

Therefore
$$\frac{kH}{J(kH)} = \frac{\mathbb{F}_p H \otimes_{\mathbb{F}_p} k}{J(\mathbb{F}_p H) \otimes_{\mathbb{F}_p} k} = \frac{\mathbb{F}_p H}{J(\mathbb{F}_p H)} \otimes_{\mathbb{F}_p} k$$
is a direct sum of matrix algebras over commutative fields.

LEMMA 4·4. *Let G be a locally finite group, V an irreducible kG-module which is finite dimensional as a module over $E = \mathrm{End}_R\, V$. Suppose that either*

(i) *k is a splitting field for all finite subgroups of G, or*

(ii) *characteristic $k = p > 0$.*

Then E is a field.

Proof. Case (i) is proved by Farkas and Snider (9). Case (ii): let $R = kG$, and
$$R_0 = \mathrm{Ann}_R\, V.$$

If $\dim V = n$, then $\bar{R} = R/R_0 \cong M_n(E)$ by the Jacobson density theorem (6), (26·8).

Therefore there is an idempotent $\bar{e}$ in $\bar{R}$ such that $\overline{eRe} \cong E$. By Lemma 3·6 we can lift $\bar{e}$ to an idempotent e in R.

Let $\bar{x}, \bar{y} \in \overline{eRe}$, $x = ere$, and $y = ese$ say, and choose a finite subgroup H of G containing the supports of e, r and s.

Now let e^* denote the image of e in $kH/J(kH)$, and set

$$A = \frac{ekHe}{ekHe \cap J(kH)} = \frac{ekHe + J(kH)}{J(kH)} \leqslant \frac{kH}{J(kH)}.$$

Then $A \cong e^*(kH/J(kH))e^*$, and $kH/J(kH)$ is a direct sum of matrix algebras over fields by Lemma 4·3. It follows that A is also a direct sum of matrix algebras over fields.

Now consider the combined map θ: $ekHe \hookrightarrow ekGe \to E$, given by the inclusion of H in G followed by the natural projection of R onto $\bar{R}$. Since E has no zero divisors and $ekHe \cap J(kH)$ is nilpotent we have $\theta(ekHe \cap J(kH)) = 0$.

Hence we have a map $\bar{\theta}: A \to E$ and by construction the elements x and y lie in the image of A. However, since A is a direct sum of matrix algebras over fields and E has no zero divisors, it follows that the image of A under $\bar{\theta}$ is a field. In particular, $\bar{x}$ and $\bar{y}$ commute, but $\bar{x}$ and $\bar{y}$ were arbitrary elements of E. Hence E is a field as claimed.

We now obtain the characterization of Σ-injective kG-modules promised in the introduction.

Theorem 4·5. *Let G be a locally finite group, k a field of* char $p \geqslant 0$. *If $p = 0$, we assume that k contains all roots of* 1.

If V is any kG-module the following are equivalent:

(1) *V is injective and satisfies* min-ann.

(2) *V is Σ-injective.*

(3) *$\exists H \trianglelefteq G$, such that H is a p' subgroup of $C_G(V)$ and G/H is a finite extension of an abelian p' group, and finitely many isomorphism types $W_1, W_2, \ldots, W_s$, of irreducible $k[G/H]$ modules such that V is isomorphic to a direct sum of the injective hulls of the $W_1, W_2, \ldots, W_s$. (Notice that by the results of Section 2 it is immaterial whether we form these injective hulls in the category of kG modules or of $k[G/H]$ modules.)*

Moreover, if (1)–(3) *hold then kG induces an artinian ring of transformations on V, that is $kG/\mathrm{Ann}_{kG} V$ is artinian.*

Proof. (1) and (2) are equivalent by 3·1.

(1) $\Rightarrow$ (3). Let $H = C_G(V)$. Then H is a normal p' subgroup of G by Lemma 2·2. Also V is a Σ-injective module for $k[G/H]$ and we may assume that $H = 1$ and then require to prove that G is a finite extension of an abelian p' group.

If $R = kG$, $R_0 = \mathrm{Ann}_{kG} V$, $\bar{R} = R/R_0$, then as in the proof of Theorem 3·7 we see that $S = \bar{R}/J(\bar{R})$ is semisimple artinian.

If U is an irreducible S module we can regard U as an irreducible kG module and since U is finite dimensional over $\mathrm{End}_S U$ and $\mathrm{End}_S U = \mathrm{End}_{kG} U$ we conclude by Lemma 4·4 that $\mathrm{End}_S U$ is a field.

Hence, by Wedderburn's theorem $S \cong \oplus M_{n_i}(k_i)$ a direct sum of matrix rings over fields. Let $U(\bar{R})$, $U(S)$ denote the groups of units of $\bar{R}$ and S respectively. Then as $C_G(V) = 1$, G embeds in $U(\bar{R})$ and we have a group homomorphism

$$G \hookrightarrow U(\bar{R}) \xrightarrow{\psi} U(S) \hookrightarrow \Pi GL(n_i, k_i).$$

Here ψ is obtained from the natural homomorphism of $\bar{R}$ onto S.

If char $k = p > 0$, then G contains no infinite p subgroups by Lemma 4·2, and so it follows from a theorem of Brauer and Feit (3) and (12), theorem 1.L.4, that the image of G in each factor $GL(n_i, k_i)$ is abelian-by-finite and hence so too is the image of G in $\Pi GL(n_i, k_i)$. Clearly the kernel of the homomorphism $G \to \Pi GL(n_i, k_i)$ is $G \cap \ker \psi$. Suppose that $\alpha \in \ker \psi$.

Then $\alpha = 1 - j$, for some $j \in J(\bar{R})$ and since $\bar{R}$ is locally artinian $J(\bar{R})$ is locally nilpotent and $\exists\ r$ such that $j^r = 0$.

Choose $s \geqslant 1$ such that $p^s \geqslant r$, then $j^{p^s} = 0$ and so $(1-j)^{p^s} = 1 - j^{p^s} = 1$. That is $\alpha^{p^s} = 1$ and the kernel is a p group P which must be finite.

Let A be a normal subgroup of finite index in G such that A/P is an abelian p' group and set $C = C_A(P)$. Then $|G: C| < \infty$ and C is nilpotent (of class two) and so

$$C = O_p(C) \times O_{p'}(C).$$

Again since G has no infinite p subgroups we have

$$|C: O_{p'}(C)| < \infty \quad \text{and so} \quad |G: O_{p'}(G)| < \infty.$$

Set $K = O_{p'}(G)$.

Then $P \cap K = 1$ and so $K = K/P \cap K \cong PK/P \leqslant G/P$ and K is abelian-by-finite. Hence G is abelian-by-finite and so is a finite extension of an abelian p' group.

We show next that $\bar{R}$ is artinian. Since K is a locally finite p' group kK is von Neumann regular and hence so is

$$T = kK/kK \cap R_0 \cong kK + R/R_0 \leqslant R/R_0 = \bar{R}.$$

Now $\bar{R}$ contains no infinite set of orthogonal idempotents by Lemma 3·3 and so by Lemma 3·5, T is semisimple artinian, but $\bar{R}$ is a finitely generated module over T (it can be generated by the image of a transversal to K in G) and therefore $\bar{R}$ is artinian.

A result of Cailleau (5) states that any Σ-injective module is a direct sum of indecomposable Σ-injective modules. This is easily seen in the present case where V is a Σ-injective module for the artinian ring $\bar{R}$, for the socle M of V is a direct sum of irreducible submodules and since V is injective $E(M) \leqslant V$. If the inclusion were strict then as $E(M)$ is injective it would have a complement N in V, but then since $\bar{R}$ is artinian N would have an irreducible submodule intersecting M trivially. This contradiction shows that V is a direct sum of injective hulls of irreducible $\bar{R}$ modules. Again since R is artinian only finitely many isomorphism types $W_1, W_2, \ldots, W_s$ can occur. Clearly each W_i is an irreducible module for $k[G]$.

(3) $\Rightarrow$ (2). Since H is a normal p' subgroup of G, any injective $k[G/H]$ module is injective when regarded as a $k[G]$ module with H acting trivially by Lemma 2·4. Hence we may assume that $H = 1$.

Therefore G is a finite extension of an abelian p' group K. Let W be any irreducible kG module. Then the restriction W_K is a direct sum of finitely many irreducible kK-modules by Clifford's theorem. Now any irreducible kK-module has countable dimension over k (see (10), p. 121) and so $\dim_k W$ is also countable. In addition any irreducible kK module is injective by the proof of (9), theorem 3, and hence W_K is injective since it is the direct sum of finitely many injective modules.

Again, since the index $|G: K|$ is finite the induced module $W_K \otimes_{kK} kG$ is an injective module containing W by Lemma 2·1 and has countable dimension over k. Hence $\dim_k E(W)$ is countable and by 3·2 and 3·1, $E(W)$ satisfies min-ann. and so is Σ-injective.

Clearly any direct sum of copies of $E(W)$ is Σ-injective, and if $W_1, W_2, \ldots, W_s$ are finitely many irreducible kG modules and V is any direct sum of their injective hulls, then V is Σ-injective. This completes the proof of Theorem 4·5.

Notice that if we have uncountably many isomorphic copies $\{W_i\}$ of an irreducible kG module, then $\oplus E(W_i)$ is a Σ-injective module with uncountable dimension over k, so the converse of Lemma 3·2 fails to hold.

We also record the following result.

Corollary 4·6. *If G is a locally finite group and k is a field of* char $p > 0$, *then kG has a Σ-injective module if and only if $|G: O_{p'}(G)| < \infty$.*

We now study the group algebra kG of a locally finite group, which has the property that the injective hull of every irreducible kG-module has countable dimension over k. We show that any such group is abelian-by-finite. This provides a generalization of (11), theorem A. The technique will be to reduce to linear groups.

Lemma 4·7. (i) *Suppose H is a subgroup of G and that there is an irreducible kH-module such that $H/C_H(W)$ is not abelian-by-finite. Then there is an irreducible kG-module V such that $G/C_G(V)$ is not abelian-by-finite.*

(ii) *Suppose G has a locally finite p' subgroup H of finite index and that G is not abelian-by-finite. Then there is an irreducible kG-module V such that $G/C_G(V)$ is not abelian-by-finite.*

Proof. (i) By (15), lemma 10·2 (i), there is an irreducible kG-module V such that W is a kH-submodule of V_H.

Hence $C_G(V) \cap H = C_H(V) \subset C_H(W)$.

Now $H/C_H(W)$ is not abelian-by-finite and so neither is

$$H/C_G(V) \cap H \cong H.C_G(V)/C_G(V).$$

Therefore $G/C_G(V)$ is not abelian-by-finite.

(ii) This follows from part (i) and (11), lemma 2·3.

Let k be a field of characteristic $p \geqslant 0$. For brevity we say that a locally finite group G is restricted over k, if every irreducible kG-module is finite dimensional over its endomorphism ring.

Theorem 4·8. *Let G be a locally finite group, k a field of characteristic $p \geqslant 0$. The injective hull of every irreducible kG-module has countable dimension over k if and only if G is abelian-by-finite and has no infinite p subgroups.*

Proof. We have shown in the proof of Theorem 4·5 that if G is a periodic abelian-by-finite group with no infinite p subgroups then the injective hull of every irreducible kG-module has countable dimension over k.

Now suppose that G is a locally finite group such that the injective hull of every irreducible kG-module has countable dimension over k. Then G is restricted over k by Theorem 3·7. If char $k = 0$, the result follows from (11), theorem B. If char $k = p > 0$, then by Lemmas 3·2 and 4·2, G has no infinite p subgroups and the result follows from the following theorem.

THEOREM 4·9. *Suppose* char $k = p > 0$ *and that* G *is a locally finite restricted group over* k *with no infinite* p *subgroups.*

Then G *is abelian-by-finite.*

Proof. Step 1. *If* G *has a* p' *subgroup of finite index then* G *is abelian-by-finite.*

Otherwise by Lemma 4·7 (ii) kG has an irreducible module V such that $G/C_G(V)$ is not abelian-by-finite.

But then if $E = \mathrm{End}_{kG} V$ and $n = \dim_E V$, $G/C_G(V)$ embeds in $GL(n, E)$ and E is a field by Lemma 4·4(ii). This is impossible by the Brauer–Feit theorem.

Step 2. *We may suppose that* $O_p(G) = 1$.

Notice that $G/O_p(G)$ is a restricted group with no infinite p subgroups. Suppose that we have shown $G/O_p(G)$ to be abelian-by-finite and let $A/O_p(G)$ be an abelian normal subgroup with finite index and $B/O_p(G) = O_{p'}(A/O_p(G))$.

Then $O_p(G)$ is a finite normal maximal p subgroup of B and so $C = C_B\,(O_p(G))$ has finite index in B. We claim that C has a p' subgroup Q of finite index. Then Q will have finite index in G and G will be abelian-by-finite by Step 1.

Set $P = C \cap O_p(G)$, then P is a central Sylow p subgroup of any finite subgroup S such that $P \leqslant S \leqslant C$ and therefore $S = P \times O_{p'}(S)$ for any such S by the Schur–Zassenhaus theorem. It follows that the p' elements of S form a subgroup and hence the p' elements of C also form a subgroup.

Therefore $C = P \times O_{p'}(C)$ and $O_{p'}(C)$ has finite index in C and so in G.

Step 3. *If* G *is residually finite and satisfies the conditions of the theorem, then* G *is abelian-by-finite.*

Let P be a maximal p subgroup of G. For each $x \in P$, $x \neq 1$, we choose a normal subgroup N_x with finite index in G such that $x \notin N_x$.

Set $N = \bigcap_{x \in P} N_x$, then N is a normal subgroup with finite index in G since P is finite and $P \cap N = 1$ by construction. Therefore since any two maximal p subgroups are conjugate by (12), 1.D.12, N is a p' subgroup of G and G is abelian-by-finite by Step 1.

Step 4. *Completion of the proof.*

Let G be any group satisfying the conditions of the theorem and suppose in addition that $O_p(G) = 1$.

If $g - 1 \in J(kG)$, then $(g-1)^{p^n} = 0$ for some n. Therefore $g^{p^n} = 1$ and so

$$G \cap 1 + J(kG) \leqslant O_p(G) = 1.$$

Hence there exist irreducible kG-modules $\{V_i \mid i \in I\}$ such that $\bigcap_{i \in I} C_G(V_i) = 1$.

Now each $G/C_G(V_i)$ is a linear group over a field and so is abelian-by-finite as in Step 1.

Let J_i be a normal subgroup with finite index in G such that $J_i/C_G(V_i)$ is abelian.

If $J = \bigcap_{i \in I} J_i$, then $J' \leqslant \bigcap_{i \in I} C_G(V_i) = 1$, so J is abelian.

Clearly G/J is residually finite and so abelian-by-finite by Step 3. Hence G is metabelian-by-finite. Let L be a metabelian subgroup of finite index in G, P a maximal p subgroup of L and Σ the local system consisting of all finite subgroups S of L containing P.

If $S \in \Sigma$, P is a Sylow p subgroup of S and, by Hall's theorem, S has a p' complement with index $|P|$.

Hence $|S: O_{p'}(S)| \leqslant |P|!$ for all $S \in \Sigma$, and an inverse limit argument such as (12), 1.K.2, shows that $|L: O_{p'}(L)| \leqslant |P|!$.

Therefore G has a p' subgroup of finite index and is abelian-by-finite by Step 1.

Remark. The above proof is substantially due to B. Hartley. Without the assumption that G has no infinite p subgroups he has shown that if G is a locally finite restricted group over a field of characteristic $p > 0$, then $G/O_p(G)$ is abelian-by-finite.

We also have the following variant of theorem 4·8.

THEOREM 4·10. *Let G be a countable locally finite group and k a field of characteristic $p \geqslant 0$. The injective hull of every irreducible kG-module has finite composition length if and only if G is abelian-by-finite and has no infinite p subgroups.*

Proof. G is a countable group and so any irreducible kG-module has countable dimension, since it is a difference module of kG. Hence any kG-module with a finite (or even countable) composition series has countable dimension over k. The result follows from Theorem 4·5 with minor modifications.

5. The injective hull of the regular module

In this section we depart from our previous notation by writing $E(G)$ for the injective hull of the right regular module kG_{kG}.

We apply the techniques of Section 3 to obtain a short proof of Lawrence's result (14), that a countable dimensional self-injective ring is quasi-Frobenius, and to show that if G is locally finite and $\dim_k E(G)$ is countable then G is finite.

For non-locally finite groups the situation may be rather complex – we have the following result.

PROPOSITION 5·1. *Let $C_\infty = < x|- >$ denote the infinite cyclic group. Then $\dim_k E(C_\infty)$ is countable if and only if k is a countable field.*

$E(C_\infty)$ is always Σ-injective.

Proof. Notice that kC_∞ is an Ore domain whose quotient ring Q may be identified with the ring of rational functions $k(x)$ in the variables x and x^{-1}. Hence $Q \cong E(C_\infty)$ by (13), §4·6, proposition 2, and §4·3, proposition 3 and $E(C_\infty)$ is Σ-injective by (7), corollary 8·4.

Now if k is a countable field the kC_∞ and Q are countable rings and so $\dim_k E(C_\infty)$ is countable.

On the other hand, if k is uncountable it is easily checked that the elements $1/(x+a)$ as a ranges over k, are linearly independent.

THEOREM 5·2. *If R is a k algebra with a countable dimensional, faithful injective right module V, then R has the maximum condition on right annihilator ideals.*

Proof. By Lemma 3·2, V satisfies min-ann. and so by Lemma 3·3, R satisfies max. on right annihilator ideals.

By a result of Faith (7), theorem 5·2, a right self-injective ring satisfying max. on right annihilator ideals is quasi-Frobenius. Hence we have the following corollary.

COROLLARY 5·3 (Lawrence (14)). *A countable dimensional right self-injective ring is quasi-Frobenius.*

For group algebras we have a slightly stronger result.

THEOREM 5·4. *Suppose kG has a faithful, injective module with countable dimension over k. Then G has no infinite locally finite subgroups.*

Proof. By Lemmas 3·2 and 3·3 kG has max. on right ideals generated by an idempotent.

If G has an infinite locally finite subgroup then by (12), corollary 2·5, G has an infinite locally finite abelian subgroup A. Since by Lemma 4·2, G has no infinite locally finite p subgroups, it follows that $H = O_{p'}(A)$ is an infinite locally finite p' subgroup of G.

Therefore H has a strictly ascending chain $H_1 < H_2 < \ldots$ of finite p' subgroups of G. Write

$$\hat{H}_i = \sum_{h \in H_i} h, \quad e_i = \hat{H}_i/|H_i|.$$

Then $(1-e_1)kG < (1-e_2)kG < \ldots$ is a strictly ascending chain of right ideals generated by idempotents. This contradiction establishes the result.

COROLLARY 5·5 (Renault (17)). *If kG is self-injective, then G is finite.*

Proof. G is easily seen to be locally finite as in (16), lemmas 3.2.5 and 3.2.7.

If kG is self-injective then so is kH whenever H is a subgroup of G, since kG_{kH} is injective and free as a kH module, and so kH_{kH} is a direct summand of an injective module and hence injective.

Therefore by Corollary 5·3 or Theorem 5·4 every countable subgroup of G is finite. If G is not finite, let S be a countably infinite subset G, then $\langle S \rangle$ is a countable subgroup of G and so finite. This is a contradiction. Hence G is finite.

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