Combinatorics of Character Formulas for the Lie Superalgebra $\mathfrak{gl}(m,n)$.

Ian M. Musson
Department of Mathematical Sciences
University of Wisconsin–Milwaukee
and
Vera V. Serganova
Department of Mathematics
University of California–Berkeley.

August 10, 2009

**Abstract**

Let $\mathfrak{g}$ be the Lie superalgebra $\mathfrak{gl}(m,n)$. Algorithms for computing the composition factors and multiplicities of Kac modules for $\mathfrak{g}$ were given by the second author, [Ser96] and by J. Brundan [Bru03].

We give a combinatorial proof of the equivalence between the two algorithms. The proof uses weight and cap diagrams introduced by Brundan and C. Stroppel, and cancelations between paths in a graph $\mathcal{G}$ defined using these diagrams. Each vertex of $\mathcal{G}$ corresponds to a highest weight of a finite dimensional simple module, and each edge is weighted by a nonnegative integer. If $\mathcal{E}$ is the subgraph of $\mathcal{G}$ obtained by deleting all edges of positive weight, then $\mathcal{E}$ is the graph that describes non-split extensions between simple highest weight modules.

We also give a procedure for finding the composition factors of any Kac module, without cancelation. This procedure leads to a second proof of the main result.

# 1 Introduction.

The problem of finding the characters of the finite dimensional simple modules for the complex Lie superalgebra $\mathfrak{g} = \mathfrak{gl}(m,n)$ was first posed by V. Kac in 1977, [Kac77]. Let $X^+(m,n)$ denote the set of dominant integral weights for $\mathfrak{g}$. In [Kac78] Kac introduced a certain finite dimensional highest weight module $K(\lambda)$, now known as a Kac module, with highest weight $\lambda \in X^+(m,n)$, whose character is given by an analog of the Weyl character formula. Furthermore any composition factor of $K(\lambda)$ is a simple module $L(\mu)$ with highest weight $\mu \in X^+(m,n)$, and the multiplicities of the composition factors of Kac modules can be expressed using an upper triangular matrix with diagonal entries equal to 1. Therefore the determination of this multiplicity matrix leads to a solution of the problem raised by Kac.

Combinatorial formulas for the multiplicity of $L(\lambda)$ as a composition factor of $K(\mu)$ were given in [Ser96] and [Bru03], using completely different methods. We give a combinatorial proof of the equivalence between these two formulas, Theorem A. Let $F$ be the set of all functions from $\mathbb{Z}$ to the set $\{\times, \circ, <, >\}$ such that $f(a) = \circ$ for all but finitely many $a \in \mathbb{Z}$. Let $\mathbb{Z}F$ be the free abelian group with basis $F$. The idea of the proof is to express the formula from [Ser96] as a signed sum of terms in $\mathbb{Z}F$. The terms from this sum correspond to paths in a certain graph $\mathcal{G}$ . We define an involution on the paths occurring in this sum such that paths that are paired by the involution have opposite signs. After these terms are canceled, what remains is the formula from [Bru03] in a form communicated to the first author by Brundan. This reformulation of the result from [Bru03] uses diagrams called weight and cap diagrams that originate in the work of Brundan and Stroppel on Khovanov's diagram algebra, [BS08a], [BS08b], [BS08c], [BS08d]. We remark that our notation for these diagrams is different from theirs. We note also that character formulas for the irreducible representations of the orthosymplectic Lie superalgebras were announced in [Ser98a]. These results are expressed in terms of weight diagrams and proved in [GS09].

Since the category of finite dimensional $\mathbb{Z}_2$-graded weight modules $\mathcal{F}$ is not semisimple, an important problem in representation theory is to determine the non trivial extensions between simple modules. This problem is related to the graph $\mathcal{G}$ as follows. Each vertex $f$ of $\mathcal{G}$ corresponds to a highest weight of a finite dimensional simple module $L(f)$, and each edge of $\mathcal{G}$ is weighted by a nonnegative integer. In Theorem B we show that if $\mathcal{E}$ is the subgraph of $\mathcal{G}$ obtained by deleting all edges of positive weight, then $Ext^1_{\mathcal{F}}(L(f), L(g)) \neq 0$ if and only if $f \longrightarrow g$ or $g \longrightarrow f$ is an edge of $\mathcal{E}$.

This paper is organized as follows. In the next section we give a formal statement of the main combinatorial result, Theorem A. Some work is necessary to derive the equivalence of the character formulae from the combinatorial statement, and this is done in Section 3. The graph $\mathcal{G}$ is introduced and Theorem A is proved in Section 4. In Section 5 we outline a procedure for finding the composition factors of any Kac module, without cancelation. This procedure leads to a second proof of the main result. Theorem B relating extensions to the subgraph $\mathcal{E}$ is presented in Section 6.

The authors would like to express their gratitude to Jon Brundan for sharing his ideas with them.

## 2 A Combinatorial Formula.

Let $F$ be the set of all functions from $\mathbb{Z}$ to the set $\{\times, \circ, <, >\}$ such that $f(a) = \circ$ for all but finitely many $a \in \mathbb{Z}$. For $f \in F$ we set $\#f = |f^{-1}(\times)|$, and

$$core_L(f) = f^{-1}(<), \quad core_R(f) = f^{-1}(>).$$

We call $\#f$ the *degree of atypicality* of $f$, and define the the *core* of $f$ to be

$$core(f) = (core_L(f), core_R(f)).$$

Let $\mathbb{Z}F$ be the free abelian group with basis $F$. Our main result is an identity for certain $\mathbb{Z}$-linear operators on $\mathbb{Z}F$. If $f \in F$ we define the *weight diagram* $D_{wt}(f)$ to be a number line with the symbol $f(a)$ drawn at each $a \in \mathbb{Z}$. Next let $C_L$ and $C_R$ be disjoint finite subsets of $\mathbb{Z}$ and consider a number line with symbols $<$ (resp. $>$) located at all $a \in C_L$ (resp. $a \in C_R$). A *cap* $C$ is the upper half of a circle joining two integers $a$ and $b$ which are not in $C_L \cup C_R$. If $b < a$ we say that $C$ *begins* at $b$ and *ends* at $a$ and we write $b(C) = b$, and $e(C) = a$. A finite set of caps, together with the symbols $<$ , $>$ located as above, is called a *cap diagram* if no two caps intersect, and the only integers inside the caps 9which are not ends of some other caps) are located at points in $C_L \cup C_R$.

If $D$ is a cap diagram there is a unique $f \in F$ such that $core(f) = (C_L, C_R)$, and for $a \notin C_L \cup C_R$,

$$\begin{aligned} f(a) &= \times \text{ if there is a cap in } D \text{ beginning at } a, \\ f(a) &= \circ \text{ otherwise.} \end{aligned}$$

We write $D = D_{cap}(f)$ in this situation. If $D = D_{cap}(f)$ or $D = D_{wt}(f)$ we set $core(D) = core(f)$.

We say that a weight diagram and a cap diagram *match* if they have the same core and, when superimposed on the same number line, each cap connects a $\times$ to a $\circ$. For $f \in F$, set

$$P(f) = \{g \in F | D_{cap}(g) \text{ matches } D_{wt}(f)\}. \tag{2.1}$$

Brundan's formula for the composition factors of a Kac module can be written in terms of matching cap and weight diagrams. We now turn to the combinatorics necessary to express the formula from [Ser96]. If $f \in F$ and $\#f = k$ we set

$$\times(f) = (a_1, a_2, \ldots, a_k) \tag{2.2}$$

if $f^{-1}(\times) = \{a_1, a_2, \ldots, a_k\}$ with $a_1 > a_2 > \ldots > a_k$.

Next suppose $f$ satisfies (2.2), and that $f(a) = \times$, and $f(b) = \circ$. Informally, we define $f_b \in F$, (resp. $f^a \in F$) by adding $b$ to $\times(f)$ (resp. deleting $a$ from $\times(f)$). Precisely $f_b$ and $f^a$ have the same core as $f$, and satisfy

$$\times(f_b) = (a_1, \ldots, a_j, b, a_{j+1}, \ldots, a_k),$$

$$\times(f^a) = (a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_k),$$

where $a = a_i$, and $a_j > b > a_{j+1}$. Here it is convenient to set $a_0 = \infty$, and $a_{k+1} = -\infty$. We also set $f^a_b = (f_b)^a = (f^a)_b$.

For $f \in F$, and $a, b \in \mathbb{Z}$ with $b < a$, let $l_f(b, a)$ be the number of occurrences of the symbol $\times$ minus the number of occurrences of $\circ$ strictly between $b$ and $a$ in the weight diagram of $f$. We say that $g \in F$ is obtained from $f$ by a *legal move* if $g = f^a_b$ for some $b < a$ such that $f(a) = \times$, and $l_f(b, c) > 0$ for all $c$ with $b < c < a$. We call $a$ the *start*, $b$ the *end* and $l_f(b, a)$ the *weight* of the legal move.

There is another way to think about legal moves. Suppose that $b < a$, $f(b) = \circ$ and $f(a) = \times$. Keep a tally starting at $b$ with a tally of one, and move to the right along the number line adding one to the tally every time a $\times$ is passed, and subtracting one every time a $\circ$ is passed in the weight diagram $D_{wt}(f)$ . Then $f_b^a$ is obtained from $f$ by a legal move if and only if the tally remains positive at all integers to the left of $a$. If this is the case the weight of the legal move is the value of the tally just before we arrive at $a$.

Next if $1 \le i \le k$ we define operators $\sigma_i : \mathbb{Z}F \longrightarrow \mathbb{Z}F$ as follows. If $f$ satisfies (2.2) and $a = a_i$, then

$$\sigma_i(f) = \sum_b (-1)^{l_f(b,a)} f_b^a \ , \tag{2.3}$$

where the sum is over all $b$ such that $f_b^a$ is obtained from $f$ by a legal move.

Now we can state our first main result.

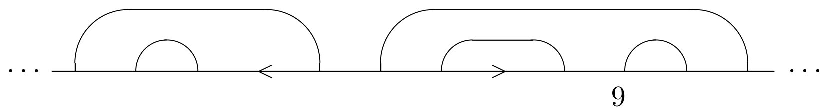
**Theorem A.** For $f \in F$ with $\#f = k$ we have

$$(1 + \sigma_1) \dots (1 + \sigma_k) f = \sum_{g \in P(f)} g. \tag{2.4}$$

**Example 2.1.** Suppose that $core(f) = (\{3\}, \{7\})$ and

$$\times(f) = (0, 1, 5, 6, 9),$$

then $D_{cap}(f)$ is pictured below. Note that $k = 5$ and $a_k = 9$.



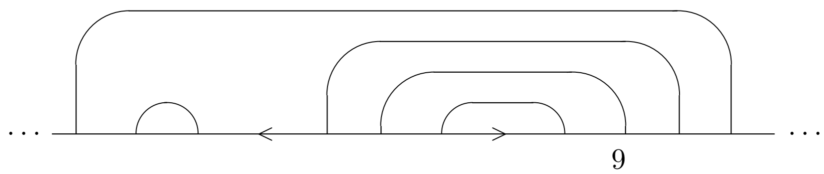

Now $f_b^9$ is obtained from $f$ by a legal move if and only if $b = -1, 4$ or 8. The weights of these legal moves are 1, 1 and 0 respectively. Thus Equation (2.3) becomes

$$\sigma_5(f) = f_8^9 - f_4^9 - f_{-1}^9.$$

Replacing 9 by $-1, 4, 8$ in $\times(f)$ we obtain

$$\times(f_{-1}^9) = (-1, 0, 1, 5, 6),$$
$$\times(f_4^9) = (0, 1, 4, 5, 6),$$
$$\times(f_8^9) = (0, 1, 5, 6, 8).$$

The cap diagram $D_{cap}(f_4^9)$ is given below.



**Remark 2.2.** The real content of Theorem A is the core-free case. Indeed since $\sigma_i f$ is a linear combination of terms $h \in F$ with the same core as $f$, and any $g \in P(f)$, has the same core as $f$, we immediately reduce to this case. The symbols $<$ and $>$ are important in the application to the Lie superalgebra $\mathfrak{gl}(m,n)$. When $f$ is core-free then $f$ is completely determined by $\times(f)$.

**Example 2.3.** An interesting case arises when $\times(f) = (2, 4, 6, \ldots, 2k-2)$. Let $A_k$ be the set of cap diagrams with $k$ caps each of which begins and ends at points in the set $\{0, 1, 2, \ldots, 2k-1\}$, and let $B_k$ be the set of cap diagrams that match the weight diagram $D_{wt}(f)$. Given a diagram in $A_k$, we obtain a diagram in $B_k$ by deleting the cap beginning at 0. This gives a bijection from $A_k$ to $B_k$. The cardinality of $A_k$ is the $kth$ Catalan number $C_k = \frac{1}{k+1}\left(\begin{array}{c} 2k \\ k \end{array}\right)$ see [Sta99] Exercise 6.19 o. In terms of representation theory this means that if $\rho$ is defined as in (3.4), then the length of a composition series for the Kac module $K(\rho)$ for $\mathfrak{gl}(k,k)$ equals $C_k$. There are further examples in [Su06] where the number of composition factors of a Kac module is a Catalan number. We conjecture that if $g$ is core-free and $|g^{-1}(\times)| = k$, then $P(g) \leq C_k$ with equality if and only if $\times(g)$ is obtained from $\times(f)$ by adding the same integer to each entry.

# 3 Character Formulas.

In this subsection $\mathfrak{g}$ will be the Lie superalgebra $\mathfrak{gl}(m,n)$ and $\mathfrak{h}$ and $\mathfrak{b}$ the Cartan and Borel subalgebras, consisting of diagonal and upper triangular matrices respectively. Let $\epsilon_i$, $\delta_j$ be the linear functionals on $\mathfrak{h}$ whose value on the diagonal matrix

$$a = diag(a_1, \ldots, a_{m+n})$$

is given by

$$\epsilon_i(a) = a_i, \quad \delta_j(a) = a_{m+j} \;\; 1 \leq i \leq m, \; 1 \leq j \leq n. \tag{3.1}$$

We define a bilinear form $( \; , \; )$ on $\mathfrak{h}^*$ by

$$(\epsilon_i, \epsilon_j) = \delta_{i,j} = -(\delta_i, \delta_j). \tag{3.2}$$

Let $X = X(m|n)$ denote the lattice of integral weights spanned by the $\epsilon_i$ and $\delta_i$. Also set

$$\Delta_0^+ = \{\epsilon_i - \epsilon_j; \delta_i - \delta_j\}_{i<j}, \; \Delta_1^+ = \{\epsilon_i - \delta_j\} \text{ and } \Delta^+ = \Delta_0^+ \cup \Delta_1^+. \tag{3.3}$$

Then the $\Delta^+$ is the set of roots of $\mathfrak{b}$. Next let

$$\rho = m\epsilon_1 + \cdots + 2\epsilon_{m-1} + \epsilon_m - \delta_1 - 2\delta_2 - \cdots - n\delta_n. \tag{3.4}$$

A weight $\lambda \in X$ is *regular* if $(\lambda + \rho, \epsilon_i - \epsilon_j) \neq 0$, and $(\lambda + \rho, \delta_i - \delta_j) \neq 0$ if $i \neq j$. Let $X_{reg}$ be the subset of $X$ consisting of regular weights, and let $S_m$ be the symmetric group of degree $m$. The Weyl group $W = S_m \times S_n$ acts on $\mathfrak{h}^*$ by permuting the $\epsilon_i$

and $\delta_i$. The dot action of $W$ is defined by $w \cdot \lambda = w(\lambda + \rho) - \rho$. We will identify $\lambda \in X(m|n)$ with the $m+n$ tuple of integers

$$(a_1, a_2, \cdots, a_m | b_1, b_2, \cdots, b_n) \tag{3.5}$$

where

$$a_1 = (\lambda + \rho, \epsilon_1), \cdots, a_m = (\lambda + \rho, \epsilon_m), \quad b_1 = (\lambda + \rho, \delta_1), \cdots, b_n = (\lambda + \rho, \delta_n). \tag{3.6}$$

Let $X^+ = X^+(m|n)$ denote the set of $\lambda = (a_1, a_2, \cdots, a_m | b_1, b_2, \cdots, b_n)$ in $X(m|n)$ with

$$a_1 > a_2 > \cdots > a_m, \quad b_1 < b_2 < \cdots < b_n. \tag{3.7}$$

In this $\rho$-shifted notation, the dot action of $W$ is represented by permutations of the entries in $\lambda$. If $\lambda \in X_{reg}$ there is a unique element $w$ of the Weyl group $W$ such that $w \cdot \lambda \in X^+$.

Given $f \in F$, denote by $\lambda = \lambda(f)$ the element of $X^+$ such that when written in the form (3.5), the entries of on the left (resp. right) side of $\lambda$ coincide with $core_L(f) \cup \times(f)$ (resp. $core_R(f) \cup \times(f)$) arranged in order as in (3.7). This defines a bijection $\lambda \longrightarrow f_\lambda$ from $X^+$ to $F$ whose inverse we write as $f \longrightarrow \lambda(f)$. Let $\mathcal{F}$ be the category of finite dimensional $\mathfrak{g}$ modules which are weight modules for $\mathfrak{h}$, and for $\lambda \in X^+$, let $K(\lambda)$ (resp. $L(\lambda)$) be the Kac module (resp. simple module) with highest weight $\lambda$. The map $f \longrightarrow L(\lambda(f))$ extends to an isomorphism from $\mathbb{Z}F$ to the Grothendieck group of $\mathcal{F}$, and we often identify these two groups. For $\lambda \in X^+$, we write $D_{wt}(\lambda)$, $D_{cap}(\lambda)$, $\#\lambda$ and $\times(\lambda)$ in place of $D_{wt}(f_\lambda)$, $D_{cap}(f_\lambda)$, $\#(f_\lambda)$ and $\times(f_\lambda)$ respectively, and set

$$\mathbf{P}(\mu) = \{\lambda | D_{cap}(\lambda) \text{ matches } D_{wt}(\mu)\}. \tag{3.8}$$

Then $\lambda \longrightarrow f_\lambda$ defines a bijection from $\mathbf{P}(\mu)$ to $P(f_\mu)$. Suppose that $\lambda \in X^+$ and

$$\times(\lambda) = (c_k, \ldots, c_1)$$

with $c_k < \ldots < c_2 < c_1$. If $\#\lambda = k$, this means that there are subsets

$$\{i_1 < \ldots < i_k\} \subseteq \{1, \ldots, m\}, \quad \{j_1 > \ldots > j_k\} \subseteq \{1, \ldots, n\}$$

such that

$$(\lambda + \rho, \epsilon_{i_p}) = (\lambda + \rho, \delta_{j_p}) = c_p,$$

for $1 \le p \le k$, and we set $\alpha_p = \epsilon_{i_p} - \delta_{j_p}$. Now suppose that the cap in $D_{cap}(\lambda)$ beginning at $c_p$ ends at $d_p = c_p + r'_p$. Next let $(r_1, \ldots, r_k)$ be the lexicographically smallest tuple of strictly positive integers such that for all $\theta = (\theta_1, \ldots, \theta_k) \in \{0,1\}^k$,

$$\mathtt{S}_\theta(\lambda) = \lambda + \sum_{p=1}^{k} \theta_p r_p \alpha_p \in X_{reg},$$

and let $\mathtt{R}_\theta(\lambda)$ denote the unique element of $X^+(m|n)$ which is conjugate under the dot action of $W$ to $\mathtt{S}_\theta(\lambda)$.

**Lemma 3.1.** *We have*

(a) $r_p = r'_p$ *for* $1 \le p \le k$.

(b) $D_{wt}(\mathtt{R}_\theta(\lambda))$ *is obtained from* $D_{wt}(\lambda)$ *by interchanging the* $\times$ *and* $\circ$ *located at* $c_p$ *and* $d_p$ *respectively for all* $p$ *such that* $\theta_p = 1$, *and leaving all other symbols unchanged.*

*Proof.* Clearly

$$(\mathtt{S}_\theta(\lambda) + \rho, \epsilon_{i_q}) = (\mathtt{S}_\theta(\lambda) + \rho, \delta_{i_q}) = c_q + r_q. \tag{3.9}$$

Assume by induction that $r_q = r'_q$ for $1 \le q \le p-1$, and set

$$Y_p = \{c_1, \dots, c_{p-1}, d_1, \dots, d_{p-1}\} \cup f_\lambda^{-1}(<) \cup f_\lambda^{-1}(>).$$

From the definition of the cap diagram $D_{cap}(\lambda)$ it follows that

$$r'_p = min\{r | r > 0, c_p + r \notin Y_p\}.$$

Using this and (3.9) we conclude that $r_p = r'_p$. This proves (a), and (b) follows since when weights are written as in equation (3.5), the dot action of $W$ is implemented by permuting the entries. □

**Corollary 3.2.** *Let* $\mathtt{R}_\theta(\lambda)$ *denote the unique element of* $X^+(m|n)$ *which is conjugate under the dot action of* $W$ *to* $\lambda + \sum_{p=1}^k \theta_p r_p \alpha_p \in X_{reg}$. *Then*

(a) $\{\mathtt{R}_\theta(\lambda) | \theta \in \{0,1\}^k\} = \{\mu \in X^+ | D_{cap}(\lambda)$ *matches the cap diagram* $D_{wt}(\mu)\}$.

(b) $\mu = \mathtt{R}_\theta(\lambda)$ *for some* $\theta \in \{0,1\}^k$ *if and only if* $\lambda \in \mathbf{P}(\mu)$.

*Proof.* This follows at once from the Lemma and equation (3.8). □

The following reformulation of the main theorem in [Bru03] was shown to the first author by Jon Brundan.

**Theorem 3.3.** *In the Grothendieck group of the category* $\mathcal{F}$ *we have*

$$K(\mu) = \sum_{\lambda \in \mathbf{P}(\mu)} L(\lambda).$$

*Proof.* The Main Theorem in [Bru03] states that for each $\mu \in X^+(m|n)$,

$$[K(\mu) : L(\lambda)] = \begin{cases} 1 & \text{if } \mu = \mathtt{R}_\theta(\lambda) \text{ for some } \theta = (\theta_1, \dots, \theta_k) \in \{0,1\}^k, \\ 0 & \text{otherwise.} \end{cases}$$

The result now follows immediately by Corollary 3.2. □

We now state the main result of [Ser96] in terms of diagrams.

**Theorem 3.4.** *If*

$$(1+\sigma_1)\dots(1+\sigma_k)f_\lambda = \sum_\mu c_{\lambda,\mu} f_\mu,$$

*then in the Grothendieck group of the category $\mathcal{F}$ we have*

$$K(\lambda) = \sum_\mu c_{\lambda,\mu} L(\mu).$$

In the rest of this section we explain how to deduce Theorem 3.4, and some further results that we will require in Section 6, from results in [Ser96]. An equivalence of categories allows us to focus our attention on the category $\mathcal{F}^k$ of all finite dimensional modules which have the degree of atypicality $k$, see [Ser98b] or [GS09]. From now on we will make this assumption. Let $F^k$ be the set of core-free $f \in F$ such that $\#f = k$. As before we may identify $\mathbb{Z}F^k$ with the Grothendieck group of $\mathcal{F}^k$.

Let $\Delta$ be the set of roots of $\mathfrak{g}$, and for any $\alpha \in \Delta$ denote by $\mathfrak{g}^\alpha \subset \mathfrak{g}$ the corresponding root space. Let $\gamma \in \mathfrak{h}^*$ be a weight such that $(\alpha, \gamma) \geq 0$ for all positive roots $\alpha$. Set

$$\Delta_\gamma = \{\alpha \in \Delta | (\alpha, \gamma) \geq 0\}.$$

We say that $\gamma$ *defines the parabolic subalgebra* $\mathfrak{q} \subseteq \mathfrak{g}$ where

$$\mathfrak{q} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta_\gamma} \mathfrak{g}^\alpha.$$

Note that $\mathfrak{b} \subseteq \mathfrak{q}$.

Let $\mathfrak{l}$ be the ad-$\mathfrak{h}$ stable Levi subalgebra of $\mathfrak{q}$ and note that $\mathfrak{l}$ has a $\mathbb{Z}$-grading $\mathfrak{l} = \mathfrak{l}_{-1} \oplus \mathfrak{l}_0 \oplus \mathfrak{l}_1$, similar to the $\mathbb{Z}$-grading of $\mathfrak{g}$. Every $\mathfrak{l}$-module can be made into a $\mathfrak{q}$-module with trivial action of the nilpotent radical of $\mathfrak{q}$. In particular, $\mathbb{Z}$-grading on $\mathfrak{l}$ allows us to construct a Kac module $K_\mathfrak{q}(\lambda)$ for $\mathfrak{q}$ and we denote the unique simple factor module of $K_\mathfrak{q}(\lambda)$ by $L_\mathfrak{q}(\lambda)$. Note that one can write

$$\mathrm{ch}\, L_\mathfrak{q}(\lambda) = \sum_{\mu \leq \lambda} a_\mathfrak{q}(\lambda, \mu) \mathrm{ch}\, K_\mathfrak{q}(\mu), \tag{3.10}$$

and we denote the matrix with coefficients $a_\mathfrak{q}(\lambda, \mu)$ by $A_\mathfrak{q}$.

Now let $\mathfrak{q} \subset \mathfrak{p}$ be a pair of parabolic subalgebras, and $V$ be a finite-dimensional $\mathfrak{q}$-module. Let $\Gamma_{\mathfrak{p},\mathfrak{q}}(V)$ be the maximal finite-dimensional quotient of the induced module $U(\mathfrak{p}) \otimes_{U(\mathfrak{q})} V$. Then clearly $\Gamma_{\mathfrak{p},\mathfrak{q}}$ is a functor from the category of finite-dimensional $\mathfrak{q}$-modules to the category of finite-dimensional $\mathfrak{p}$-modules and this functor is exact on the right. In general, the functor $\Gamma_{\mathfrak{p},\mathfrak{q}}$ is not exact, but it was proven in [Ser96] that $\Gamma_{\mathfrak{p},\mathfrak{q}}$ is exact on $\mathfrak{q}$-modules free over $U(\mathfrak{l}_{-1})$. It is not hard to see that any $\mathfrak{q}$-module free over $U(\mathfrak{l}_{-1})$ has a filtration with quotients isomorphic to Kac modules $K_\mathfrak{q}(\mu)$. (By definition, $K_\mathfrak{q}(\mu) = K_\mathfrak{l}(\mu)$ with trivial action of the nilpotent radical of $\mathfrak{q}$). Moreover,

$$\Gamma_{\mathfrak{p},\mathfrak{q}} K_\mathfrak{q}(\mu) = K_\mathfrak{p}(\mu). \tag{3.11}$$

We construct derived functors $\Gamma^i_{\mathfrak{p},\mathfrak{q}}(V)$, of $\Gamma_{\mathfrak{p},\mathfrak{q}}$ as follows. Take a resolution a resolution

$$\cdots \to M^1 \to M^0 \to 0$$

of $V$ by $\mathfrak{q}$-modules free over $U(\mathfrak{l}_{-1})$, and define $\Gamma^i_{\mathfrak{p},\mathfrak{q}}(V)$ to be the $i^{th}$ cohomology group of the complex

$$\cdots \to \Gamma_{\mathfrak{p},\mathfrak{q}}(M^1) \to \Gamma_{\mathfrak{p},\mathfrak{q}}(M^0) \to 0.$$

The result does not depend on a choice of resolution since $\Gamma_{\mathfrak{p},\mathfrak{q}}$ is exact on $\mathfrak{q}$-modules which are free over $U(\mathfrak{l}_{-1})$. Clearly, we have a natural homomorphism of $\mathfrak{p}$-modules $\gamma : \Gamma^0_{\mathfrak{p},\mathfrak{q}}(L_{\mathfrak{q}}(\lambda)) \to L_{\mathfrak{p}}(\lambda)$. Define

$$U^i_{\mathfrak{p},\mathfrak{q}}(\lambda) = \Gamma^{i-1}_{\mathfrak{p},\mathfrak{q}}(L_{\mathfrak{q}}(\lambda))$$

for $i > 1$ and

$$U^1_{\mathfrak{p},\mathfrak{q}}(\lambda) = \operatorname{Ker}\gamma.$$

Put

$$U^i_{\mathfrak{p},\mathfrak{q}}(\lambda,\mu) = [U^i_{\mathfrak{p},\mathfrak{q}}(\lambda) : L_{\mathfrak{p}}(\mu)].$$

Take a resolution $M^\bullet$ of $L_{\mathfrak{q}}(\lambda)$ such that $M^0 = K_{\mathfrak{q}}(\lambda)$, and for $i > 0$, $M^i$ is free over $U(\mathfrak{l}_{-1})$ and has all weights strictly less than $\lambda$. Then we have

$$U^i_{\mathfrak{p},\mathfrak{q}}(\lambda,\mu) \neq 0 \text{ implies } \lambda > \mu. \tag{3.12}$$

Clearly, we have

$$\operatorname{ch} L_{\mathfrak{q}}(\lambda) = \sum_{i\geq 0}(-1)^i \operatorname{ch} M^i_\lambda, \tag{3.13}$$

and

$$\operatorname{ch} L_{\mathfrak{p}}(\lambda) - \sum_{i\geq 1}(-1)^i \operatorname{ch} U^i_{\mathfrak{p},\mathfrak{q}}(\lambda) = \sum_{i\geq 0}(-1)^i \operatorname{ch} \Gamma_{\mathfrak{p},\mathfrak{q}}(M^i_\lambda). \tag{3.14}$$

Combine (3.10) and (3.13), and then apply $\Gamma_{\mathfrak{p},\mathfrak{q}}$, using (3.11) to obtain

$$\sum_{i\geq 0}(-1)^i \operatorname{ch} \Gamma_{\mathfrak{p},\mathfrak{q}}(M^i_\lambda) = \sum_{\mu\leq\lambda} a_{\mathfrak{q}}(\lambda,\mu) \operatorname{ch} K_{\mathfrak{p}}(\mu).$$

From this and (3.14) we deduce the following important identity

$$\operatorname{ch} L_{\mathfrak{p}}(\lambda) - \sum_{\nu,i}(-1)^i U^i_{\mathfrak{p},\mathfrak{q}}(\lambda,\nu) \operatorname{ch} L_{\mathfrak{p}}(\nu) = \sum_{\mu\leq\lambda} a_{\mathfrak{q}}(\lambda,\mu) \operatorname{ch} K_{\mathfrak{p}}(\mu). \tag{3.15}$$

Set $U_{\mathfrak{p},\mathfrak{q}}(\lambda,\mu) = \sum_{i\geq 1}(-1)^i U^i_{\mathfrak{p},\mathfrak{q}}(\lambda,\mu)$. Let $U_{\mathfrak{p},\mathfrak{q}}$ be the matrix with coefficients $U_{\mathfrak{p},\mathfrak{q}}(\lambda,\mu)$. Then using (3.10), the identity (3.15) can be rewritten in the form

$$(1 - U_{\mathfrak{p},\mathfrak{q}})A_{\mathfrak{p}} = A_{\mathfrak{q}}. \tag{3.16}$$

For $1 \leq s \leq k$ let

$$\gamma_s = s\varepsilon_1 + \cdots + \varepsilon_s + s\delta_k + \cdots + \delta_{k-s+1},$$

and let $\mathfrak{q}^{(s)}$ be the parabolic subalgebra defined by $\gamma_s$. Consider the flag of parabolic subalgebras

$$\mathfrak{g} = \mathfrak{q}^{(0)} \supset \mathfrak{q}^{(1)} \supset \cdots \supset \mathfrak{q}^{(k)} = \mathfrak{b}. \tag{3.17}$$

Consecutive application of (3.16) to the pairs $\mathfrak{p} = \mathfrak{q}^{(i)} \supset \mathfrak{q} = \mathfrak{q}^{(i+1)}$ and the fact that $A_{\mathfrak{b}} = 1$ give us

$$A_{\mathfrak{g}} = (1 - U_{\mathfrak{q}^{(0)},\mathfrak{q}^{(1)}})^{-1} \dots (1 - U_{\mathfrak{q}^{(k-1)},\mathfrak{q}^{(k)}})^{-1}. \tag{3.18}$$

The matrix $C$ with coefficients $c_{\lambda,\mu}$ as in Theorem 3.4 is the inverse of $A_{\mathfrak{g}}$. Hence we have

$$C = (1 - U_{\mathfrak{q}^{(k-1)},\mathfrak{q}^{(k)}}) \dots (1 - U_{\mathfrak{q}^{(0)},\mathfrak{q}^{(1)}}). \tag{3.19}$$

Note that this equality of operators on the Grothendieck group of the category $\mathcal{F}^k$. We define analogous linear operators $U_{\mathfrak{p},\mathfrak{q}}$ and $C$ on $\mathbb{Z}F^k$ defined by first setting

$$U_{\mathfrak{p},\mathfrak{q}}(f,g) = U_{\mathfrak{p},\mathfrak{q}}(\lambda(f),\lambda(g)), \quad c(f,g) = c(\lambda(f),\lambda(g)).$$

and then

$$U_{\mathfrak{p},\mathfrak{q}}(f) = \sum_{g\in F} U_{\mathfrak{p},\mathfrak{q}}(f,g)g, \quad C(f) = c(f,g)g. \tag{3.20}$$

Then (3.19) can be also be viewed as an equality of linear operators on linear operators in $\mathbb{Z}F^k$.

The equation (3.19) reduces the problem of finding the composition factors of Kac modules to the problem of calculating $U^i_{\mathfrak{q}^{(j)},\mathfrak{q}^{(j+1)}}(\lambda,\mu)$. Concerning the latter problem, the next result summarizes Theorems 6.15 and 6.22 from [Ser96].

**Theorem 3.5.**

*(a) If $\lambda-\alpha$ is $\mathfrak{q}^{(j)}$-dominant then*

$$U^i_{\mathfrak{q}^{(j)},\mathfrak{q}^{(j+1)}}(\lambda) = U^{i+1}_{\mathfrak{q}^{(j)},\mathfrak{q}^{(j+1)}}(\lambda-\alpha)$$

*for $i > 1$, and*

$$U^1_{\mathfrak{q}^{(j)},\mathfrak{q}^{(j+1)}}(\lambda) = L_{\mathfrak{q}^{(j)}}(\lambda-\alpha) \oplus U^2_{\mathfrak{q}^{(j)},\mathfrak{q}^{(j+1)}}(\lambda-\alpha).$$

*(b) If $\lambda-\alpha$ is not $\mathfrak{q}^{(j)}$-dominant then*

$$[U^i_{\mathfrak{q}^{(j)},\mathfrak{q}^{(j+1)}}(\lambda) : L_{\mathfrak{q}^{(j)}}(\mu)] = [U^{i-1}_{\mathfrak{q}^{(j+1)},\mathfrak{q}^{(j+2)}}(\lambda-\alpha) : L_{\mathfrak{q}^{(j+1)}}(\mu)].$$

*and $U^1_{\mathfrak{q}^{(j)},\mathfrak{q}^{(j+1)}}(\lambda) = 0$.*

*(c)*

$$U^1_{\mathfrak{q}^{(k-1)},\mathfrak{q}^{(k)}}(\lambda) = L_{\mathfrak{q}^{(k-1)}}(\lambda-\alpha), \quad U^i_{\mathfrak{q}^{(k-1)},\mathfrak{q}^{(k)}}(\lambda) = 0, \text{ if } i > 1.$$

To prove Theorem 3.4 it remains to interpret the above result in terms of diagrams. Below we use an induction argument on $k$, and for this purpose we define, by analogy with (3.17), the flag of parabolic subalgebras in $\mathfrak{gl}(k-1,k-1)$

$$\mathfrak{gl}(k-1,k-1) = \mathfrak{p}^{(0)} \supset \mathfrak{p}^{(1)} \cdots \supset \mathfrak{p}^{(k-1)} = \mathfrak{b}',$$

where $\mathfrak{b}'$ is the Borel subalgebra consisting of upper triangular matrices. Let $\mathfrak{g}_{(s)}$ be the subalgebra of $\mathfrak{g}$ consisting of all matrices with zero entries in rows $s, 2k-s+1$, and zero entries in columns $s, 2k-s+1$. We have an obvious isomorphism from $\mathfrak{gl}(k-1,k-1)$ to $\mathfrak{g}_{(s)}$, and we denote the image of $\mathfrak{p}^{(j)}$ under this isomorphism by $\mathfrak{p}^{(j)}_{(s)}$. If $l(\mathfrak{p})$ is the quotient of $\mathfrak{p}$ by the nilpotent radical, then because we deleted two diagonal entries from $\mathfrak{g}$ to get $\mathfrak{g}_{(s)}$ we have

$$l(\mathfrak{q}^{(j+1)}) \simeq l(\mathfrak{p}^{(j)}_{(s)}) \oplus \mathbb{C} \oplus \mathbb{C}.$$

Let $\lambda = (a_1, \ldots, a_k | a_k, \ldots, a_1)$ and

$$\lambda' = (a_1, \ldots, a_{s-1}, a_{s+1}, \ldots, a_k | a_k, \ldots, a_{s+1}, a_{s-1}, \ldots, a_1).$$

If we regard $L_{\mathfrak{q}^{(j+1)}}(\lambda)$ as a $\mathfrak{p}^{(j)}_{(s)}$-module, via the above isomorphism, then it remains irreducible with highest weight $\lambda'$. This implies

$$[U^i_{\mathfrak{q}^{(j+1)},\mathfrak{q}^{(j+2)}}(\lambda) : L_{\mathfrak{q}^{(j+1)}}(\mu)] = [U^i_{\mathfrak{p}^{(j)},\mathfrak{p}^{(j+1)}}(\lambda') : L_{\mathfrak{p}^{(j)}}(\mu')]. \tag{3.21}$$

**Lemma 3.6.** *Let $U^i_{\mathfrak{q}^{(j)},\mathfrak{q}^{(j+1)}}(f) = U^i_{\mathfrak{q}^{(j)},\mathfrak{q}^{(j+1)}}(\lambda(f))$ and $L_{\mathfrak{q}}(f) = L_{\mathfrak{q}}(\lambda(f))$. Next let*

$$f^{-1}(\times) = \{a_1, \ldots, a_k\} \tag{3.22}$$

*with $a_1 > a_2 > \cdots > a_k$ and $a = a_{j+1}$.*

*Then the relations of Theorem 3.5 can be rewritten in the following way in terms of weight diagrams*

*(a) If $f(a-1) = \circ$ then*

$$U^i_{\mathfrak{q}^{(j)},\mathfrak{q}^{(j+1)}}(f) = U^{i+1}_{\mathfrak{q}^{(j)},\mathfrak{q}^{(j+1)}}(f^a_{a-1}) \tag{3.23}$$

*for $i > 1$, and*

$$U^1_{\mathfrak{q}^{(j)},\mathfrak{q}^{(j+1)}}(f) = L_{\mathfrak{q}^{(j)}}(f^a_{a-1})) \oplus U^2_{\mathfrak{q}^{(j)},\mathfrak{q}^{(j+1)}}(f^a_{a-1}). \tag{3.24}$$

*(b) If $f(a-1) = \times$*

$$[U^i_{\mathfrak{q}^{(j)},\mathfrak{q}^{(j+1)}}(f) : L_{\mathfrak{q}^{(j)}}(g)] = [U^{i-1}_{\mathfrak{p}^{(j)},\mathfrak{p}^{(j+1)}}(f^a) : L_{\mathfrak{q}^{(j)}}(g^{a-1})]. \tag{3.25}$$

*In addition, $U^1_{\mathfrak{q}^{(j)},\mathfrak{q}^{(j+1)}}(f) = 0$.*

*(c)*

$$U^1_{\mathfrak{q}^{(k-1)},\mathfrak{q}^{(k)}}(f) = L_{\mathfrak{q}^{(k-1)}}(f^a_{a-1}), \quad U^i_{\mathfrak{q}^{(k-1)},\mathfrak{q}^{(k)}}(f) = 0, \text{ if } i > 1.$$

*Proof.* (a) and (c) follow immediately from the identity $\lambda(f^a_{a-1}) = \lambda(f) - \alpha$. To prove (b) note that by (3.21) we have

$$[U^{i-1}_{\mathfrak{q}^{(j+1)},\mathfrak{q}^{(j+2)}}(\lambda(f) - \alpha) : L_{\mathfrak{q}^{(j+1)}}(g)] = [U^{i-1}_{\mathfrak{p}^{(j)},\mathfrak{p}^{(j+1)}}(f^a) : L_{\mathfrak{p}^{(j)}}(g^{a-1})]. \tag{3.26}$$

By Theorem 3.5 (b) we have

$$[U^{i}_{\mathfrak{q}^{(j)},\mathfrak{q}^{(j+1)}}(\lambda(f)) : L_{\mathfrak{q}^{(j)}}(\lambda(g))] = [U^{i-1}_{\mathfrak{q}^{(j+1)},\mathfrak{q}^{(j+2)}}(\lambda(f) - \alpha) : L_{\mathfrak{q}^{(j+1)}}(\lambda(g))],$$

and combining this with (3.26) we deduce (3.25). □

We introduce two related pieces of notation. Sometimes one is more convenient than the other. First suppose $f \in F^k$ with $f(a) = \times$, set

$$\mathrm{LM}_k(f, a, i) = \{b \in \mathbb{Z} | f^a_b \text{ is obtained from } f \text{ by a legal move of weight } i\}.$$

Next define with the notation of (3.22),

$$\mathbf{LM}(f, p) = \{g | g \text{ is obtained from } f \text{ by a legal move of weight 0 } f \text{ starting at } a_p\}.$$

**Lemma 3.7.** *Let $f \in F^k$, $f(a) = \times$, then we have*

*(a) If $f(a-1) = \circ$, and $i > 0$, then $\mathrm{LM}_k(f, a, i) = \mathrm{LM}_k(f^a_{a-1}, a-1, i+1)$.*

*(b) If $f(a-1) = \circ$, then $\mathrm{LM}_k(f, a, 0) = \mathrm{LM}_k(f^a_{a-1}, a-1, 1) \cup \{a-1\}$.*

*(c) If $f(a-1) = \times$, and $h = f^a$, then $\mathrm{LM}_k(f, a, 0) = \emptyset$ and*

$$\mathrm{LM}_k(f, a, i) = \mathrm{LM}_{k-1}(h, a-1, i-1),$$

*for $i > 0$.*

*(d) If $f(a-1) = \circ$, and $h = f^a_{a-1}$, or $f(a-1) = \times$, and $h = f^a$, then $f^a_b = h^{a-1}_b$ for all $b \in \mathrm{LM}_k(f, a, i)$.*

*Proof.* Straightforward. □

**Corollary 3.8.** *Let $U^{i+1}_{\mathfrak{q}^{(j)},\mathfrak{q}^{(j+1)}}(f) = \sum_{g \in F} U^{i+1}_{\mathfrak{q}^{(j)},\mathfrak{q}^{(j+1)}}(\lambda(f), \lambda(g))g$. Then*

$$U^{i+1}_{\mathfrak{q}^{(j)},\mathfrak{q}^{(j+1)}}(f) = \sum_{b \in \mathrm{LM}_k(f,a,i)} f^a_b$$

*Proof.* It is sufficient to prove the statement for $j = 0$, since none of the terms in $U^{i+1}_{\mathfrak{q}^{(j)},\mathfrak{q}^{(j+1)}}(f)$ depends on the $j$ rightmost $\times$-s in $D_{wt}(f)$.

The proof goes by induction on $k$, and for $k$ fixed a second induction on the distance between the leftmost $\times$ and the rightmost $\times$ of $D_{wt}(f)$. The case $k = 1$ immediately follows from Lemma 3.6(c). Let $a = a_1$ be the position of the rightmost $\times$ of $D_{wt}(f)$.

First, assume that $f(a-1)=\circ$. Let $h=f^a_{a-1}$. Then if $i>0$, we have using parts (a), (d) of Lemma 3.7, the second induction hypothesis applied to $h$, and then (3.23) we have

$$\begin{aligned}\sum_{b\in \mathrm{LM}_k(f,a,i)} f^a_b &= \sum_{b\in \mathrm{LM}_k(h,a-1,i+1)} h^{a-1}_b \\ &= \sum_g U^{i+2}_{\mathfrak{q}^{(j)},\mathfrak{q}^{(j+1)}}(h,g)g \\ &= \sum_g U^{i+1}_{\mathfrak{q}^{(j)},\mathfrak{q}^{(j+1)}}(f,g)g.\end{aligned}$$

If $i=0$, the result follows similarly, using part (b) of the Lemma and (3.24). Finally if $f(a-1)=\times$, let $h=f^a$. Then using parts (c) , (d) of the Lemma, induction on $k$ and (3.25) we have

$$\begin{aligned}\sum_{b\in \mathrm{LM}_k(f,a,i)} f^a_b &= \sum_{b\in \mathrm{LM}_{k-1}(h,a-1,i-1)} h^{a-1}_b \\ &= U^i_{\mathfrak{p}^{(j)},\mathfrak{p}^{(j+1)}}(h) = U^{i+1}_{\mathfrak{q}^{(j)},\mathfrak{q}^{(j+1)}}(f).\end{aligned}$$

□

**Corollary 3.9.** *We have*

$$\sigma_{j+1} = -U_{\mathfrak{q}^{(j)},\mathfrak{q}^{(j+1)}}. \tag{3.27}$$

*Proof.* The result follows from the previous Corollary since

$$U_{\mathfrak{q}^{(j)},\mathfrak{q}^{(j+1)}} = \sum_i U_{\mathfrak{q}^{(j)},\mathfrak{q}^{(j+1)}}.$$

□

Together Equations (**??**) and (3.27) yield Theorem 3.4. Combining Theorem A, Theorem 3.3 and Theorem 3.4, we obtain a combinatorial proof of the equivalence of the algorithms from [Bru03] and [Ser96].
We remark that by Corollary 6.25 from [Ser96], the modules $U^i_{\mathfrak{q}^{(j)},\mathfrak{q}^{(j+1)}}(\lambda)$ are semisimple. Thus Corollary 3.8 determines their decompositions into simple modules. In particular this gives us the first part of the next result. The second part will be used in Section 6 of this paper.

**Corollary 3.10.** *For $f$ be as in* (2.2) *we have*

*(a)*

$$U^1_{\mathfrak{q}^{(p-1)},\mathfrak{q}^{(p)}}(\lambda(f)) = \bigoplus_{g\in \mathbf{LM}(f,p)} L_{\mathfrak{q}^{(p-1)}}(\lambda(g)).$$

*(b) $\Gamma_{\mathfrak{q}^{(p-1)},\mathfrak{q}^{(p)}}(L_{\mathfrak{q}^{(p)}}(\lambda(f)))$ is generated by a highest weight vector of weight $\lambda$ and its structure can be described by the exact sequence*

$$0\to \bigoplus_{g\in \mathbf{LM}(f,p)} L_{\mathfrak{q}^{(p-1)}}(\lambda(g)) \to \Gamma_{\mathfrak{q}^{(p-1)},\mathfrak{q}^{(p)}}(L_{\mathfrak{q}^{(p)}}(\lambda(f))) \to L_{\mathfrak{q}^{(p-1)}}(\lambda(f)) \to 0.$$

*Proof.* By what we said above, it is enough to note that (b) follows from (a) and Lemma 4.11 in [Ser96]. □

# 4 The Graph $\mathcal{G}$ and the Involution on Irregular Paths.

From now on we consider only elements of $F$ that are core-free. Define $\mathcal{G}$ to be the oriented graph whose vertices are elements of $F$, and we join $f$ and $g$ by an edge $f \longrightarrow g$ if $g$ is obtained from $f$ by a legal move. We put the label $[s,t]$ on this edge if the corresponding legal move has start $s$ and end $t$, in other words, $g = f_t^s$ (always $s > t$). The weight of an edge is the weight of the corresponding legal move, and if $g = f_t^s$ as above we set $l([t,s]) = l_f(t,s)$

It is easy to check that $\mathcal{G}$ does not have oriented loops. A *path* in $\mathcal{G}$ is a sequence $[s_1,t_1],\ldots,[s_q,t_q]$ where for $1 \le i \le q$, $[s_i,t_i]$ is a legal move from $f_{i-1}$ to $f_i$. We say that the path is *increasing* if $s_1 < \ldots < s_q$. (It follows immediately from the definition that in any path $s_i \neq s_{i+1}$.) Often we refer to a path by listing only the legal moves. The weight $l(P)$ of a path $P$ is the sum of weights of all edges in $P$.

**Lemma 4.1.** *Let $\mathcal{P}_{f,g}(\mathcal{G})$ denote the set of all increasing paths in $\mathcal{G}$ leading from $f$ to $g$, and let*

$$(1+\sigma_1)\ldots(1+\sigma_k)f = \sum_g c_{f,g}g.$$

*Then*

$$c_{f,g} = \sum_{P\in\mathcal{P}_{f,g}(\mathcal{G})} (-1)^{l(P)}. \tag{4.1}$$

*Proof.* Write

$$(1+\sigma_1)\ldots(1+\sigma_k)f = \sum_{i_1<\ldots<i_r} \sigma_{i_1}\ldots\sigma_{i_r}(f).$$

Using (2.3) we see that each increasing path $P$ with edges $[a_{i_r},b_{i_r}],\ldots,[a_{i_1},b_{i_1}]$ which leads from $f$ to $g$ gives the term $(-1)^{l(P)}g$ in $\sigma_{i_1}\ldots\sigma_{i_r}(f)$. □

We call an increasing path from $f$ to $g$ in $\mathcal{G}$ *irregular* if one of the following conditions hold

(a) The path contains an edge with positive weight

(b) There are repetitions among the labels on the path, in other words there are edges with label $[c,d]$ and $[b,c]$ in the path.

An edge $[c,d]$ of an irregular path is called *irregular* if it has a positive weight or there is an edge with label $[b,c]$ later in the path. A path which is not irregular is *regular.*

**Lemma 4.2.** *Suppose an increasing path has edges $[b,c]$ and $[a,s]$ with $c < s < b < a$. Then the edge $[b,c]$ has positive weight.*

*Proof.* If the result is false, then with $b, c$ fixed choose a counterexample with $s$ as large as possible. Suppose that the edge $[b, c]$ connects vertex $f$ to $f'$ and that $D = D_{wt}(f')$ has $P$ $\times'$s and $p$ $\circ'$s in the interval $(c, s)$. Similarly suppose that the edge $[a, s]$ connects vertex $g'$ to $g$ and that $D_{wt}(g')$ has $Q$ $\times'$s and $q$ $\circ'$s in the interval $(s, b)$. We claim that $D$ also has $Q$ $\times'$s and $q$ $\circ'$s in the interval $(s, b)$. Indeed, consider the part of the path between the edges $[b, c]$ and $[a, s]$. Since the path is increasing no $\times$ in the interval $(s, b)$ can be moved. Also by the choice of the counterexample, there can be no edge with label $[d, e]$ where $d > b$ and $s < e < b$. The claim follows from this. We deduce that $P \geq p$ and $Q \geq q$, since $[b, c]$ and $[a, s]$ are legal moves. Now $D$ has $p + q + 1$ $\circ'$s in the interval $(c, b)$ since, in addition to those counted before there is also a $\circ$ at $s$. Because $[b, c]$ has weight zero, and $D$ has $P + Q$ $\times'$s in the interval $(c, b)$, we have $P + Q = p + q + 1$. This implies that either $P = p$, in which case the cap in $D_{cap}(f')$ beginning at $c$ would end at $s$, or $Q = q$, in which case the cap in $D_{cap}(g')$ beginning at $s$ would end at $b$. Either way we reach a contradiction. $\square$

**Lemma 4.3.** *Let $\mathcal{R}_{f,g}(\mathcal{G})$ denote the set of all increasing regular paths in $\mathcal{G}$ leading from $f$ to $g$. Then*

$$\sum_{P \in \mathcal{P}_{f,g}(\mathcal{G})} (-1)^{l(P)} = |\mathcal{R}_{f,g}(\mathcal{G})|. \tag{4.2}$$

*Proof.* Define an involution $*$ on the set of all increasing irregular paths by the following procedure. Let $P$ be an irregular increasing path. Consider the irregular edge $[s, t]$ of $P$ with maximal possible end $t$. There are exactly two possibilities: either $P$ contains a regular edge $[a, s]$, or $s$ is not the end of any edge in $P$.

In the former case, define $P^*$ to be the path obtained from $P$ by removing $[s, t]$ and $[a, s]$ and inserting the edge $[a, t]$. If there were an edge $[b, c]$ in $P$ with $s < b < a$ and $t < c < s$, then $[b, c]$ would be irregular by Lemma 4.2, contradicting the choice of $s$. Since the edge $[a, s]$ is regular, it has zero weight. Therefore $l([a, t]) = l([s, t]) + 1$ and $l(P^*) = l(P) + 1$. Note also that $P^*$ is again irregular and the edge $[a, t]$ is the irregular edge with maximal possible end.

In the latter case, let $f' \to g'$ be the edge with label $[s, t]$. Then $g'(s) = \circ$, and the symbol $\times$ occurs more often than $\circ$ in the part of the weight diagram $D_{wt}(g')$ strictly between $t$ and $s$. In other words we can find a cap in $D_{cap}(g')$ beginning at $b > t$ and ending at $s$. Then we define $P^*$ to be the path obtained from $P$ by removing the edge $[s, t]$ and inserting the edges $[b, t]$ and $[s, b]$. Note that $[s, b]$ is regular, $[b, t]$ is irregular and $l([b, t]) = l([s, t]) - 1$. Hence $l(P^*) = l(P) - 1$. It is clear that $P^*$ is irregular and $[b, t]$ is the irregular edge with maximal possible end.

It is obvious that $*$ is an involution and since $(-1)^{l(P)} = -(-1)^{l(P^*)}$, all irregular paths in the left hand side of (4.2) cancel. Hence we have

$$\sum_{P \in \mathcal{P}_{f,g}(\mathcal{G})} (-1)^{l(P)} = \sum_{P \in \mathcal{R}_{f,g}(\mathcal{G})} (-1)^{l(P)}.$$

Now the statement follows since $l(P) = 0$ for any regular path $P$.

$\square$

We have two immediate consequences of the above work, namely

$$c_{f,g} = |\mathcal{R}_{f,g}(\mathcal{G})|, \tag{4.3}$$

and

$$(1+\sigma_1)\dots(1+\sigma_k)f = \sum_g |\mathcal{R}_{f,g}(\mathcal{G})|g. \tag{4.4}$$

**Lemma 4.4.** *Suppose $f, g \in F$.*

*(a) If $g \in P(f)$, then $|\mathcal{R}_{f,g}(\mathcal{G})| = 1$.*

*(b) If $g \notin P(f)$ then $\mathcal{R}_{f,g}(\mathcal{G})$ is empty.*

*Proof.* Suppose $g \in P(f)$, and that $\times(f) = (a_1, a_2, \dots, a_k)$. Then let

$$I = \{i \in \{1, \dots, k\} | D_{cap}(g) \text{ has a cap ending at } a_i\},$$

and for $i \in I$, suppose that ending at $a_i$ begins at $b_i$. Then there is a regular path from $f$ to $g$ given by $\overrightarrow{\prod}_{i \in I}[a_i, b_i]$ where the arrow means that we take the product in the order that gives an increasing path. It follows easily from the definitions that this is the only way to get a regular increasing path from $f$ to $g$. $\square$

The main theorem immediately follows from (4.4) and the previous Lemma.

**Example 4.5.** Let $k = 2$. For $a < b \in \mathbb{Z}$ define $f_{(a,b)} \in F$ so that $\times(f_{(a,b)}) = (a, b)$. Below we give the part of the graph $\mathcal{G}$ used to show that

$$(1+\sigma_1)(1+\sigma_2)f_{(2,3)} = f_{(2,3)} + f_{(1,3)} + f_{(0,1)}. \tag{4.5}$$

Legal moves are represented by arrows together with their labels. All edges have weight zero except the edge with label $[3, 1]$ which has weight 1.

$$\begin{tikzcd}
 & f_{(1,2)} & \\
f_{(2,3)} \arrow[ur, "{[3,1]}"] \arrow[dr, "{[2,1]}"'] & & \\
 & f_{(1,3)} \arrow[uu, "{[3,2]}"'] \arrow[r, "{[3,0]}"] & f_{(0,1)}
\end{tikzcd}$$

There are two irregular paths starting from $f_{(2,3)}$, both ending at $f_{(1,2)}$. These paths are interchanged by the involution $*$. Summing over the remaining paths and using (4.4) gives (4.5).

# 5 Composition factors of Kac modules.

We describe a procedure for determining the composition factors of any Kac module, without cancelation. By Brundan's Theorem we need a procedure for finding the set in $P(f)$ in Equation (2.1). This is a problem in enumerative combinatorics, similar to the problem of describing the set $B_n$ in Example 2.3. We give a solution based on the Lemma below. By Remark 2.2 we can restrict our attention to the core-free case.

Suppose $f \in F$ with $\times(f)$ as in Equation (2.2) and set $a = a_1$. Let $f' \in F$ be given by $\times(f') = (a_2, \ldots, a_k)$, and set

$$P_{k-1}(f') = \{ \, g' \in F \mid D_{cap}(g') \text{ matches } D_{wt}(f')\},$$

$$Q(f) = \{ \, g \in P(f) \mid D_{cap}(g) \text{ has a cap joining } a \text{ to } a+1\}.$$

**Lemma 5.1.** *There is a bijection $P_{k-1}(f') \longrightarrow Q(f)$ such that $g' \in P_{k-1}(f')$ maps to $g$ where $\times(g) = (a, b_1, \ldots, b_{k-1})$ if $\times(g') = (b_1, \ldots, b_{k-1})$.*

*Proof.* Straightforward. □

Given $f$ as above, we can assume by induction that we have found $P_{k-1}(f')$ and hence $Q(f)$. Now suppose $g \in P(f) \backslash Q(f)$. Then $D_{cap}(g)$ has a cap joining $b$ to $a$ for some $b < a$. Replacing this cap with a cap beginning at $a$, we obtain a cap diagram $D_{cap}(h)$ for some $h \in Q(f)$ such that $g = h^a_b$. Moreover we can determine the set $P(f) \backslash Q(f)$ as follows. For each $h \in Q(f)$ list the cap diagrams $D_{cap}(h^a_b)$ that match $D_{wt}(f)$. Then it is easy to see, for example by Proposition 5.4 below, that every cap diagram $D_{cap}(g)$ with $g \in P(f) \backslash Q(f)$ will have been listed exactly once.

The above procedure suggests another proof of Theorem A. By induction we may assume that

$$(1 + \sigma_1) \ldots (1 + \sigma_{k-1}) f' = \sum_{h \in P_{k-1}(f')} h.$$

Since $\sigma_2, \ldots, \sigma_k$ do not move the rightmost $\times$ in $D_{wt}(f)$, it follows that

$$(1 + \sigma_2) \ldots (1 + \sigma_k) f = \sum_{h \in Q(f)} h.$$

Now set $\sigma = \sigma_1$. It remains to show that

$$\sigma \sum_{h \in Q(f)} h = \sum_{g \in P(f) \backslash Q(f)} g, \tag{5.1}$$

but this follows at once from the Proposition below. Note that $g \in P(f) \backslash Q(f)$ implies that

$$\times(g) = (b_1, \ldots, b_k) \tag{5.2}$$

with $b_1 < a$. For $m \in \mathbb{Z}F$, write $m = \sum_{f \in F} |m : f| f$, with $|m : f| \in \mathbb{Z}$. Suppose that $g \in F, a \in \mathbb{Z}$, and set

$$\mathbf{Y}(g, a) = \{b \in \mathbb{Z} | g = f^a_b \text{ is obtained from } f \text{ by a legal move}\}.$$

Next suppose $f$ satisfies (2.2), and that $f(a) = \times$, and $f(b) = \circ$.

**Lemma 5.2.** *Suppose $g \in F$, and $\times(g) = (b_1, \ldots, b_k)$ with $b_1 < a$.*

$$\{h \in F | h(a) = \times \text{ and } |\sigma h : g| \neq 0\} = \{g_a^b | b \in \mathbf{Y}(g, a)\}.$$

*Proof.* This follows by considering legal moves that end at $g$. □

Next we compare the caps in $D_{cap}(h)$ and $D_{cap}(h_b^a)$.

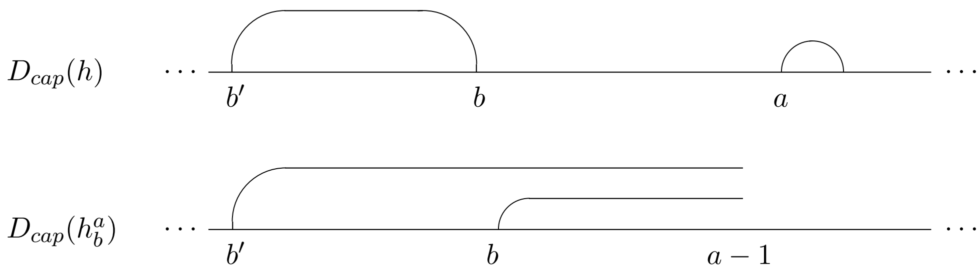
**Lemma 5.3.** *Suppose that $g = h_b^a$, where $b \in \mathbf{Y}(g, a)$.*

*(a) If $D_{cap}(h)$ has a cap $C$ joining $b'$ to $b$, for some $b' < b$, then in place of $C$ and the cap in $D_{cap}(h)$ beginning at $a$, $D_{cap}(g)$ has caps $C_1, C_2$ with $b(C_1) = b'$, $b(C_2) = b$, and $e(C_1) > e(C_2) \geq a$.*

*(b) If $D_{cap}(h)$ has no cap ending at $b$, then in place of the cap in $D_{cap}(h)$ beginning at $a$, $D_{cap}(g)$ has a cap $C$ with $b(C) = b$ and $e(C) \geq a$.*

*(c) Apart from the different endpoints of the caps resulting from (a) and (b), the caps in $D_{cap}(h)$ have the same left endpoints as $D_{cap}(g)$. They also have the same right endpoints except that if a cap in $D_{cap}(h)$ ends at $c > a$, then in $D_{cap}(g)$ the corresponding cap ends at $c - 2$.*

The relevant parts of the cap diagrams in case (a) are shown below. In case (b) the diagrams are the same except that there are no caps beginning at $b'$.




**Proposition 5.4.** *Suppose $g$ satisfies Equation (5.2), and $c_k < a$. Set*

$$R_{f,g} = \{h \in Q(f) | h(a) = \times \text{ and } |\sigma h : g| \neq 0\}$$

*Then one of the following holds*

*(a) $g \in P(f) \backslash Q(f)$. In this case $R_{f,g} = \{h\}$ is a singleton and $|\sigma h : g| = 1$.*

*(b) $g \notin P(f)$. In this case either $R_{f,g}$ is empty or $R_{f,g} = \{h^{(1)}, h^{(2)}\}$ consists of two elements and*

$$|\sigma h^{(1)} : g| + |\sigma h^{(2)} : g| = 0.$$

*Proof.* Assume that $h \in R_{f,g}$. Then by Lemma 5.2, $g = h^a_b$, (equivalently $h = g^b_a$) for some $b \in \mathbf{Y}(g,a)$. We use the comparison between in $D_{cap}(h)$ and $D_{cap}(g)$. given in Lemma 5.3. Suppose that $\mathbf{Y}(g,a) = \{c_1 < c_2 < \ldots < c_r\}$, and that $b = c_i$ with $1 \leq i \leq r$.

(a) If $g \in P(f)$, then since $g(a) \neq \times$, there must be a cap $C$ in $D_{cap}(g)$ ending at $a$. By Lemma 5.3 this can only happen if $C$ begins at $b = c_r$. Thus $b$ and hence $h$ are uniquely determined by the pair $f, g$. Since $b = c_r$, there are no caps $C$ in $D_{cap}(g)$ with $b(C) > b$ and $e(C) \geq a$, so $|\sigma h : g| = 1$.

(b) We claim that one of the following two cases holds

(i) $f(c_{i-1}) = \circ$ and $f(c_j) = \times$ if $j \neq i-1$,

(ii) $f(c_i) = \circ$ and $f(c_j) = \times$ if $j \neq i$.

Suppose first that $f(b) = \times$. Then since $h(b) = \circ$, there is no cap in $D_{cap}(h)$ beginning at $b$, so as $D_{cap}(h)$ matches $D_{wt}(f)$, there is a cap $C$ in $D_{cap}(h)$ ending at $b$. This implies that $i > 1$, and $C$ begins at $c_{i-1}$. Because $D_{cap}(h)$ matches $D_{wt}(f)$, we have $f(c_{i-1}) = \circ$. Now if $j \neq i-1, i$ then $c_j \in \mathbf{Y}(h,a)$ , so the cap $C$ in $D_{cap}(g)$ with $b(C) = c_j$ has $e(C) \geq a$. Again since $D_{cap}(h)$ matches $D_{wt}(f)$, it follows that $f(c_j) = \times$, so (i) holds.

Now suppose that $f(b) = \circ$. Since $g = h^a_b$, and $D_{cap}(g)$ does not match $D_{wt}(f)$, the cap $C$ in $D_{cap}(g)$ with $b(C) = b$ has $e(C) > a$. It follows that $i < r$, and $h'' = g^{c_{i+1}}_a \in R_{f,g}$. Since $D_{cap}(h)$ has a cap joining $c_i$ and $c_{i+1}$, we have $f(c_{i+1}) = \times$. Thus replacing $h$ by $h''$ and $i$ by $i+1$ in the case already considered we see that (ii) holds. This proves the claim.

The argument of the preceding paragraph shows that there is no loss in assuming that (i) holds. Then $b$ and hence $h$ are again uniquely determined by the pair $f, g$. The conclusion in (b) therefore follows with $h^{(1)} = g^{c_{i-1}}_a$ and $h^{(2)} = g^{c_i}_a$. $\square$

**Lemma 5.5.**

*(a) Given $g \in F$ and $a \in \mathbb{Z}$ there is at most one $f \in F$ such that there is legal move with weight zero from $f$ to $g$ starting at $a$.*

*(b) For each $g \in P(f)$ there is a unique regular path from $f$ to $g$.*

*Proof.* (a) If there is a cap $C$ in $D_{cap}(g)$ with $e(C) = a$, then the unique $f$ in the statement is $g^b_a$ where $b(C) = b$.
(b) follows at once from (a). $\square$

It is easy to see how the above proof is related to our first proof of Theorem A. Indeed let * be the involution of the set of all increasing paths in $\mathcal{G}$ leading from $f$ to $g$ defined in the proof of Lemma 4.3. Then * preserves the set $S$ of paths with last label of the form $[a,t]$ all of whose edges are regular except possibly the last.

Suppose the set $R_{f,g}$ is defined as in the proof of Proposition 5.4 is nonempty. If $g \in P(f) \backslash Q(f)$ and $R_{f,g} = \{h\}$, then on the unique regular path from $f$ to $g$, $h$ is the vertex before $g$ and $g$ is obtained from $h$ by a legal move with start $a$. On the other hand if $g \in P(f)$, then $R_{f,g} = \{h^{(1)}, h^{(2)}\}$ and $S$ consists of two paths which are interchanged by $*$. The vertices before the last in these paths are $h^{(1)}$ and $h^{(2)}$.

# 6 The Subgraph $\mathcal{E}$ and Extensions.

The goal if this section is to prove

**Theorem B.** We have

(a)

$$\dim \mathrm{Ext}^1(L(\lambda(f)), L(\lambda(g))) \leq 1.$$

(b) $\mathrm{Ext}^1(L(\lambda(f)), L(\lambda(g))) \neq 0$ if and only if $f \longrightarrow g$ or $g \longrightarrow f$ is an edge of $\mathcal{E}$.

Let $\tau$ be the automorphism of $\mathfrak{g}$ defined by $\tau(X) = -X^{st}$, where $X^{st}$ is the super-transpose of $X$, and for any $\mathfrak{g}$-module $M$, let $M^\tau$ denote the twist by $\tau$. Thus as a set $M^\tau = \{m^\tau | m \in M\}$ and the module structure is given by

$$xm^\tau = (\tau(x)m)^\tau$$

for $x \in \mathfrak{g}$ and $m \in M$. The superHopf algebra structure of $U(\mathfrak{g})$ allows us to make the dual $N^*$ of any $\mathbb{Z}_2$-graded module $N$ in $\mathcal{F}$ into a module in $\mathcal{F}$, and we set $\check{M} = (M^*)^\tau$. Then $M \to \check{M}$ is a contravariant exact functor on $\mathcal{F}$ which maps a simple finite-dimensional module to itself. Hence we have

$$\mathrm{Ext}^1(L(\lambda), L(\mu)) = \mathrm{Ext}^1(\check{L}(\mu), \check{L}(\lambda)) = \mathrm{Ext}^1(L(\mu), L(\lambda)). \tag{6.1}$$

**Remark 6.1.** Equation (6.1) reflects a more general phenomenon. Indeed there is well developed theory of links between prime ideals in a Noetherian ring $R$, see for example, [GW04]. The *graph of links* is the directed graph whose vertices are the prime ideals of $R$, with arrows between linked prime ideals. It is shown in [Mus93] that if $\mathfrak{g}$ is a classical simple Lie superalgebra and $\mathfrak{g} \neq P(n)$, then for prime ideals $P, Q$ of $U(\mathfrak{g})$ there is a link from $P$ to $Q$ if and only if there is a link from $Q$ to $P$. Equation (6.1) follows from this fact by taking $P$ and $Q$ to be coartinian. In the case where $\mathfrak{g} = \mathfrak{sl}(2,1)$ graph of links between primitive ideals is described in [Mus93].

Define an order on the set $X(m,n)$ by putting $\mu \leq \lambda$ if $\lambda - \mu$ is a sum of positive roots.

**Lemma 6.2.** *Let* $\mathrm{Ext}^1(L(\lambda), L(\mu)) \neq 0$*, then either* $\lambda \leq \mu$ *or* $\mu \leq \lambda$*.*

*Proof.* Assume that $\lambda$ and $\mu$ are not compatible. Consider an exact sequence

$$0 \to L(\lambda) \to M \to L(\mu) \to 0.$$

Since $\mu$ has multiplicity one as a weight of $M$, a non-zero vector of weight $\mu$ generates a proper submodule in $M$. Hence the exact sequence splits. $\square$

**Lemma 6.3.** *Let $\mathfrak{p} \subset \mathfrak{q}$ be a pair of parabolic subalgebras, and suppose $\gamma \in \mathfrak{h}^*$ defines $\mathfrak{p}$. If $\mu \leq \lambda$ and $\mathrm{Ext}^1_{\mathfrak{q}}(L_{\mathfrak{q}}(\lambda), L_{\mathfrak{q}}(\mu)) \neq 0$, then one of the following holds*

*(a)* $(\mu, \gamma) = (\lambda, \gamma)$ *and* $\mathrm{Ext}^1_{\mathfrak{p}}(L_{\mathfrak{p}}(\lambda), L_{\mathfrak{p}}(\mu)) \neq 0$

*(b)* $(\mu, \gamma) < (\lambda, \gamma)$ *and $L_{\mathfrak{q}}(\mu)$ is a subquotient in $\Gamma_{\mathfrak{q},\mathfrak{p}}(L_{\mathfrak{p}}(\lambda))$.*

*Proof.* The condition $\mu \leq \lambda$ implies that $(\mu, \gamma) \leq (\lambda, \gamma)$. Consider a non-split exact sequence

$$0 \to L_{\mathfrak{q}}(\mu) \to M \to L_{\mathfrak{q}}(\lambda) \to 0.$$

Let $z \in \mathfrak{h}$ be the element such that $\beta(z) = (\beta, \gamma)$ for every $\beta \in \mathfrak{h}^*$. For every $\mathfrak{q}$-module $N$ let

$$N' = \{x \in N | zx = (\lambda, \gamma)x\}.$$

If $(\mu, \gamma) = (\lambda, \gamma)$ we have $L_{\mathfrak{q}}(\lambda)' = L_{\mathfrak{p}}(\lambda)$, $L_{\mathfrak{q}}(\mu)' = L_{\mathfrak{p}}(\mu)$ and an exact sequence of $\mathfrak{p}$-modules

$$0 \to L_{\mathfrak{p}}(\mu) \to M' \to L_{\mathfrak{p}}(\lambda) \to 0.$$

We claim that this exact sequence does not split. Indeed, $M$ is a quotient of $U(\mathfrak{q}) \otimes_{U(\mathfrak{p})} M'$. If $M' = L_{\mathfrak{p}}(\lambda) \oplus L_{\mathfrak{p}}(\mu)$, then we have an exact sequence

$$U(\mathfrak{q}) \otimes_{U(\mathfrak{p})} L_{\mathfrak{p}}(\lambda) \oplus U(\mathfrak{q}) \otimes_{U(\mathfrak{p})} L_{\mathfrak{p}}(\mu) \to M \to 0,$$

hence $M = L_{\mathfrak{q}}(\lambda) \oplus L_{\mathfrak{q}}(\mu)$, a contradiction.

If $(\mu, \gamma) < (\lambda, \gamma)$, then $L_{\mathfrak{q}}(\lambda)' = L_{\mathfrak{p}}(\lambda)$ and $L_{\mathfrak{q}}(\mu)' = 0$. Therefore $M' = L_{\mathfrak{p}}(\lambda)$. The homomorphism $M' \to M$ of $\mathfrak{p}$-modules induces a homomorphism $U(\mathfrak{q}) \otimes_{U(\mathfrak{p})} M' \to M$, which is surjective because $U(\mathfrak{q})M' = M$. Thus, $M$ is a quotient of $U(\mathfrak{q}) \otimes_{U(\mathfrak{p})} L_{\mathfrak{p}}(\lambda)$, hence of $\Gamma_{\mathfrak{q},\mathfrak{p}}(L_{\mathfrak{p}}(\lambda))$. □

Lemma 6.3, (6.1) and Corollary 3.10 imply the following two corollaries.

**Corollary 6.4.** *If $\mathrm{Ext}^1(L(\lambda(f)), L(\lambda(g))) \neq 0$ and $\lambda(f) < \lambda(g)$, then there is a legal move of weight zero from $f$ to $g$.*

*Proof.* We have $\mathrm{Ext}^1(L_{\mathfrak{q}^{(k)}}(\lambda(f)), L_{\mathfrak{q}^{(k)}}(\lambda(g))) = 0$. So if $p$ is chosen minimal with

$$\mathrm{Ext}^1(L_{\mathfrak{q}^{(p)}}(\lambda(f)), L_{\mathfrak{q}^{(p)}}(\lambda(g))) = 0,$$

then by Lemma 6.3 with $\mathfrak{q} = \mathfrak{q}^{(p-1)}$ and $\mathfrak{p} = \mathfrak{q}^{(p)}$, $L_{\mathfrak{q}}(\lambda(g))$ is a subquotient of $\Gamma_{\mathfrak{q},\mathfrak{p}}(L_{\mathfrak{p}}(\lambda(f)))$. Hence the result follows from Corollary 3.10. □

**Corollary 6.5.** $\dim \mathrm{Ext}^1(L(\lambda(f)), L(\lambda(g))) \leq 1$.

*Proof.* There is at most one legal move joining $f$ and $g$. Indeed, if $g$ is obtained from $f$ by a legal move,then $g = f^a_b$, and we have

$$g(a) = \circ, f(a) = \times, g(b) = \times, f(b) = \circ,$$

and $f(s) = g(s)$ if $s \neq a, b$. In other words, $f$ and $g$ are different exactly in two positions which define the start and the end of a legal move. □

**Lemma 6.6.** *Let $g$ be obtained from $f$ by a legal move of weight zero with start $s_p$. Then*

$$\operatorname{Ext}^1_{\mathfrak{q}^{(p-1)}}(L_{\mathfrak{q}^{(p-1)}}(\lambda(g)), L_{\mathfrak{q}^{(p-1)}}(\lambda(f))) \neq 0.$$

*Proof.* To simplify notation we set

$$\Gamma^{(p)} = \Gamma_{\mathfrak{q}^{(p-1)},\mathfrak{q}^{(p)}}.$$

To construct a non-trivial extension consider the exact sequence from Lemmma 3.10 (b), and set

$$M := \frac{\Gamma^{(p)}(L_{\mathfrak{q}^{(p)}}(\lambda(f)))}{\bigoplus_{\substack{h\in\mathbf{LM}(f,p),\\ h\neq g}} L_{\mathfrak{q}^{(p-1)}}(\lambda(h))}.$$

Then $M$ is indecomposable and can be included as the middle term in the exact sequence

$$0 \to L_{\mathfrak{q}^{(p-1)}}(\lambda(g)) \to M \to L_{\mathfrak{q}^{(p-1)}}(\lambda(f)) \to 0.$$

$\square$

For $f$ as in (2.2), define $|f| = a_1 + \cdots + a_k$.

**Lemma 6.7.** *Let $g \in \mathbf{LM}(f,p), h \in \mathbf{LM}(f,r)$ and $p < r$.*

*(a)* $|f| - |g| \equiv 1 \bmod 2$.

*(b) $g$ and $h$ are not connected by a legal move of weight zero.*

*Proof.* (a) follows immediately from the definition of a legal move. For (b), assume the opposite. Since $h(s_p) = \times$ and $g(s_p) = \circ$ the only possibility is $g \in \mathbf{LM}(h,p)$, but this cannot happen by (a). $\square$

**Lemma 6.8.** *Let $g$ be obtained from $f$ by a legal move of weight zero. Then*

$$\operatorname{Ext}^1(L(\lambda(g)), L(\lambda(f))) \neq 0.$$

*Proof.* Let the legal move have start $s_p$. By Lemma 6.6 we have

$$\operatorname{Ext}^1_{\mathfrak{q}^{(p-1)}}(L_{\mathfrak{q}^{(p-1)}}(\lambda(g)), L_{\mathfrak{q}^{(p-1)}}(\lambda(f))) \neq 0.$$

We will prove

$$\operatorname{Ext}^1_{\mathfrak{q}^{(i)}}(L_{\mathfrak{q}^{(i)}}(\lambda(g)), L_{\mathfrak{q}^{(i)}}(\lambda(f))) \neq 0$$

for all $i \leq p-1$ by reverse induction in $i$. So we assume that the statement is true for $i$ and prove it for $i-1$. Consider a non-split exact sequence

$$0 \to L_{\mathfrak{q}^{(i)}}(\lambda(g)) \to V \to L_{\mathfrak{q}^{(i)}}(\lambda(f)) \to 0.$$

Apply $\Gamma^{(i)}$ to the sequence to obtain

$$0 \to \Gamma^{(i)}(L_{\mathfrak{q}^{(i)}}(\lambda(g))) \xrightarrow{\phi} \Gamma^{(i)}(V) \to \Gamma^{(i)}(L_{\mathfrak{q}^{(i)}}(\lambda(f))) \to 0.$$

This sequence is not exact, but $\Gamma^{(i)}$ is exact on the right. Moreover, $\Gamma^{(i)}(V)/\operatorname{Im}\phi$ is a quotient of $U(\mathfrak{q}^{(i-1)})\otimes_{U(\mathfrak{q}^{(i)})} L_{\mathfrak{q}^{(i)}}(\lambda(f))$, since by construction it is generated by the $\mathfrak{q}^{(i)}$-submodule $L_{\mathfrak{q}^{(i)}}(\lambda(f))$. Since $\Gamma^{(i)}(L_{\mathfrak{q}^{(i)}}(\lambda(f)))$ is the maximal finite-dimensional quotient of the parabolically induced module, we have an isomorphism

$$\Gamma^{(i)}(V)/\operatorname{Im}\phi \simeq \Gamma^{(i)}(L_{\mathfrak{q}^{(i)}}(\lambda(f))).$$

Thus, the above sequence is exact at the two right-most non-zero terms.

Now let $M$ and $N$ be the proper maximal submodules in $\Gamma^{(i)}(L_{\mathfrak{q}^{(i)}}(\lambda(g)))$ and $\Gamma^{(i)}(L_{\mathfrak{q}^{(i)}}(\lambda(f)))$ respectively. Let $X = \Gamma^{(i)}(V)/\phi(M)$. We have an exact sequence

$$0 \to L_{\mathfrak{q}^{(i-1)}}(\lambda(g)) \to X \xrightarrow{\pi} \Gamma^{(i)}(L_{\mathfrak{q}^{(i)}}(\lambda(f))) \to 0.$$

From Theorem 3.10 we have that

$$N = \bigoplus_{h\in\mathbf{LM}(f,i)} L_{\mathfrak{q}^{(i-1)}}(\lambda(h)).$$

By Lemma 6.7 and Corollary 6.4

$$\operatorname{Ext}^1_{\mathfrak{q}^{(i-1)}}(N, L_{\mathfrak{q}^{(i-1)}}(\lambda(g))) = 0.$$

Therefore

$$\pi^{-1}(N) = L_{\mathfrak{q}^{(i-1)}}(\lambda(g)) \oplus \bigoplus_{h\in\mathbf{LM}(f,i)} L_{\mathfrak{q}^{(i-1)}}(\lambda(h)).$$

So $X/(\bigoplus_{h\in\mathbf{LM}(f,i)} L_{\mathfrak{q}^{(i-1)}}(\lambda(h)))$ gives a non-trivial extension between $L_{\mathfrak{q}^{(i-1)}}(\lambda(g))$ and $L_{\mathfrak{q}^{(i-1)}}(\lambda(f))$. The case $i=0$ implies the statement. $\square$

Corollary 6.4, Corollary 6.5, Lemma 6.8 and (6.1) imply Theorem B.

# References


[Bru03] J. Brundan, *Kazhdan-Lusztig polynomials and character formulae for the Lie superalgebra $\mathfrak{gl}(m|n)$*, J. Amer. Math. Soc. **16** (2003), no. 1, 185–231 (electronic). MR1937204 (2003k:17007) ↑1, 2, 7, 13

[BS08a] J Brundan and C Stroppel, *Highest weight categories arising from Khovanov's diagram algebra I: cellularity*, arXiv:0806.1532 (2008). ↑2

[BS08b] ———, *Highest weight categories arising from Khovanov's diagram algebra II: Koszulity*, arXiv:0806.3472 (2008). ↑2

[BS08c] ———, *Highest weight categories arising from Khovanov's diagram algebra III: Category $\mathcal{O}$*, arXiv:0812.1090 (2008). ↑2

[BS08d] ———, *Highest weight categories arising from Khovanov's diagram algebra IV: the general linear supergroup*, arXiv:0907.2543 (2008). ↑2

[GW04] K. R. Goodearl and R. B. Warfield Jr., *An introduction to noncommutative Noetherian rings*, 2nd ed., London Mathematical Society Student Texts, vol. 61, Cambridge University Press, Cambridge, 2004. MR2080008 (2005b:16001) ↑20

[GS09] C. Gruson and V. Serganova, *Cohomology of generalized supergrassmannians and character formulae for basic classical Lie superalgebras*, arXiv:0906.0918v1 (2009). ↑2, 8

[Kac77] V. G. Kac, *Characters of typical representations of classical Lie superalgebras*, Comm. Algebra **5** (1977), no. 8, 889–897. MR0444725 (56 #3075) ↑1

[Kac78] ______, *Representations of classical Lie superalgebras*, Differential geometrical methods in mathematical physics, II (Proc. Conf., Univ. Bonn, Bonn, 1977), Lecture Notes in Math., vol. 676, Springer, Berlin, 1978, pp. 597–626. MR519631 (80f:17006) ↑1

[Mus93] I. M. Musson, *Primitive ideals in the enveloping algebra of the Lie superalgebra* $\mathrm{sl}(2,1)$, J. Algebra **159** (1993), no. 2, 306–331. MR1231215 (94g:17016) ↑20

[Ser96] V. Serganova, *Kazhdan-Lusztig polynomials and character formula for the Lie superalgebra* $\mathfrak{gl}(m|n)$, Selecta Math. (N.S.) **2** (1996), no. 4, 607–651. MR1443186 (98f:17007) ↑1, 2, 3, 7, 8, 10, 13, 14

[Ser98a] ______, *Characters of irreducible representations of simple Lie superalgebras*, Proceedings of the International Congress of Mathematicians, Vol. II (Berlin, 1998), 1998, pp. 583–593 (electronic). MR1648107 (99j:17010) ↑2

[Ser98b] ______, *Blocks in the category of finite dimensional representations of* $\mathfrak{gl}(m|n)$, *(preprint)* (1998). ↑8

[Sta99] R. P. Stanley, *Enumerative combinatorics. Vol. 2*, Cambridge Studies in Advanced Mathematics, vol. 62, Cambridge University Press, Cambridge, 1999. With a foreword by Gian-Carlo Rota and appendix 1 by Sergey Fomin. MR1676282 (2000k:05026) ↑5

[Su06] Y. Su, *Composition factors of Kac modules for the general linear Lie superalgebras*, Math. Z. **252** (2006), no. 4, 731–754. MR2206623 (2007b:17006) ↑5