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A Classification of Primitive Ideals in the Enveloping Algebra of a Classical Simple Lie Superalgebra

IAN M. MUSSON

University of Wisconsin-Milwaukee, Milwaukee, Wisconsin 53201

For a Lie superalgebra $\mathfrak{g}$ we denote the even and odd parts of $\mathfrak{g}$ by $\mathfrak{g}_0$ and $\mathfrak{g}_1$, respectively. The simple Lie superalgebra $\mathfrak{g}$ is called *classical* if $\mathfrak{g}_0$ is reductive. For $\mathfrak{g}$ classical simple we study primitive ideals in the enveloping algebra $U(\mathfrak{g})$. Our main result is that any graded primitive ideal is the annihilator of a graded simple quotient of a Verma module. This is an analogue of the well-known theorem of Duflo [D] on primitive ideals in the enveloping algebra of a semisimple Lie algebra. The proof is based on Duflo's theorem and some work of E. Letzter [L1, L2] on primitive ideals in finite ring extensions.

The definition of a Verma module depends on the existence of a triangular decomposition in $\mathfrak{g}$. This is dicussed in Section 1. A more precise statement of the main theorem is given Section 2. In Section 3 we discuss some corollaries, for example we show that if $\mathfrak{g} \neq Q(n)$ then graded prime ideals are prime (Corollary 3.1), and if $\mathfrak{g} \neq P(n)$, then any factor ring of $U(\mathfrak{g})$ has the same left and right Krull dimension (Corollary 3.3).

Classical simple Lie superalgebras which are not Lie algebras have been classified by Kac [K1, Theorem 2, p. 44] (see also [Sch, Theorem 1, p. 140]). In the notation of Kac these algebras are as follows. Scheunert's notation, if different is given in parentheses.

$$A(m, n) = sl(m+1, n+1), \qquad m \neq n,\ m, n \geqslant 0\ (spl(m+1, n+1))$$

$$A(n, n) = sl(n+1, n+1)/\langle I_{2n+2}\rangle, \qquad n > 0\ (spl(n+1, n+1)/\mathbb{C}I_{2n+2})$$

$$B(m, n) = osp(2m+1, 2n), \qquad m \geqslant 0, n > 0$$

$$D(m, n) = osp(2m, 2n), \qquad m \geqslant 2, n > 0$$

$$C(n) = osp(2, 2n-2), \qquad n \geqslant 2$$

$$G(3), F(4) \qquad (\Gamma_2, \Gamma_3, \text{respectively})$$

$$D(2, 1; \alpha) \qquad \alpha \in \mathbb{C}\backslash\{0, -1\}\ ((\Gamma(1, -1-\alpha, \alpha)))$$

$$P(n) \qquad n \geqslant 2\ (b(n+1))$$

$$Q(n) = \tilde{Q}(n)/\langle I_{2n+2}\rangle, \qquad n \geqslant 2$$

$(d(n+1)/\mathbb{C}I_{2n+2}$, the f, d algebras of Gell–Mann, Michel, Radicati).

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We refer to [K1, 2.1] or [Sch, Chap. II, Sect. 4] for the construction and properties of these Lie superalgebras.

The superalgebras $P(n)$ and $Q(n)$ are called strange and the others are known as basic classical Lie superalgebras.

Throughout this paper the adjective graded refers to the $\mathbb{Z}_2$-grading on $U(\mathfrak{g})$. If $M = M_0 \oplus M_1$ and $N = N_0 \oplus N_1$ are graded $U(\mathfrak{g})$-modules, a graded homomorphism $\phi: M \to N$ is a module homomorphism such that for some $j \in \mathbb{Z}_2$, $\phi(M_i) \subseteq N_{i+j}$. In particular this means that M is isomorphic to the module M' defined by reversing the grading, that is, by setting $(M')_0 = M_1$ and $(M')_1 = M_0$. An element of M is homogeneous if it is contained in $M_0 \cup M_1$.

1. Triangular Decompositions and Verma Modules

1.1. We need to know that every classical simple Lie superalgebra $\mathfrak{g}$ has a *triangular decomposition*. By this we mean that there is a direct sum decomposition

$$\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$$

such that

(1) $\mathfrak{n}^-$, $\mathfrak{n}^+$, and $\mathfrak{h}$ are graded subalgebras of $\mathfrak{g}$ with $\mathfrak{n}^\pm$ nilpotent.

(2) $\mathfrak{g}_0 = \mathfrak{n}_0^- \oplus \mathfrak{h}_0 \oplus \mathfrak{n}_0^+$ is a triangular decomposition of $\mathfrak{g}_0$ in the usual sense (see [Dix, 1.10.14]).

(3) $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}^+$ is a solvable subalgebra of $\mathfrak{g}$.

If M is an $\mathfrak{h}_0$-module and $\alpha \in \mathfrak{h}_0^*$ we define $M^\alpha = \{x \in M \mid hx = \alpha(h)x$ for all $h \in \mathfrak{h}_0\}$. Note that, if M is a graded $\mathfrak{h}_0$-module, then M^α is a graded subspace of M. We say M is *diagonalizable* if $M = \bigoplus M^\alpha$. We also require

(4) $\mathfrak{n}_0^\pm$, $\mathfrak{n}_1^\pm$, $\mathfrak{h}_0$, and $\mathfrak{h}_1$ are diagonalizable $\mathfrak{h}_0$-modules via the adjoint action of $\mathfrak{h}_0$.

We shall see that every classical simple Lie superalgebra has a triangular decomposition. In contrast to the Lie algebra case, triangular decompositions are not in general unique up to an automorphism of $\mathfrak{g}$. In applications of the main theorem it is convenient to allow different triangular decompositions of $\mathfrak{g}$. For example if $\mathfrak{g} = A(n, n)$ we use different triangular decompositions of $\mathfrak{g}$ in Corollary 3.2 and Theorem 3.4.

Suppose we are given a triangular decomposition $\mathfrak{g} = n^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$. As in [J, 4.3] we denote by $\mathcal{O}$ the category of $\mathfrak{g}_0$-modules M with the following properties

(a) $M = \bigoplus_{\mu \in \mathscr{h}_0^*} M^{\mu}$.

(b) For all $v \in M$, $\dim U(n_0^+)v < \infty$.

(c) M is a finitely generated $U(\mathscr{g}_0)$-module.

In addition we let $\tilde{\mathscr{O}}$ denote the category of graded $\mathscr{g}$-modules M which belong to $\mathscr{O}$ when regarded as $\mathscr{g}_0$-modules by restriction. The morphisms in $\tilde{\mathscr{O}}$ are graded homomorphisms of $\mathscr{g}$-modules.

For $\lambda \in \mathscr{h}_0^*$, we let $\mathbb{C}v_\lambda$ be the one dimensional $\mathscr{b}_0$-module with $\mathscr{n}_0^+ v_\lambda = 0$ and $hv_\lambda = \lambda(h)v_\lambda$ for $h \in \mathscr{h}_0$. We show below that there is a unique finite dimensional graded simple $\mathscr{b}$-module V_λ such that $\mathscr{n}^+ V_\lambda = 0$ and $hv = \lambda(h)v$ for all $h \in \mathscr{h}_0$, $v \in V_\lambda$. Furthermore any finite dimensional graded simple $\mathscr{b}$-module is isomorphic to V_λ for some $\lambda \in \mathscr{h}_0^*$.

In all cases except $\mathscr{g} = Q(n)$ we will have $\mathscr{h} = \mathscr{h}_0$ and $\dim V_\lambda = 1$. We define Verma modules for $\mathscr{g}_0$ and $\mathscr{g}$ by

$$M(\lambda) = U(\mathscr{g}_0) \otimes_{U(\mathscr{b}_0)} \mathbb{C}v_\lambda$$

$$\tilde{M}(\lambda) = U(\mathscr{g}) \otimes_{U(\mathscr{b})} V_\lambda.$$

Then $M(\lambda)$ has a unique maximal submodule $M(\lambda)^0$ and if $L(\lambda) = M(\lambda)/M(\lambda)^0$, then $M(\lambda)$ has a finite composition series with factors $L(\mu)$ for various $\mu \in \mathscr{h}_0^*$ [Dix, 7.1.11 and 7.6.1]. We establish similar properties for $\tilde{M}(\lambda)$:

PROPOSITION. (a) *$\tilde{M}(\lambda)$ has a composition series of finite length.*

(b) *$\tilde{M}(\lambda)$ has a unique maximal graded submodule $\tilde{M}(\lambda)^0$.*

(c) *If $\mathscr{g} \neq Q(n)$, then $\tilde{M}(\lambda)^0$ is actually the unique maximal submodule of $\tilde{M}(\lambda)$.*

(d) *If M is a nonzero module in $\tilde{\mathscr{O}}$ then, for some $\lambda \in \mathscr{h}_0^*$, $\operatorname{Hom}(\tilde{M}(\lambda), M) \neq 0$.*

The proof is given later in this section.

We set $\tilde{L}(\lambda) = \tilde{M}(\lambda)/\tilde{M}(\lambda)^0$.

COROLLARY. (a) *A graded $\mathscr{g}$-module M belongs to $\tilde{\mathscr{O}}$ if and only if M is diagonalizable as an $\mathscr{h}_0$-module and there is a finite series $0 = M_0 \subset M_1 \subset \cdots \subset M_r = M$ of graded submodules of M such that each M_i/M_{i-1} is a homomorphic image of some $\tilde{M}(\lambda_i)$ for $1 \leqslant i \leqslant r$.*

(b) *The Verma module $\tilde{M}(\lambda)$ has a finite graded composition series with factors isomorphic to $\tilde{L}(\mu)$ for various $\mu \in \mathscr{h}_0^*$.*

Proof of the Corollary. (a) We note that the category $\tilde{\mathscr{O}}$ is closed under taking graded homomorphic images and submodules since $\mathscr{O}$ is closed under these operations. If $M \in \tilde{\mathcal{O}}$, we can construct a series of the required form using (d) of the Proposition and the fact that M is a Noetherian $U(\mathfrak{g})$-module. For the converse it suffices to show that $\tilde{M}(\lambda) \in \tilde{\mathcal{O}}$, but this is traightforward (see the remark after Lemma 1.3).

(b) This now follows easily.

1.2. Let $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ be any Lie superalgebra. If V is a $\mathfrak{g}_0$-module the exterior algebra on V is denoted ΛV. The action of $\mathfrak{g}_0$ on V extends to a homomorphism $\mathfrak{g}_0 \to \mathrm{Der}(\Lambda V)$. Thus we can view ΛV as a $\mathfrak{g}_0$-module. We need the following observation.

LEMMA. *As $\mathfrak{g}_0$-modules via the adjoint action $U(\mathfrak{g}) \cong \Lambda \mathfrak{g}_1 \otimes_{U(\mathfrak{g}_0)} U(\mathfrak{g}_0)$.*

The proof is elementary.

1.3. Let $\mathfrak{g}_0 = \mathfrak{n}_0^- \oplus \mathfrak{h}_0 \oplus \mathfrak{n}_0^+$ be a triangular decomposition of $\mathfrak{g}_0$, R_0 the set of roots of $\mathfrak{g}_0$, and R_0^+ the set of positive roots with respect to this decomposition. Write $\mathfrak{h}_0 = \mathfrak{h}_0' \oplus \mathfrak{z}$ where $\mathfrak{h}_0'$ is a Cartan subalgebra of $[\mathfrak{g}_0, \mathfrak{g}_0]$ and $\mathfrak{z}$ is the center of $\mathfrak{g}_0$. The Weyl group W of $[\mathfrak{g}_0, \mathfrak{g}_0]$ acts naturally on $(\mathfrak{h}_0')^*$ which we identify with $\{\lambda \in \mathfrak{h}_0^* \mid \lambda(\mathfrak{z}) = 0\}$. We extend this action to an action of W on $\mathfrak{h}_0^*$ by requiring W to fix the set $\{\lambda \in \mathfrak{h}_0^* \mid \lambda(\mathfrak{h}_0') = 0\}$. Define the translated action $w \cdot \lambda$ by

$$w \cdot \lambda = w(\lambda + \rho_0) - \rho_0,$$

where $\rho_0 = \frac{1}{2} \sum_{\alpha \in R_0^+} \alpha$. Since $\mathfrak{g}$ is classical simple, $\mathfrak{g}_1$ is a semisimple $\mathfrak{g}_0$-module, by [Sch, Theorem 1, p. 101]. Hence $\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in R} \mathfrak{g}^\alpha$, where $R = \{\alpha \in \mathfrak{h}_0^* \mid \mathfrak{g}^\alpha \neq 0 \text{ and } \alpha \neq 0\}$ is the set of roots of $\mathfrak{g}$ and $\mathfrak{h}$ is the centralizer of $\mathfrak{h}_0$. We set $Q = \mathbb{Z}R$, $Q_0 = \mathbb{Z}R_0$, and $Q_0^+ = \mathbb{N}R_0^+$. The set of weights of a $\mathfrak{g}_0$-module V is denoted by $\Pi(V)$.

Part (a) of Proposition 1.1 follows from the next result which will also give more detailed information on the relationship between primitive ideals in $U(\mathfrak{g}_0)$ and primitive ideals in $U(\mathfrak{g})$.

LEMMA. *Suppose $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$ is a triangular decomposition, and V is a finite dimensional $\mathfrak{h}$-module such that $\mathfrak{n}^+ V = 0$ and $hv = \lambda(h)v$ for all $h \in \mathfrak{h}_0$, $v \in V$, where $\lambda \in \mathfrak{h}_0^*$. Then the modules $\tilde{M} = U(\mathfrak{g}) \otimes_{U(\mathfrak{h})} V$ and $U(\mathfrak{g}) \otimes_{U(\mathfrak{g}_0)} M(\lambda)$ have finite length as $U(\mathfrak{g}_0)$-modules with composition factors of the form $L(\nu)$ where*

$$\nu \in \bigcup_{\mu \in \Pi(\Lambda \mathfrak{g}_1)} (W \cdot (\lambda + \mu) \cap (\lambda + \mu - Q_0^+)).$$

Proof. It is enough to prove the statement about $U(\mathfrak{g}) \otimes_{U(\mathfrak{g}_0)} M(\lambda)$ since $\tilde{M}$ is a factor module of $N = U(\mathfrak{g}) \otimes_{U(\mathfrak{b}_0)} V$ where V is a $\mathfrak{b}_0$-module by restriction and N is a direct sum of copies of

$$U(\mathfrak{g}) \otimes_{U(\mathfrak{b}_0)} \mathbb{C}v_\lambda = U(\mathfrak{g}) \otimes_{U(\mathfrak{g}_0)} M(\lambda).$$

Now as a $U(\mathfrak{g}_0)$-module, $U(\mathfrak{g}) \otimes_{U(\mathfrak{g}_0)} M(\lambda) \cong \Lambda\mathfrak{g}_1 \otimes_{U(\mathfrak{g}_0)} M(\lambda)$ by Lemma 1.2, and this has a finite series with factors of the form $M(\lambda + \mu)$ where $\mu \in \Pi(\Lambda\mathfrak{g}_1)$ by [Dix, 7.6.14]. Therefore the result follows from [Dix, 7.6.1].

Remark. Since the category $\mathcal{O}$ is closed under tensor products with finite dimensional modules, it is clear that the modules $\tilde{M}$ and $U(\mathfrak{g}) \otimes_{U(\mathfrak{g}_0)} M(\lambda)$ belong to $\tilde{\mathcal{O}}$.

1.4. Let $\mathfrak{g}$ be a classical simple Lie superalgebra. We say that the elements $\beta_1, \ldots, \beta_n$ of $\mathfrak{h}_0^*$ form a basis of simple roots for $\mathfrak{g}$ if

(1) $\beta_1, \ldots, \beta_n$ are linearly independent.

(2) For any root $\beta \in R$ we have either $\beta \in Q^+$ or $-\beta \in Q^+$ where $Q^+ = \sum_{i=1}^n \mathbb{N}\beta_i$.

When $\mathfrak{g}$ has a basis of simple roots, we set $\mathfrak{n}^+ = \bigoplus_{\alpha \in Q^+ \setminus \{0\}} \mathfrak{g}^\alpha$, $\mathfrak{n}^- = \bigoplus_{-\alpha \in Q^+ \setminus \{0\}} \mathfrak{g}^\alpha$, and let $\mathfrak{h}$ be the centralizer of $\mathfrak{h}_0$ in $\mathfrak{g}$.

If M is a module in $\tilde{\mathcal{O}}$, and $v \in M^\mu$ then v is a highest weight vector if $\mathfrak{n}^+ v = 0$. In this case μ is a *highest weight* of M.

LEMMA. *Suppose $\mathfrak{g}$ has a basis of simple roots. Then*

(a) $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$ *is a triangular decomposition.*

(b) *Any nonzero module in $\tilde{\mathcal{O}}$ has a highest weight. If M is a simple module in $\tilde{\mathcal{O}}$, then M has a unique highest weight.*

(c) *Suppose in addition that $\mathfrak{h} = \mathfrak{h}_0$. For $\lambda \in \mathfrak{h}_0^*$, let $V_\lambda = \mathbb{C}v_\lambda$ be the one dimensional graded $\mathfrak{b}$-module with v_λ homogeneous, $\mathfrak{n}^+ v_\lambda = 0$, and $hv_\lambda = \lambda(h)v_\lambda$ for all $h \in \mathfrak{h}_0$. Then any finite dimensional graded simple $\mathfrak{b}$-module is isomorphic to V_λ for some λ. Furthermore the Verma module $\tilde{M}(\lambda) = U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} V_\lambda$ satisfies properties (a)–(d) of Proposition 1.1.*

Proof. (a) Note that $[\mathfrak{h}_1, \mathfrak{h}_1] \subseteq \mathfrak{h}_0$ which is central in $\mathfrak{h}$. Therefore $\mathfrak{h}$ is nilpotent. Clearly $\mathfrak{n}^+$ is a nilpotent ideal in $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}^+$, so $\mathfrak{b}$ is solvable.

Now $\mathfrak{h}_0 = \mathfrak{h}_0' \oplus \mathfrak{z}$ where $\mathfrak{h}_0'$ is a Cartan subalgebra of $[\mathfrak{g}_0, \mathfrak{g}_0]$ and $\mathfrak{z}$ is the center of $\mathfrak{g}_0$. Let E be the real vector space spanned by $\beta_1, \ldots, \beta_n$ and $E_1 = \{\beta \in E \mid \beta(\mathfrak{z}) = 0\}$, $E_2 = \{\beta \in E \mid \beta(\mathfrak{h}_0') = 0\}$, so that $E = E_1 \oplus E_2$. For $\beta \in E_1$, let t_β be the element of $\mathfrak{h}_0'$ such that $\beta(h) = K(t_\beta, h)$ for $h \in \mathfrak{h}_0'$ where $K(\ ,\ )$ is the Killing form, and set $(\alpha, \beta) = K(t_\alpha, t_\beta)$ for $\alpha, \beta \in E_1$. We extend

$(\ ,\ )$ to a nondegenerate symmetric bilinear form on E such that E_1 and E_2 are orthogonal. Then there exists $\gamma \in E$ such that $(\beta_i, \gamma) > 0$ for $i = 1, ..., n$. Let $R_0^+ = \{\alpha \in (\mathfrak{h}_0')^* \mid \alpha$ is a root of $\mathfrak{g}_0$ with $(\alpha, \gamma) > 0\}$ and note that R_0^+ is precisely the set of roots of $\mathfrak{n}_0^+$. Write $\gamma = \gamma_1 + \gamma_2$ where $\gamma_i \in E_i$. Since $R_0^+ \subseteq E_1$ we have $(\alpha, \gamma) = (\alpha, \gamma_1) > 0$ for $\alpha \in R_0^+$. By the proof of [H, Theorem 10.1] the set of indecomposable roots in R_0^+ is a basis of simple roots of $\mathfrak{g}_0$. Hence $\mathfrak{g}_0 = \mathfrak{n}_0^- \oplus \mathfrak{h} \oplus \mathfrak{n}_0^+$ is a triangular decomposition. The rest now follows easily.

(b) This is the same proof as for semisimple Lie algebras.

(c) Note that $[\mathfrak{b}_1, \mathfrak{b}_1] = [\mathfrak{n}_1, \mathfrak{n}_1] \subseteq \mathfrak{n}_0^+ = [\mathfrak{h}, \mathfrak{n}_0^+] \subseteq [\mathfrak{b}_0, \mathfrak{b}_0]$. Therefore by [K1, Proposition 5.2.4] every finite dimensional graded simple $\mathfrak{b}$-module has dimension one, and so is annihilated by $[\mathfrak{b}, \mathfrak{b}] \supseteq \mathfrak{n}^+$. It follows that any such module is isomorphic to V_λ for some λ. As in the proof of [Dix, 7.1.11], every proper $\mathfrak{g}$-submodule of $\tilde{M}(\lambda)$ is contained in $M^+ = \bigoplus_{\mu \in \lambda - Q^+} \tilde{M}(\lambda)^\mu$. Note that M^+ is a graded subspace of $\tilde{M}(\lambda)$. It follows that $\tilde{M}(\lambda)$ has a unique maximal submodule which is graded. The rest follows from (b) and Lemma 1.3.

1.5. *Contragredient Lie Superalgebras.* Our definition of contragredient Lie superalgebras is the same as that given in [vdL1] (see also [K2]).

Let $A = (a_{ij})$ be an $n \times n$ matrix of rank l with complex entries. A *realization* of A consists of a complex vector space $\mathfrak{h}$ together with subsets $\Pi = \{\alpha_1, ..., \alpha_n\} \subseteq \mathfrak{h}^*$ and $\Pi^\vee = \{h_1, ..., h_n\} \subseteq \mathfrak{h}$ such that

(1) The sets Π and $\Pi^\vee$ are linearly independent.

(2) $\langle h_i, \alpha_j \rangle = a_{ij}$ $(i, j = 1, ..., n)$.

(3) $\dim \mathfrak{h} = 2n - l$.

By [K2, Proposition 1.1], every $n \times n$ matrix has a realization which is unique up to isomorphism.

Given a matrix A, a realization as above, and a subset τ of $I = \{1, 2, ..., n\}$ we define the Lie superalgebra $\tilde{\mathfrak{g}}(A, \tau)$ to have generators e_i, f_i $(i = 1, ..., n)$, and $\mathfrak{h}$ and defining relations

$$\begin{aligned}
&[e_i, f_j] = \delta_{ij} h_i && i, j = 1, ..., n\\
&[h, h'] = 0 && \text{for} \quad h, h' \in \mathfrak{h}\\
&[h, e_i] = \langle \alpha_i, h \rangle e_i &&\\
&[h, f_i] = -\langle \alpha_i, h \rangle f_i. &&
\end{aligned}$$

The $\mathbb{Z}_2$-grading on $\tilde{\mathfrak{g}}(A, \tau)$ is given by

$$\begin{aligned}
&\deg h_i = 0 \qquad \deg e_i = \deg f_i = 0 && \text{for} \quad i \notin \tau\\
& \deg e_i = \deg f_i = 1 && \text{for} \quad i \in \tau.
\end{aligned}$$

Now there exists a unique maximal ideal $\imath$ of $\tilde{\mathfrak{g}}(A, \tau)$ intersecting $\mathfrak{h}$ trivially. The proof of this statement is the same as for Lie algebras [K2, Theorem 1.2e)].

Set $\mathfrak{g}(A, \tau) = \tilde{\mathfrak{g}}(A, \tau)/\imath$. We call $\mathfrak{g}(A, \tau)$ the *contragredient Lie superalgebra associated to* (A, τ). It should be noted that the contragredient Lie superalgebra $G(A, \tau)$ of [K1, Sect. 2.5] is isomorphic to the derived algebra $\mathfrak{g}'$ of $\mathfrak{g}(A, \tau)$. The pairs (A, τ) for which $\mathfrak{g}(A, \tau)$ is finite dimensional and $\mathfrak{g}'/C$ is simple, where C is the center of $\mathfrak{g}'$, are classified in [K1, Theorem 3] (see also [vdL2, Sect. 5]). The algebras $\mathfrak{g}'/C$ which arise are exactly the basic classical simple Lie superalgebras. We have $\dim \mathfrak{g}(A, \tau)/\mathfrak{g}' = \dim C = n - l$. Also $C \neq 0$ if and only if $\det A = 0$ and this occurs exactly when $\mathfrak{g}'/C \cong A(n, n)$ (see [K1, Proposition 2.5.6]). Henceforth we assume that $\det A \neq 0$, so $\mathfrak{g}(A, \tau) = \mathfrak{g}$ is classical simple.

Since $\det A \neq 0$, $\alpha_1, ..., \alpha_n$ forms a basis for $\mathfrak{h}^*$. Moreover if $\mathfrak{n}^+$ (resp. $\mathfrak{n}^-$) is the subalgebra of $\mathfrak{g}$ generated by $e_1, ..., e_n$ (resp. $f_1, ..., f_n$) then $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$ is a triangular decomposition. Also if $Q^+ = \sum_{i=1}^n \mathbb{N}\alpha_i$ then $\mathfrak{n}^+ = \bigoplus_{\alpha \in Q^+ \setminus \{0\}} \mathfrak{g}^\alpha$ and $\mathfrak{n}^- = \bigoplus_{-\alpha \in Q^+ \setminus \{0\}} \mathfrak{g}^\alpha$. The proofs of these statements are the same as for Kac–Moody Lie algebras (see [K2, Sect. 1.3]). Hence $\mathfrak{g}$ has a basis of simple roots and $\mathfrak{h} = \mathfrak{h}_0$ so we obtain Verma modules $\tilde{M}(\lambda)$ satisfying properties (a)–(d) of Proposition 1.1.

1.6. The advantage of using contragredient Lie superalgebras is that it provides a unified way in which the existence of a basis for the simple roots can be established. However, it is difficult to examine the structure of the algebras $\mathfrak{g}(A, \tau)$ directly from the definition. Fortunately it is possible to give direct constructions for the basic Lie superalgebras [K1, Sect. 2.1; Sch, Chap. II, Sect. 4] and then show the existence of a basis via a case by case examination. A list of possible bases, up to W-equivalence is given in [K1, 2.5.4]. Some omissions are corrected in [vdL2, Sect. 5].

We take this opportunity to correct an error in [K1, 2.5.4]. For the case $\mathfrak{g} = A(n, n) = sl(n+1, n+1)/\langle I_{2n+2} \rangle$ the set of roots which are claimed to be a basis are not in fact linearly independent. However, we have the following.

LEMMA. *The Lie superalgebra $\mathfrak{g} = A(n, n)$ has a basis of simple roots. Furthermore if $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$ is the corresponding triangular decomposition, and $x \to {}^t x$ is the antiautomorphism of $\mathfrak{g}$ induced by sending every matrix in $sl(n+1, n+1)$ to its transpose then*

$$ {}^t\mathfrak{n}^- = \mathfrak{n}^+, \qquad {}^t\mathfrak{n}^+ = \mathfrak{n}^-, \qquad {}^t h = h \qquad \textit{for all} \quad h \in \mathfrak{h}. $$

Proof. First we introduce some notation. Let V be the space of all column vectors with standard basis e_i, $1 \leqslant i \leqslant 2n+2$, and e_{ij} the matrix with $e_{ij}e_k = \delta_{jk}e_i$. For $1 \leqslant i \leqslant n$ let $h_i = e_{ii} - e_{i+1\,i+1}$ and for $n+1 \leqslant i \leqslant 2n$, $h_i = e_{i+1\,i+1} - e_{i+2\,i+2}$. We denote the images of the elements h_i, e_{ij} in $\mathfrak{g}$ by the same symbol. Let $\mathfrak{h} = \mathfrak{h}_0$ be the subalgebra of $\mathfrak{g}$ spanned by the h_i. Let λ_i, $1 \leqslant i \leqslant 2n$, be the dual basis of $\mathfrak{h}^*$ defined by $\lambda_i(h_j) = \delta_{ij}$. For $1 \leqslant i \leqslant n$ set $x_i = e_{i,n+1+i}$ and set $x_0 = e_{2(n+1),n+1}$. Let $\mu_i = -\lambda_{i-1} + \lambda_i + \lambda_{n+i-1} - \lambda_{n+i}$ for $2 \leqslant i \leqslant n$, $\mu_0 = \lambda_n - \lambda_{2n}$, and $\mu_1 = \lambda_1 - \lambda_{n+1}$. Note that $x_i \in \mathfrak{g}^{\mu_i}$ for $0 \leqslant i \leqslant n$. The crucial point of the proof is the observation that $\mu_0 = \sum_{i=1}^n \mu_i$.

If $F: 0 = U_0 \subset U_1 \subset \cdots \subset U_{2n+2} = U$ is a flag in V we set

$$\mathfrak{n}(F) = \{x + \langle I_{2n+2} \rangle \mid x \in sl(n+1, n+1),\ xU_i \subseteq U_{i-1},\ 1 \leqslant i \leqslant 2n+2\}.$$

Set $v_0 = e_{2(n+1)}$, $v_i = e_i$ for $1 \leqslant i \leqslant n$, and $w_i = e_{n+1+i}$ for $0 \leqslant i \leqslant n+1$. Now let

$$V_i = \operatorname{span}\{v_0, w_0, v_1, ..., w_{i-1}, v_i\}, \qquad W_i = \operatorname{span}\{v_0, w_0, ..., v_i, w_i\}$$

$$V_i' = \operatorname{span}\{w_n, v_n, ..., w_{n-i}, v_{n-i}\}, \qquad W_i' = \operatorname{span}\{w_n, v_n, ..., v_{n-i+1}, w_{n-i}\}$$

and consider the flags

$$F^+ : 0 \subset V_0 \subset W_0 \subset \cdots V_i \subset W_i \subset V_{i+1} \subset \cdots \subset W_n = V$$

$$F^- : 0 \subset W_0' \subset V_0' \subset \cdots W_i' \subset V_i' \subset W_{i+1}' \subset \cdots \subset V_n' = V.$$

We set $\mathfrak{n}^{\pm} = \mathfrak{n}(F^{\pm})$, and $R^{\pm} = \{\alpha \in \mathfrak{h}^* \mid (\mathfrak{n}^{\pm})^{\alpha} \neq 0\}$. It is easy to see that $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$ is a triangular decomposition of $\mathfrak{g}$. Note that $x_i w_j = \delta_{ij} v_j$ and $x_i v_j = 0$ for $0 \leqslant i \leqslant n$, $0 \leqslant j \leqslant n+1$, so $x_i \in \mathfrak{n}^+$. We set $y_i = e_{n+i,i}$ for $1 \leqslant i \leqslant n$. Then $y_i v_j = \delta_{ij} w_{i-1}$, $y_i w_j = 0$, and $y_i \in \mathfrak{g}^{\nu_i}$ for certain $\nu_i \in \mathfrak{h}^*$. Then $\mathfrak{n}^+$is generated by $x_0, x_1, ..., x_n$ and $y_1, ..., y_n$ and $\mathfrak{n}^-$ is generated by ${}^t x_0$, ${}^t x_1, ..., {}^t x_n$, ${}^t y_1, ..., {}^t y_n$. Hence any $\alpha \in R^+$ (resp. R^-) can be expressed as an integral linear combination of $\mu_0, \mu_1, ..., \mu_n, \nu_1, ..., \nu_n$ with all coefficients nonnegative (resp. nonpositive). Since $\mu_0 = \sum_{i=1}^n \mu_i$, α can be expressed as such a linear combination of $\mu_1, ..., \mu_n, \nu_1, ..., \nu_n$. It is new easy to show that $\{\mu_1, ..., \mu_n, \nu_1, ..., \nu_n\}$ is a basis of simple roots of $\mathfrak{g}$ with the required property.

1.7. Suppose now that $\mathfrak{g}_1 = \mathfrak{g}_1^+ + \mathfrak{g}_1^-$ is a sum of two proper $\mathfrak{g}_0$-submodules. This applies when $\mathfrak{g} = A(m, n)$, $C(n)$, or $P(n)$ [K1, Proposition 2.1.2]. Of course the choice of which submodule to call $\mathfrak{g}_1^+$ is arbitrary. By checking each case, or using [Sch, Proposition 3, p. 96] we have $\mathfrak{g}_1 = \mathfrak{g}_1^+ \oplus \mathfrak{g}_1^-$, $[\mathfrak{g}_1^+, \mathfrak{g}_1^+] = [\mathfrak{g}_1^-, \mathfrak{g}_1^-] = 0$, and $\mathfrak{g}_1^+$, $\mathfrak{g}_1^-$ are simple $\mathfrak{g}_0$-modules. Let $\mathfrak{g}_0 = \mathfrak{n}_0^- \oplus \mathfrak{h}_0 \oplus \mathfrak{n}_0^+$ be a triangular decomposition of $\mathfrak{g}_0$ and set $\mathfrak{n}^- = \mathfrak{n}_0^- \oplus \mathfrak{g}_1^-$, $\mathfrak{n}^+ = \mathfrak{n}_0^+ \oplus \mathfrak{g}_1^+$, $\mathfrak{h} = \mathfrak{h}_0$, and $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}^+$. For $\lambda \in \mathfrak{h}^*$, let $V_\lambda = \mathbb{C} v_\lambda$ be the one-dimensional graded $\mathfrak{b}$-module with v_λ

homogeneous, $\mathfrak{n}^+ v_\lambda = 0$, and $hv_\lambda = \lambda(h)v_\lambda$ for $h \in \mathfrak{h}_0$. Note that $[\mathfrak{b}_1, \mathfrak{b}_1] = [\mathfrak{g}_1^+, \mathfrak{g}_1^+] = 0$ so by [K1, Proposition 5.2.4] any finite dimensional graded simple $\mathfrak{b}$-module has dimension one and hence is isomorphic to V_λ for some λ. We set $\tilde{M}(\lambda) = U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} V_\lambda$.

LEMMA. (a) $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$ *is a triangular decompsition.*

(b) $I = \mathfrak{g}_1^- U(\mathfrak{n}^-) = U(\mathfrak{n}^-)\mathfrak{g}_1^-$ *is a nilpotent ideal of* $U(\mathfrak{n}^-)$ *with* $U(\mathfrak{n}^-)/I \cong U(\mathfrak{n}_0^-)$.

(c) *Properties* (a)–(d) *of Proposition* 1.1 *hold for* $\tilde{M}(\lambda)$ *and* $\tilde{L}(\lambda)$ *is a simple* $U(\mathfrak{g})$*-module.*

Proof. (a) Set $\mathfrak{g}_1^+(0) = 0$ and define $\mathfrak{g}_1^+(i)$ inductively by $\mathfrak{g}_1^+(i) = \{x \in \mathfrak{g}_1^+ \mid [x, \mathfrak{n}_0^+] \subseteq \mathfrak{g}_1^+(i-1)\}$ for $i \geqslant 1$. Since $\mathfrak{g}_1^+$ is a simple $\mathfrak{g}_0$-module and $\mathfrak{g}_0$ is reductive we see that $\mathfrak{g}_1^+(m) = \mathfrak{g}_1^+$ for some m. Since $\mathfrak{n}_0^+$ is nilpotent and $[\mathfrak{g}_1^+, \mathfrak{g}_1^+] = 0$, it follows that $\mathfrak{n}^+$ is nilpotent. The rest is easy.

(b) Let $x \in \mathfrak{g}_1^-$, $y \in \mathfrak{n}_0^-$. Then $xy = yx + [x, y] \in U(\mathfrak{n}^-)\mathfrak{g}_1^-$. Since also $[x, \mathfrak{g}_1^-] = 0$, we see that $\mathfrak{g}_1^- U(n^-) = U(n^-)\mathfrak{g}_1^- = I$ is an ideal of $U(\mathfrak{n}^-)$.

Since $[\mathfrak{g}_1^-, \mathfrak{g}_1^-] = 0$ we have $(\mathfrak{g}_1^-)^m = 0$ in $U(\mathfrak{n}^-)$ if $m > \dim \mathfrak{g}_1$, so I is nilpotent. By the PBW theorem $I + U(\mathfrak{n}_0^-) = U(\mathfrak{n}^-)$ and $I \cap U(\mathfrak{n}_0^-) = 0$ so the rest follows.

(c) We have $\tilde{M}(\lambda) = U(\mathfrak{n}^-)v_\lambda = (I \oplus U(\mathfrak{n}_0^-))v_\lambda$. Let $\Gamma = -Q_0^+ \setminus \{0\}$. We show that if N is any proper submodule (not necessarily graded) of $\tilde{M}(\lambda)$ then $N \subseteq M^+ = (I \oplus \bigoplus_{\mu \in \Gamma} U(\mathfrak{n}_0^-)^\mu)v_\lambda$. Since $N = \bigoplus N^\mu$ and $N^\mu \subseteq M^+$ if $\mu \neq \lambda$, we may assume $N^\lambda \neq 0$. By Lemma 1.2, $U(\mathfrak{n}^-) \cong \Lambda\mathfrak{g}_1^- \otimes U(\mathfrak{n}_0^-)$ as an $\mathfrak{h}$-module and so $U(\mathfrak{n}^-)^\alpha = \bigoplus_{\alpha = \beta + \gamma} (\Lambda\mathfrak{g}_1^-)^\beta \otimes U(\mathfrak{n}_0^-)^\gamma$. Therefore $T = U(\mathfrak{n}^-)^0 = \{x \in U(\mathfrak{n}^-) \mid [h, x] = 0 \text{ for all } h \in \mathfrak{h}\}$ is a finite dimensional subalgebra of $U(\mathfrak{n}^-)$. Now $J = T \cap I$ is a nilpotent ideal of T and $T/J \cong (T+I)/I \subseteq U(\mathfrak{n}^-)/I \cong U(\mathfrak{n}_0^-)$ so $T/J \cong \mathbb{C}$. Therefore T is a local ring whose maximal ideal J is maximal as a left ideal. Now N^λ is a T-submodule of $\tilde{M}(\lambda)^\lambda$ and $\tilde{M}(\lambda)^\lambda = Tv_\lambda = (\mathbb{C} + J)v_\lambda$ is a free left T-module. Therefore any proper T-submodule of $\tilde{M}(\lambda)^\lambda$ is contained in $Jv_\lambda \subseteq M^+$. It follows that $N \subseteq M^+$, and $\tilde{M}(\lambda)$ has a unique graded simple quotient $\tilde{L}(\lambda)$ which is a simple $U(\mathfrak{g})$-module. Finally, if M is a nonzero module in $\tilde{\mathcal{O}}$ we can find a nonzero homogeneous $u \in M^\mu$ such that $\mathfrak{n}_0^+ u = 0$. Let $\mathfrak{g}_1^+ = \mathfrak{a}_m \supseteq \cdots \supseteq \mathfrak{a}_1 \supseteq \mathfrak{a}_0 = 0$ be a series of $\mathfrak{n}_0^+$-submodules of $\mathfrak{g}_1^+$ with $[\mathfrak{n}_0^+, \mathfrak{a}_i] \subseteq \mathfrak{a}_{i-1}$ and $\mathfrak{a}_i = \mathfrak{a}_{i-1} \oplus \mathbb{C}x_i$ for $i = 1, ..., m$. Let $\mathfrak{b}_i = \mathfrak{n}_0^+ \oplus \mathfrak{a}_i$. Suppose we can find a nonzero homogeneous u such that $\mathfrak{b}_{i-1}u = 0$. Then either $\mathfrak{b}_i u = 0$ or $x_i u = v \neq 0$. In the latter case we have for $y \in \mathfrak{n}_0^+$, $yv = x_i yu + [y, x_i]u = 0$ since $[y, x_i] \in \mathfrak{a}_{i-1}$. Since also $\mathfrak{a}_i v = 0$, we obtain $\mathfrak{b}_i v = 0$. Proceeding in this way we can find a nonzero homogeneous $w \in M^\lambda$

for some λ such that $\mathfrak{n}^+ w = 0$. It then follows that $U(\mathfrak{g})w \subseteq M$ and $U(\mathfrak{g})w$ is a homomorphic image of $\tilde{M}(\lambda)$.

Remark. Let $\mathfrak{g} = A(n, n)$ and let $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$ be a triangular decomposition as described in this subsection. We note that there is no basis of simple roots corresponding to this decomposition. For suppose $S = \{\alpha \in \mathfrak{h}^* \mid (g_1^+)^\alpha \neq 0\}$. A simple calculation shows that $\sum_{\alpha \in S} \alpha = 0$. Therefore if $\alpha_1, ..., \alpha_l$ is a set of roots such that every $\alpha \in S$ can be expressed as an nonnegative integer linear combination of $\alpha_1, ..., \alpha_l$. then $\alpha_1, ..., \alpha_l$ are linearly dependent.

1.8. The only classical simple Lie superalgebra which has not been treated above is $\mathfrak{g} = Q(n)$. This is the Lie superalgebra $\tilde{Q}(n)/I$ where $\tilde{Q}(n)$ consists of matrices $\left(\begin{smallmatrix} a & b \\ b & a \end{smallmatrix}\right)$ with $a \in gl(n+1)$, $b \in sl(n+1)$, and $I = \mathbb{C}I_{2n+2}$. Complications arise here because $\mathfrak{h} \neq \mathfrak{h}_0$ and there exist finite dimensional graded simple $U(\mathfrak{h})$-modules with dimension greater than one.

Denote by N^-, H, N^+ the strictly lower triangular, diagonal, and strictly lower triangular matrices in $sl(n+1)$, respectively. We define

$$\mathfrak{h}_0 = \left\{ \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \middle| \, a \in H \right\} \Big/ I, \qquad \mathfrak{h}_1 = \left\{ \begin{pmatrix} 0 & b \\ b & 0 \end{pmatrix} \middle| \, b \in H \right\} \Big/ I$$

$$\mathfrak{n}_0^\pm = \left\{ \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \middle| \, a \in N^\pm \right\} \Big/ I, \qquad \mathfrak{n}_1^\pm = \left\{ \begin{pmatrix} 0 & b \\ b & 0 \end{pmatrix} \middle| \, b \in N^\pm \right\} \Big/ I$$

$$\mathfrak{h} = \mathfrak{h}_0 \oplus \mathfrak{h}_1, \qquad \mathfrak{n}^\pm = \mathfrak{n}_0^\pm \oplus \mathfrak{n}_1^\pm, \qquad \mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}^+.$$

PROPOSITION. (a) *$\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$ is a triangular decomposition of $\mathfrak{g}$.*

(b) *For any $\lambda \in \mathfrak{h}_0^*$, there exists a unique finite dimensional graded simple $\mathfrak{b}$-module V_λ such that $\mathfrak{n}^+ V_\lambda = 0$ and $hv = \lambda(h)v$ for all $h \in \mathfrak{h}_0$. Any finite dimensional graded simple $\mathfrak{b}$-module is isomorphic to V_λ for some $\lambda \in \mathfrak{h}_0^*$.*

(c) *The Verma module $\tilde{M}(\lambda) = U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} V_\lambda$ satisfies properties* (a), (b), (d) *of Proposition* 1.1.

Proof. (a) Since $\mathfrak{g}_1 \cong \mathfrak{g}_0$ as $\mathfrak{g}_0$-modules it follows that the basis R_0^+ of simple roots with respect to the decomposition $\mathfrak{g}_0 = \mathfrak{n}_0^- \oplus \mathfrak{h}_0 \oplus \mathfrak{n}_0^+$ is in fact a basis of simple roots for $\mathfrak{g}$. Thus (a) follows from Lemma 1.4.

(b) This follows from [K1, Theorem 7]. However, we can give a more elementary proof as follows. For $\lambda \in \mathfrak{h}_0^*$ define a symmetric bilinear form f_λ on $\mathfrak{h}_1$ by $f_\lambda(x, y) = \lambda([x, y])$. Let $\mathfrak{h}_1^\perp = \{x \in \mathfrak{h}_1 \mid f_\lambda(x, \mathfrak{h}_1) = 0\}$ be the radical of this form, and $\mathfrak{a}_\lambda = \operatorname{Ker} \lambda \oplus \mathfrak{h}_1^\perp$. Then $\mathfrak{a}_\lambda$ is an ideal in $\mathfrak{h}$ and we set $\mathfrak{c}_\lambda = \mathfrak{h}/\mathfrak{a}_\lambda$. If $\lambda \neq 0$ we can find $z \in (\mathfrak{c}_\lambda)_0$ such that $\lambda(z) = 1$. The factor algebra $A_\lambda = U(c_\lambda)/(z-1)$ of $U(h)$ depends only on λ. Let q be the nonsingular quadratic form on $(c_\lambda)_1$ defined by

$$q(x + h_1^\perp) = \tfrac{1}{2} f_\lambda(x, x).$$

Then q is nonsingular, and A_λ is isomorphic to the Clifford algebra of q. By [Lam, Chap. V, Sect. 2], this Clifford algebra is graded simple, so it has a unique graded simple module V_λ. Clearly we can regard V_λ as a $U(b)$-module, and it has the required properties. If $\lambda = 0$, then $a_\lambda = h$, and we let V_0 be the trivial $U(b)$-module, and $A_0 = U(h)/hU(h)$.

Now let V be any finite dimensional graded simple b-module and $\phi : b \to gl(V)$ the representation afforded by V. We claim that if $x \in n^+$ then $\phi(x)$ is nilpotent. It suffices to show this for $x \in n_0^+$ since if $y \in n_1^+$ then $[y, y] \in n_0^+$ and $\phi(y)^2 = 1/2\phi([y, y])$. Now V is a finite dimensional b_0-module and by Lie's theorem every b_0-composition factor of V is annihilated by n_0^+ so the claim follows. By Engel's theorem for Lie superalgebras [Sch, p. 236], we have $V' \neq 0$ where $V' = \{v \in V \mid n^+ v = 0\}$. Since n^+ is an ideal of b, V' is a graded submodule of V, so $V' = V$, and V may be regarded as a $h = b/n^+$-module. Since h_0 is central in h, there exists $\lambda \in h_0^*$ such that $hv = \lambda(h)v$ for all $h \in h_0$, $v \in V$ by Schur's Lemma for Lie superalgebras [K1, p. 18]. Therefore V is a $h/\mathrm{Ker}\,\lambda$-module. If $h_1^\perp$ is defined as in the first part of the proof then the image of $h_1^\perp$ in $h/\mathrm{Ker}\,\lambda$ is supercentral, so $h_1^\perp$ acts as zero on V, since V is graded simple. It follows easily that V is a graded simple A_λ-module and $V \cong V_\lambda$.

(c) As we noted in the proof of (a) the basis of simple roots R_0^+ with respect to the decomposition $g_0 = n_0^- \oplus h_0 \oplus n^+$ is in fact a basis of simple roots of g. Let $Q^+ = \mathbb{N}R_0^+$ be the subsemigroup of h_0^* generated by R_0^+. We have $\tilde{M}(\lambda) = \bigoplus_{\mu \in \lambda - Q^+} \tilde{M}(\lambda)^\mu$. Let $\tilde{M}(\lambda)^+ = \bigoplus_{\mu \neq \lambda} \tilde{M}(\lambda)^\mu$. If U is any graded submodule of $\tilde{M}(\lambda)$ not contained in $\tilde{M}(\lambda)^+$ we have $U^\lambda = \tilde{M}(\lambda)^\lambda \cap U \neq 0$, but $\tilde{M}(\lambda)^\lambda = V_\lambda$ and this is a graded-simple $U(b)$-module. Hence $U^\lambda = V_\lambda$ and $U \supseteq U(g)V_\lambda = \tilde{M}(\lambda)$. It follows that $\tilde{M}(\lambda)$ has a unique maximal graded submodule.

Finally if M is a nonzero module in $\tilde{\mathcal{O}}$, then by Lemma 1.4, M has a highest weight λ. If $v \in M^\lambda$ is homogeneous, then $n^+ v = 0$ so $U(b)v$ is a finite dimensional graded A_λ-module, so it contains a copy of V_λ. Hence $U(g)V_\lambda \subseteq M$ is a homomorphic image of $\tilde{M}(\lambda)$.

Remark. From the proof we see that the graded simple $U(g)$-module $\tilde{L}(\lambda)$ is simple if and only if V_λ is a simple $U(b)$-module and this is equivalent to the condition that rank f_λ is even. When $g = Q(3)$ a simple calculation shows that the possibilities are rank $f_\lambda = 0$, 1, or 2.

2. The Classification Theorem

2.1. Let $\mathfrak{g}$ be a Lie superalgebra and σ the automorphism of $\mathfrak{g}$ with $\sigma(x+y)=x-y$ for $x\in\mathfrak{g}_0$, $y\in\mathfrak{g}_1$. Then σ extends to an automorphism of $U(\mathfrak{g})=U$. We denote the set of prime, primitive, graded prime, and graded primitive ideals of U by Spec U, Prim U, Gr Spec U and Gr Prim U, respectively. Because of the following lemma the primitive ideals of U are easily described in terms of the graded primitive ideals.

Lemma. (a) *If* $P\in$ Spec U, *then* $P\cap\sigma(P)\in$ Gr Spec U.

(b) *If* $p\in$ Gr Spec U, *then* $p=P\cap\sigma(P)$ *where* $P\in$ Spec U *is minimal over* p.

(c) *If* p *and* P *are as in* (a), (b) *then* $p\in$ Gr Prim U *if and only if* $P\in$ Prim U.

Proof. (a) is obvious. For (b) see [CM, Theorem 6.3] and for (c) [L1, Theorem 3.1].

2.2. Let $\mathfrak{g}$ be classical simple and set

$$I(\lambda)=\operatorname{ann}_{U(\mathfrak{g}_0)} L(\lambda),\qquad J(\lambda)=\operatorname{ann}_{U(\mathfrak{g})} \tilde{L}(\lambda).$$

We use the following result of Duflo [D] (see also [J, 7.4]).

Theorem. *The map* $\lambda\to I(\lambda)$ *from* $\mathfrak{h}_0^*$ *to* Prim $U(\mathfrak{g}_0)$ *is surjective.*

This result is usually only stated for $\mathfrak{g}_0$ semisimple, but the extension to the reductive case is routine.

Our main result is an analogue for classical simple Lie superalgebras.

Main Theorem. *The map* $\lambda\to J(\lambda)$ *from* $\mathfrak{h}_0^*$ *to* Gr Prim $U(\mathfrak{g})$ *is surjective.*

2.3. Assume that R is a Noetherian subring of S such that S is finitely generated as both a left and right R-module. In addition we require R and S to be algebras of finite GK-dimension over a field.

Definitions. Let Q be a prime ideal of R and T_Q the submodule of ${}_SS_R$ such that $SQ\subseteq T_Q$, and T_Q/SQ is the torsion submodule of S/SQ as a right R/Q-module. Let J be the left annihilator in S of S/T_Q, and X_Q the set of prime ideals of S minimal over J. We need the following results of Letzter:

THEOREM. (1) $\operatorname{Spec} S = \bigcup_{Q \in \operatorname{Spec} R} X_Q$.

(2) $\operatorname{Prim} S = \bigcup_{Q \in \operatorname{Prim} R} X_Q$.

Proof. See [L2, Proposition 4.2].

2.4. We need a graded version of Theorem 2.3. Let $R = U(\mathfrak{g}_0)$ and $S = U(\mathfrak{g})$. For Q a prime ideal of R, S/SQ is a free right R/Q-module, so $T_Q = SQ$. We let $\operatorname{Gr} X_Q$ be the set of graded prime ideals of S which are minimal over $J = l\operatorname{ann}_S(S/SQ)$. Using Lemma 2.1 we have $\operatorname{Gr} X_Q = \{P \cap \sigma(P) \mid P \in X_Q\}$. Then from Theorem 2.3 we obtain

COROLLARY. (1) $\operatorname{Gr} \operatorname{Spec} S = \bigcup_{Q \in \operatorname{Spec} R} \operatorname{Gr} X_Q$.

(2) $\operatorname{Gr} \operatorname{Prim} S = \bigcup_{Q \in \operatorname{Spec} R} \operatorname{Gr} X_Q$.

2.5. LEMMA. *Let R be a subring of S such that S is a free right R-module. If M is a left R-module with* $\operatorname{ann}_R M = Q$, *then* $\operatorname{ann}_S(S \otimes_R M) = \operatorname{ann}_S(S/SQ)$.

Proof. See the proof of [BGR, Lemma 10.4a].

2.6. For $\lambda \in \mathfrak{h}_0^*$ we set $\chi(\lambda) = \bigcup \{w \cdot (\lambda + \mu) \cap (\lambda + \mu - Q_0^+) \mid \mu \in \Pi(\Lambda \mathfrak{g}_1),$ $w \in W\}$. Note this is a finite set depending only on λ. If $P \in \operatorname{Gr} \operatorname{Prim} U(\mathfrak{g})$ then by Corollary 2.4 and Duflo's theorem we have $P \in \operatorname{Gr} X_{I(\lambda)}$ for some $\lambda \in \mathfrak{h}_0^*$. Hence the main theorem follows from the following result.

THEOREM. *We have* $\operatorname{Gr} X_{I(\lambda)} \subseteq \{J(\nu) \mid \nu \in \chi(\lambda)\}$.

Proof. Let $R = U(\mathfrak{g}_0)$ and $S = U(\mathfrak{g})$. If $P \in \operatorname{Gr} X_{I(\lambda)}$, then by Lemma 2.5 P is minimal over $\operatorname{ann}_S(S \otimes_R L(\lambda))$. By Lemma 1.3, $S \otimes_R M(\lambda)$ has a finite graded composition series. Therefore since $S \otimes_R L(\lambda)$ is a factor module of $S \otimes_R M(\lambda)$, P is the annihilator of some graded composition factor of $S \otimes_R M(\lambda)$. Now

$$\begin{aligned} S \otimes_R M(\lambda) &= U(\mathfrak{g}) \otimes_{U(\mathfrak{g}_0)} U(\mathfrak{g}_0) \otimes_{U(\mathfrak{b}_0)} \mathbb{C}v_\lambda \\ &= U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} (U(\mathfrak{b}) \otimes_{U(\mathfrak{b}_0)} \mathbb{C}v_\lambda), \end{aligned}$$

and $U(\mathfrak{b}) \otimes_{U(\mathfrak{b}_0)} \mathbb{C}v_\lambda$ has a finite graded composition series with factors V_μ, for various $\mu \in \mathfrak{h}_0^*$. Hence $S \otimes_R M(\lambda)$ has a finite series with factors $U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} V_\mu = \tilde{M}(\mu)$. By Corollary 1.1, $\tilde{M}(\mu)$ has a finite composition series with factors $\tilde{L}(\nu)$ for various $\nu \in \mathfrak{h}_0^*$. Hence $P = J(\nu)$ for some ν. We have $\tilde{L}(\nu) = U(\mathfrak{g})v$ where $\mathfrak{n}_0^+ v = 0$ and $hv = \nu(h)v$ for all $h \in \mathfrak{h}_0^*$. Therefore $U(\mathfrak{g}_0)v$ is a $U(\mathfrak{g}_0)$-subfactor of $S \otimes_R M(\lambda)$ and $L(\nu)$ is an image of $U(\mathfrak{g}_0)v$. It follows that $L(\nu)$ is a $U(\mathfrak{g}_0)$-composition factor of $U(\mathfrak{g}) \otimes_{U(\mathfrak{g}_0)} M(\lambda)$. By Lemma 1.3 we have $\nu \in \chi(\lambda)$ and this proves the result.

3. Consequences and Further Remarks

3.1. Corollary. *Let $\mathfrak{g}$ be a classical simple Lie superalebra, $\mathfrak{g} \neq Q(n)$. If I is a semiprime ideal of $U(\mathfrak{g})$, then I is graded. Furthermore* Spec $U(\mathfrak{g}) =$ Gr Spec $U(\mathfrak{g})$ *and* Prim $U(\mathfrak{g}) =$ Gr Prim $U(\mathfrak{g})$.

Proof. First let P be a primitive ideal of $U(\mathfrak{g})$. Then $P \cap \sigma(P)$ is graded primitive, so $P \cap \sigma(P) = \operatorname{ann} \tilde{L}(\lambda)$ for some λ. However by Proposition 1.1 $\tilde{L}(\lambda)$ is irreducible if $\mathfrak{g} \neq Q(n)$, so $\operatorname{ann} \tilde{L}(\lambda)$ is primitive. Therefore $P = \sigma(P) = \operatorname{ann} \tilde{L}(\lambda)$. Since $U(\mathfrak{g})$ is a Jacobson ring by [L1, Lemma 2.5] or [CS, Theorem 1] any semiprime ideal I is an intersection of primitive ideals. Hence $\sigma(I) = I$. The last statements follows easily.

3.2. For $\mathfrak{g}$ classical simple, $\mathfrak{g} \neq P(n)$, we have seen in Section 1 that there is a basis of simple roots and a corresponding triangular decomposition $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$. We next show that there is an antiautomorphism $x \to {}^t x$ of $\mathfrak{g}$ such that

$$ {}^t\mathfrak{n}^- = \mathfrak{n}^+, \qquad {}^t\mathfrak{n}^+ = \mathfrak{n}^-, \qquad {}^t h = h \qquad \text{for all} \quad h \in \mathfrak{h}. $$

If $\mathfrak{g} = A(n, n)$ this is done in Lemma 1.6. If $\mathfrak{g} = \mathfrak{g}(A, \tau)$ is a contragredient Lie superalgebra, with $\det A \neq 0$, we define a map $x \to {}^t x$ on the Lie superalgebra $\tilde{\mathfrak{g}}(A, \tau)$ by the rules

$$ e_i \to f_i, \qquad f_i \to e_i \qquad 1 \leqslant i \leqslant n $$
$$ h \to h \qquad \text{for} \quad h \in \mathfrak{h}. $$

If $\mathfrak{r}$ is the unique maximal ideal of $\tilde{\mathfrak{g}}(A, \tau)$ intersecting $\mathfrak{h}$ trivially then ${}^t\mathfrak{r} = \mathfrak{r}$ by the same proof as for the Cartan involution of [K1, Sect. 1.3]. Hence, we obtain an antiautomorphism on $\mathfrak{g}(A, \tau)$ also denoted $x \to {}^t x$.

Finally if $\mathfrak{g} = Q(n)$ the map $x \to {}^t x$ is induced from a similar map on $\tilde{Q}(n)$ sending a matrix to its transpose.

Note that if $\mathfrak{g} = P(n)$ with the triangular decomposition given in 1.7, no such map $x \to {}^t x$ as above can exist since $\dim \mathfrak{n}^- \neq \dim \mathfrak{n}^+$.

We extend the map $x \to {}^t x$ to an antiautomorphism of $U(\mathfrak{g})$. If $M \in \tilde{\mathcal{O}}$, we let ${}^t M$ be the dual of the $U(\mathfrak{g}_0)$-module in the category $\mathcal{O}$ as defined in [J, 4.10]. Then ${}^t M$ is a submodule of the dual space M^* and ${}^t M$ can be regarded as a $U(\mathfrak{g})$-module via the action

$$ (x\phi)(m) = \phi({}^t x m) \qquad \text{for} \quad x \in U(\mathfrak{g}), \phi \in {}^t M, m \in M. $$

Hence ${}^t M \in \tilde{\mathcal{O}}$ and it is immediate that

$$ {}^t\operatorname{ann}_{U(\mathfrak{g})} M = \operatorname{ann}_{U(\mathfrak{g})} {}^t M. $$

By [J, 4.10] the modules M, $'M$ have the same composition length and the same character. In particular $'\tilde{L}(\lambda)$ is a simple object in $\tilde{\mathcal{O}}$, and since $\mathfrak{g}$ has a basis of simple roots this module has a unique highest weight λ. Therefore $'\tilde{L}(\lambda) \cong \tilde{L}(\lambda)$, so $'J(\lambda) = J(\lambda)$ for all $\lambda \in \mathfrak{h}_0^*$. We can now prove an analogue of [J, Cor. 7.5].

COROLLARY. *If $\mathfrak{g} \neq P(n)$ is classical simple then for any graded semiprime ideal I of $U(\mathfrak{g})$ we have $'I = I$.*

Proof. By [L1, Lemma 2.5] $U(\mathfrak{g})$ is a Jacobson ring. It follows easily that for any graded semiprime ideal I we have $I = \bigcap \{P_\alpha \mid P_\alpha$ graded primitive, $\alpha \in A\}$ for some index set A. Since each P_α has the form $J(\lambda)$, $\lambda \in \mathfrak{h}_0^*$, we obtain the result.

3.3. The preceding corollary implies a symmetry of many right and left properties of factor rings of $U(\mathfrak{g})$. We record one application here.

COROLLARY. *If $\mathfrak{g} \neq P(n)$ is classical simple and R any factor ring of $U(\mathfrak{g})$, then the left and right Krull dimensions of R coincide.*

Proof. Using [MR, Corollary 6.3.8] and Lemma 2.1 we can reduce to the case $R = U(\mathfrak{g})/I$ where I is a graded prime ideal of $U(\mathfrak{g})$. In this case the map $x + I \to {}'x + I$ is an antiautomorphism of the ring R.

Remark. A similar argument using Duflo's theorem shows that if $\mathfrak{g}$ is a semisimple Lie algebra and R a factor ring of $U(\mathfrak{g})$ then the left and right Krull dimensions of R coincide. This has been noted by Levasseur [Lev, p. 174].

3.4. We return to the situation discussed in 1.7 where $\mathfrak{g}_1 = \mathfrak{g}_1^+ \oplus \mathfrak{g}_1^-$ is a direct sum of two $\mathfrak{g}_0$-submodules. In this case we can make an improvement to the main theorem. Note that $\mathfrak{p} = \mathfrak{g}_1^+ \oplus \mathfrak{g}_0$ is a subalgebra of $U(\mathfrak{g})$, and that $\mathfrak{g}_1^+ U(\mathfrak{p}) = U(\mathfrak{p})\mathfrak{g}_1^+ = J$ is a nilpotent ideal of $U(\mathfrak{p})$ with $U(\mathfrak{p})/J \cong U(\mathfrak{g}_0)$. For $\lambda \in \mathfrak{h}_0^*$ we can regard the $U(\mathfrak{g}_0)$ modules $M(\lambda)$ and $L(\lambda)$ as $U(\mathfrak{p})$-modules with J acting as zero. Note that as $U(\mathfrak{p})$-modules $M(\lambda) = U(\mathfrak{g}_0) \otimes_{U(\mathfrak{b}_0)} \mathbb{C}v_\lambda = U(\mathfrak{p}) \otimes_{U(\mathfrak{b})} \mathbb{C}v_\lambda$ where $\mathfrak{b} = \mathfrak{b}_0 \oplus \mathfrak{g}_1^+$, and $\mathbb{C}v_\lambda$ is the $\mathfrak{b}$-module with $\mathfrak{g}_1^+ v_\lambda = \mathfrak{n}_0^+ v_\lambda = 0$ and $hv_\lambda = \lambda(h)v_\lambda$ for $h \in \mathfrak{h}$. Hence $\tilde{M}(\lambda) = U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} M(\lambda)$. Since J is nilpotent, it follows from Duflo's theorem that the primitive ideals of $U(\mathfrak{p})$ have the form $\hat{I}(\lambda) = \operatorname{ann}_{U(\mathfrak{p})} L(\lambda)$. For $\lambda \in \mathfrak{h}^*$ let $\hat{\chi}(\lambda) = \bigcup \{W \cdot (\lambda + \mu) \cap (\lambda + \mu - Q_0^+) \mid \mu \in \Pi(\Lambda\mathfrak{g}_1^-)\}$. As a $\mathfrak{g}_0$-module we have $\tilde{M}(\lambda) = \Lambda\mathfrak{g}_1^- \otimes_{\mathbb{C}} M(\lambda)$, which has a composition series with factors $L(\nu)$, $\nu \in \hat{\chi}(\lambda)$. We can apply Corollary 2.4 to the ring extension $R = U(\mathfrak{p}) \subseteq S = U(\mathfrak{g})$. Repeating the earlier arguments we obtain

THEOREM. *For $\lambda \in \mathfrak{h}_0^*$, $X_{\hat{I}(\lambda)} \subseteq \{J(\nu) \mid \nu \in \hat{\chi}(\lambda)\}$.*

The advantage of this result over Theorem 2.6 is that the set $\tilde{\chi}(\lambda)$ is much smaller than $\chi(\lambda)$, and so we have better control over Prim $U(\mathfrak{g})$. In [M] we apply this result to obtain a detailed description of Prim $U(\mathfrak{g})$ in the case $\mathfrak{g} = sl(2, 1)$.

3.5. It is an interesting problem to find necessary and sufficient conditions for the modules $\tilde{L}(\lambda)$ to be finite dimensional. This of course depends on the triangular decomposition $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$. We briefly summarize what is known about this problem.

(a) If $\tilde{L}(\lambda)$ is finite dimensional then so is $L(\lambda)$. This follows since $\mathrm{Hom}_{U(\mathfrak{g}_0)}(M(\lambda), (\tilde{M}(\lambda))) \neq 0$.

(b) If $\mathfrak{g}_1 = \mathfrak{g}_1^+ \oplus \mathfrak{g}_1^-$ and $\mathfrak{p} = \mathfrak{g}_1^+ \oplus \mathfrak{g}_0$ as in 1.7 and 3.4, then $L(\lambda)$ can be regarded as a $U(\mathfrak{p})$ module with $\mathfrak{g}_1^+$ acting as zero. Then $\tilde{L}(\lambda)$ is a homomorphic image of $U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} L(\lambda)$ and hence $\tilde{L}(\lambda)$ is finite dimensional if $L(\lambda)$ is finite dimensional.

(c) If $\mathfrak{g} = Q(n)$ with the triangular decomposition of 1.8, then necessary and sufficient conditions for $\tilde{L}(\lambda)$ to be finite dimensional are given in [K1, Theorem 8b)]. We note that it is possible for $L(\lambda)$ to be finite dimensional and $\tilde{L}(\lambda)$ infinite dimensional.

(d) If $\mathfrak{g}$ is a basic Lie superalgebra, and is realized as a contragredient Lie superalgebra with a matrix corresponding to the Dynkin diagram of [K1, Table VI], then necessary and sufficient conditions for $\tilde{L}(\lambda)$ to be finite dimensional are given in [K1, Theorem 8c)]. It is not hard to show that $L(\lambda)$ is finite dimensional if and only if conditions (1) and (2) of [K1, Theorem 8c)] hold. However, these conditions are not sufficient for $\tilde{L}(\lambda)$ to be finite dimensional. We remark that Kac has given an alternative construction of the finite dimensional simple modules for a basic classical simple Lie superalgebra in [K3, Proposition 2.4].

Note added in proof. If $\mathfrak{g} = \mathrm{osp}(1, 2)$, a classification of primitive ideals in $u(\mathfrak{g})$ has been obtained by G. Pinczon, The enveloping algebra of the Lie superalgebra osp(1, 2), *J. Algebra* **132** (1990), 219–242. In addition, a result similar to Proposition 1.8(b) has been proved by I. A. Skornyakov. His result appears as Proposition 1 in I. B. Penkov, Characters of typical irreducible finite-dimensional $q(n)$-modules, *Funct. Anal. Appl.* **20** (1986), 30–37.

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