Restricting a representation to a principally embedded $\mathfrak{sl}_2$ subalgebra

by

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ABSTRACT

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Throughout this exposition, the ground field is $\mathbb{C}$, the field of complex numbers, and all vector spaces and Lie algebras are finite dimensional over $\mathbb{C}$. Also the two expressions "$\mathfrak{g}$-module" and "representation of $\mathfrak{g}$" will be used interchangeably.
Let $\mathfrak{k}$ be a semisimple Lie subalgebra of a simple Lie algebra $\mathfrak{g}$.
Let V be an irreducible finite dimensional representation of $\mathfrak{g}$. Once the action is restricted to $\mathfrak{k}$ one gets a decomposition of the form:

$$V \cong \bigoplus_{i=1}^{\infty} \underbrace{W_i \oplus W_i \oplus \cdots \oplus W_i}_{m_i(V)}$$

with all W_i 's pairwise inequivalent irreducible representations of $\mathfrak{k}$, and $m_i(V) \geq 0$ is the multiplicity of W_i in V.
A question one might ask is:
Does there exist an upperbound $\mathrm{b}(\mathfrak{k}, \mathfrak{g})$ such that, for any irreducible $\mathfrak{g}$-representation V, there exists a W_i, an irreducible $\mathfrak{k}$-representation occurring in V with $\dim W_i \leq \mathrm{b}(\mathfrak{k}, \mathfrak{g})$? It turns out that $\mathrm{b}(\mathfrak{k}, \mathfrak{g})$ is not always finite, and if it is, another question one can ask is:
What is the sharpest upperbound, $\mathbf{b}(\mathfrak{k}, \mathfrak{g})$, of all these $\mathrm{b}(\mathfrak{k}, \mathfrak{g})$'s ?
Our result here is the following:
For $\mathfrak{k}$ being the principally embedded Lie subalgebra $\mathfrak{s}l_2$ the simple Lie algebra $\mathfrak{s}l_n$ with $n \geq 3$ we have:

$$\mathbf{b}(\mathfrak{k}, \mathfrak{g}) = n$$

Table of Contents

List of Figures

This PhD research thesis is dedicated to all members of my family, especially my brother Ahmed, and to all my friends: A.Ourourou, H. Ayyachi, A.Ourourou, M. Raghib, M. Merzouk, H. Elbouazzaoui, M. Maha, R. Hamroud, S.O. Elboubekri.

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Chapter 1

Introduction

In this chapter we first give a summary of basic concepts, notations, terminology and results. We first start by basic concepts of Lie theory such as such as: nilpotency, solvability, roots, Dynkin diagrams, highest weights, fundamental representations etc.

1.1 Basic concepts of Lie theory

Throughout this work, the ground field is $\mathbb{C}$, the field of complex numbers, and all vector spaces and Lie algebras are finite dimensional over $\mathbb{C}$. Also the two expressions "$\mathfrak{g}$-module" and "representation of $\mathfrak{g}$" will be used interchangeably.

1.1.1 Definitions and examples.

A Lie algebra is a vector space $\mathfrak{g}$ over a field $\mathbb{F}$ on which a multiplication:

$$\mathfrak{g} \times \mathfrak{g} \longrightarrow \mathfrak{g}$$
$$(x, y) \mapsto [x, y]$$

is defined satisfying the axioms:

a. $[x, y]$ is linear in both x and y
b. $[x, x] = 0$ for all x in $\mathfrak{g}$
c. $[[x, y], z] + [[y, z], x] + [[z, x], y] = 0$ for all x, y, z in $\mathfrak{g}$

Property *c*. is called the `Jacobi identity` (cf. [3]).

Note: The multiplication is not associative since in general we do not have: $[[x, y], z] = [x, [y, z]]$, therefore, the inclusion of Lie brackets in products of elements is necessary in the notation.

We also have for any pair of elements x and y in $\mathfrak{g}$

$$[x+y, x+y] = [x,x] + [x,y] + [y,x] + [y,y]$$

and since $[x,x] = 0$ for all x in $\mathfrak{g}$ it follows that $[x,y] = -[y,x]$ for all x and y in $\mathfrak{g}$. In other words the multiplication in a Lie algebra is anticommutative.

Example 1.1.1. Any associative algebra A can be made into a Lie algebra by taking $[x,y] = xy - yx$. It is easy to see that all the Lie algebra axioms are satisfied.

Definition 1.1.2. Let $\mathfrak{g}_1$ and $\mathfrak{g}_2$ be Lie algebras over a field $\mathbb{F}$. A homomorphism of Lie algebras is a linear map

$$\theta : \mathfrak{g}_1 \longrightarrow \mathfrak{g}_2 \quad \text{defined by} \quad \theta[x,y] = [\theta(x), \theta(y)]$$

for all x and y in $\mathfrak{g}_1$.
If in addition θ is a bijection, it is called an isomorphism of Lie algebras.

Definition 1.1.3. A subalgebra of $\mathfrak{g}$ is a subspace $\mathfrak{h}$ of $\mathfrak{g}$ such that $[\mathfrak{h}, \mathfrak{h}] \subset \mathfrak{h}$.
An ideal of $\mathfrak{g}$ is a subspace $\mathfrak{h}$ of $\mathfrak{g}$ such that $[\mathfrak{g}, \mathfrak{h}] \subset \mathfrak{h}$, which is equivalent to $[\mathfrak{h}, \mathfrak{g}] \subset \mathfrak{h}$ since $[\mathfrak{h}, \mathfrak{g}] = [\mathfrak{g}, \mathfrak{h}]$

Now, let $\mathfrak{h}$ be an ideal of the Lie algebra $\mathfrak{g}$, and $\mathfrak{g}/\mathfrak{h}$ be the vector space of cosets. By introducing the Lie multiplication $[x+\mathfrak{h}, y+\mathfrak{h}] = [x,y]+\mathfrak{h}$, we make $\mathfrak{g}/\mathfrak{h}$ into a Lie algebra, furthermore there is a natural homomorphism $\theta : \mathfrak{g} \to \mathfrak{g}/\mathfrak{h}$ defined by $\theta(x) = x + \mathfrak{h}$.
Conversely given any algebra homomorphism: $\theta : \mathfrak{g}_1 \longrightarrow \mathfrak{g}_2$ of Lie algebras. If θ is surjective then the factor $\mathfrak{g}_1/\ker(\theta)$ is isomorphic to $\mathfrak{g}_2$.

1.1.2 Nilpotent and Solvable Lie algebras.

Definition 1.1.4. A Lie subalgebra is called abelian if $[\mathfrak{g}, \mathfrak{g}] = 0$.

We define the sequence of ideals $\mathfrak{g}^1$, $\mathfrak{g}^2$, $\mathfrak{g}^3 \ldots$ of $\mathfrak{g}$ by

$$\mathfrak{g}^1 = \mathfrak{g}, \;\; \mathfrak{g}^{n+1} = [\mathfrak{g}^n, \mathfrak{g}]$$

for all $n = 1, 2, 3, \cdots$
We have then the descending series of ideals:

$$\mathfrak{g} = \mathfrak{g}^1 \supseteq \mathfrak{g}^2 \supseteq \mathfrak{g}^3 \ldots$$

Definition 1.1.5. The Lie algebra $\mathfrak{g}$ is called nilpotent if $\mathfrak{g}^i = 0$ for some $i \geq 2$.

Notice that $\mathfrak{g}$ is abelian if and only if $i = 2$.
Similarly, we define the sequence of ideals $\mathfrak{g}^{(0)}$, $\mathfrak{g}^{(0)}$, $\mathfrak{g}^{(2)} \ldots$ of $\mathfrak{g}$ by

$$\mathfrak{g}^{(0)} = \mathfrak{g}, \; \mathfrak{g}^{(n+1)} = [\mathfrak{g}^{(n)}, \mathfrak{g}^{(n)}].$$

for all $n = 1, 2, 3, \cdots$.
Here again we have the descending sequence of ideals.

$$\mathfrak{g} = \mathfrak{g}^{(0)} \supseteq \mathfrak{g}^{(1)} \supseteq \mathfrak{g}^{(2)} \ldots .$$

This series is also called the derived series (cf. [3]).

Definition 1.1.6. The Lie algebra $\mathfrak{g}$ is called solvable if $\mathfrak{g}^{(i)} = 0$ for some $i \geq 1$.

Proposition 1.1.7. *Every nilpotent Lie algebra is solvable, but the converse is not true*

Example 1.1.8. For any field $\mathbb{F}$, let $\mathfrak{g} = \mathbb{F}a + \mathbb{F}b$ be the Lie algebra defined by $[a, b] = b$. It is obvious that: $\mathfrak{g}^1 = \mathbb{F}b$ and $\mathfrak{g}^2 = \mathfrak{g}^3 = \mathfrak{g}^4 \ldots = 0$
Also, we have: $\mathfrak{g}^{(1)} = \mathbb{F}b$, and $\mathfrak{g}^{(i)} = 0$ for all $i \geq 2$, so $\mathfrak{g}$ is a solvable but not a nilpotent Lie algebra. (`http://math.mit.edu/classes/18.745/Notes/`)

1.1.3 Representations and modules

Let $\mathfrak{g}$ be an algebra over a field $\mathbb{F}$, a representation of $\mathfrak{g}$ is a homomorphism:

$$\rho : \mathfrak{g} \longrightarrow \mathfrak{gl}_n(\mathbb{F})$$

for some n, where $\mathfrak{gl}_n(\mathbb{F})$ denote the Lie algebra of $n \times n$-matrices over the field $\mathbb{F}$.
Two representations ρ and ρ' of $\mathfrak{g}$ of degree n are called equivalent if there is a nonsingular $n \times n$-matrix T over $\mathbb{F}$ such that $\rho'(x) = T^{-1}\rho(x)T$ for all x in $\mathfrak{g}$.
A left $\mathfrak{g}$-module is a vector space V over $\mathbb{F}$ with a multiplication

$$\mathfrak{g} \times V \longrightarrow V$$
$$(x, v) \mapsto xv$$

satisfying the axioms:

i. xv is linear both in x and v.

ii. $[x, y]v = x(yv) - y(xv)$ for x, y in $\mathfrak{g}$ and all v in V.

It is clear that every finite dimensional $\mathfrak{g}$-module gives a representation of $\mathfrak{g}$ and vice versa.

Example 1.1.9 (Adjoint representation)**.** Let $\mathfrak{g}$ be Lie algebra over any field K. The Linear map:

$$\mathfrak{g} \to \mathrm{End}(\mathfrak{g})$$
$$g \mapsto ad_g$$

where $ad_g(x) = [g, x]$ for all g and x in $\mathfrak{g}$, is a representation of $\mathfrak{g}$ called: *Adjoint representation* of $\mathfrak{g}$. The Lie Bracket in End $(\mathfrak{g})$ is given by:

$$[ad_x, ad_y] = ad_x ad_y - ad_y ad_x$$

for all x and y in $\mathfrak{g}$.
This particular representation is crucial to representation theory in many ways.

1.2 Complex semisimple Lie algebras

In this section, the notion of semisimple Lie algebras is introduced. This is a class of Lie algebras whose irreducible representation are classified. Finite dimensional simple Lie algebras are given explicitly.

1.2.1 Classification of simple Lie algebras over the complex numbers

Definition 1.2.1. The radical of a finite dimensional Lie algebra $\mathfrak{g}$ is the largest solvable ideal of $\mathfrak{g}$.

Remark 1.2.2. The radical of $\mathfrak{g}$ contains any solvable ideal of $\mathfrak{g}$ and is unique.

Definition 1.2.3. A non-abelian Lie algebra $\mathfrak{g}$ whose only ideals are $\{0\}$ and $\mathfrak{g}$ is said to be simple.
A Lie algebra that is a direct sum of simple Lie algebras is said to be semisimple.
Over a field of characteristic 0, the following conditions are equivalent:

- A finite dimensional Lie algebra $\mathfrak{g}$ is semisimple.
- The killing form, $K(x, y) = \mathrm{tr}(ad(x)ad(y))$ is nondegenerate.
- $\mathfrak{g}$ has no nonzero solvable ideal.

- The radical of $\mathfrak{g}$ is zero.
  (cf. https://en.wikipedia.org/wiki/Semisimple_Lie_algebra)

Example 1.2.4. Simple Lie algebras over the field of complex numbers are classified as follows:
$\mathcal{A}_n$: $\mathfrak{sl}_{n+1}$, $n \geq 1$; the special linear Lie algebra.
$\mathcal{B}_n$: $\mathfrak{so}_{2n+1}$, $n \geq 2$; the odd dimensional special orthogonal Lie algebra.
$\mathcal{C}_n$: $\mathfrak{sp}_{2n}$, $n \geq 3$; the symplectic Lie algebra.
$\mathcal{D}_n$: $\mathfrak{so}_{2n}$, $n \geq 4$; the even dimensional special orthogonal Lie algebra.
The Lie algebras above together with the following exceptional Lie algebras:
E_6, E_7, E_8, F_4 and G_2 are the only simple Lie algebras over the complex numbers.

This elegant result is due to both *Wilhelm Killing* (1888 - 90), and *Elie Cartan* (1894). The dimensions of the classical complex simple Lie algebras are given below:

Lie algebra	$\mathcal{A}_n$	$\mathcal{B}_n$	$\mathcal{C}_n$	$\mathcal{D}_n$	E_6	E_7	E_8	F_4	G_2
Dimension	$n(n+1)$	$n(2n+1)$	$n(2n+1)$	$n(2n-1)$	78	133	248	52	14

(cf. [7], Appendix C).

1.2.2 Cartan subalgebras, Root system and Root space decomposition

Definition 1.2.5. Let $\mathfrak{g}$ be a finite dimensional Lie algebra over $\mathbb{C}$ and let $\mathfrak{h}$ be a subalgebra of $\mathfrak{g}$. The set

$$I(\mathfrak{h}) = \{x \in \mathfrak{g};\ [y, x] \in \mathfrak{h} \text{ for all } y \in \mathfrak{h}\}$$

is a subalgebra of $\mathfrak{g}$ containing $\mathfrak{h}$, and of which $\mathfrak{h}$ is an ideal. Furthermore $I(\mathfrak{h})$ is the largest subalgebra of $\mathfrak{g}$ in which $\mathfrak{h}$ is an ideal.
$I(\mathfrak{h})$ is called the idealizer of $\mathfrak{h}$.

Definition 1.2.6. A subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ is called a Cartan subalgebra if $\mathfrak{h}$ is nilpotent and $I(\mathfrak{h}) = \mathfrak{h}$.

Example 1.2.7. A Cartan subalgebra of the Lie algebra $n \times n$ complex matrices $\mathfrak{gl}_n$ is the algebra of all diagonal matrices.

Theorem 1.2.8. *Every finite dimensional Lie algebra $\mathfrak{g}$ over $\mathbb{C}$ has a Cartan subalgebra. Any two such subalgebras are isomorphic (as Lie algebras).*

Proof. The proof can be found in most of introductory books on Lie theory, such as ([4] Appendix D, [6]). □

Definition 1.2.9. Let V be a finite-dimensional euclidean vector space, with the standard euclidean inner product denoted by $(\cdot,\cdot)$
A root system in V is a finite set Φ of nonzero vectors (called roots) such that

i. The roots span V.

ii. If $x \in \Phi$ and $\lambda x \in \Phi$ then $\lambda = 1$ or $\lambda = -1$.

iii. For every root $x \in \Phi$, the set Φ is closed under reflection R_x through the hyperplane perpendicular to x.

iv. (Integrality) If x and y are roots in Φ, then the projection of y onto the line through x is a half-integral multiple of x.

(cf. [6]).

Remark 1.2.10. The conditions (iii) and (iv) of (1.2.9) can be written as:
iii)' For all x and y in Φ, the element $y - 2\frac{(x,y)}{(x,x)}x \in \Phi$.
iv)' For all x and y in Φ, the number $2\frac{(x,y)}{(x,x)}$ is an integer.

To a semisimple Lie algebra, we attach a root system from which we can read off the structure of the Lie algebra and its representations. As every root system arises from a semisimple Lie algebra and determines it up to isomorphism, root systems classify the semisimple Lie algebras over $\mathbb{C}$.
Now let $\mathfrak{g}$ be a complex semisimple Lie algebra, choose a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$, recall that $\mathfrak{h}$ is abelian (cf. [6]). The adjoint action of $\mathfrak{h}$ on $\mathfrak{g}$ leads to a root decomposition of $\mathfrak{g}$ with respect to $\mathfrak{h}$ as follows:
For each λ of $\mathfrak{h}^\star$, the algebraic dual of $\mathfrak{h}$, we define the vector subspace of $\mathfrak{g}$ by:

$$\mathfrak{g}_\lambda = \{a \in \mathfrak{g} : [h,a] = \lambda(h)a \quad \text{for all} \quad h \in \mathfrak{h}\}$$

A nonzero λ in $\mathfrak{h}^\star$ is called a root if the subspace $\mathfrak{g}_\lambda$ is not trivial. If this is the case, the subspace $\mathfrak{g}_\lambda$ is called the *root space* of $\mathfrak{g}$.
If we denote the set of all roots by Φ then $\mathfrak{g}$ decomposes as:

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\lambda \in \Phi} \mathfrak{g}_\lambda \tag{1.1}$$

Φ is a root system of $\mathfrak{g}$.

Note that since $\mathfrak{h}$ is a Carton subalgebra $\mathfrak{h} = \mathfrak{g}_0$, and that the subspace $\mathfrak{g}_\lambda$ is one-dimensional for all nonzero λ. This decomposition of $\mathfrak{g}$ is called *the root space decomposition* (cf. [2], [5] or [4]).

Example 1.2.11. Let $\mathfrak{g} = \mathfrak{sl}_n$ for $n \geq 2$. $\mathfrak{g}$ is simple (therefore semisimple). $\mathfrak{h}$ is the subspace of $\mathfrak{g}$ of all diagonal matrices in $\mathfrak{g}$.

For each $i = 1, 2, \cdots n$, let e_i be the element of $\mathfrak{h}^\star$ defined by:

$$e_i \begin{pmatrix} h_1 & & & \\ & h_2 & & \\ & & \ddots & \\ & & & h_n \end{pmatrix} = h_i$$

The root space decomposition is then:

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{i \neq j} \mathfrak{g}_{(e_i - e_j)}$$

where

$$\mathfrak{g}_{(e_i - e_j)} = \{X \in \mathfrak{g} \mid [h, X] = (e_i - e_j)X \quad \text{for all } h \in \mathfrak{h}\}$$

1.2.3 Positive and simple roots, Weyl group

Definition 1.2.12. Given a root system Φ we can always choose a set of positive roots. These are a subset Φ^+ of Φ satisfying:

- For each root $\alpha \in \Phi$, exactly one of the roots $\alpha, -\alpha$ is in Φ^+.
- For any two distinct $\alpha, \beta \in \Phi^+$ such that $\alpha + \beta$ is a root, $\alpha + \beta \in \Phi^+$.
  If a set of positive roots Φ^+ is chosen, elements of $-\Phi^+$ are called negative roots.
- An element of Φ^+ that can not be written as a sum of two elements of Φ^+ is called a simple root. Moreover, the set Δ of simple roots forms a basis of $\mathfrak{h}^\star$ and has the property that every element in Φ is a linear combination of elements of Δ with all coefficients nonnegative or all coefficients nonpositive.

Now consider the (real) euclidean space $\mathfrak{h}^\star_{\mathbb{R}}$ whose basis is Δ. In this vector space, the lattice generated by Δ is called the root lattice of $\mathfrak{g}$.

Notice that

$$dim_{\mathbb{R}} \mathfrak{h}^\star_{\mathbb{R}} = dim_{\mathbb{C}} \mathfrak{h}$$

This dimension is called the rank of $\mathfrak{g}$ (cf. [3]).

Remark 1.2.13. The Euclidean structure of $\mathfrak{h}_{\mathbb{R}}^{\star}$ needs some explanation.
The Killing form $< x, y > = \mathrm{tr}(ad(x)ad(y))$ is nondegenerate on $\mathfrak{h}$ as well, this is inherited from its nondegeneracy on $\mathfrak{g}$.
By letting $f(x)(y) \; = < x, y >$ for all x and y in $\mathfrak{h}$, the map:

$$\mathfrak{h} \longrightarrow \mathfrak{h}^{\star}$$
$$x \mapsto f(x)$$

is a bijection.
If we then set $< f(x), f(y) > = < x, y >$ for all x and y in $\mathfrak{h}$, we have defined a bilinear form on $\mathfrak{h}^{\star}$. When this bilinear from is restricted to $\mathfrak{h}_{\mathbb{R}}^{\star}$, it gives the desired Euclidean structure.

Example 1.2.14. The case where $\mathfrak{g}$ is the special Lie algebra $\mathfrak{sl}_2$, a basis of $\mathfrak{g}$ is given by:

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \; X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \text{ and } \; Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

The following relations are clearly satisfied: $[H, X] = 2X$, $[H, Y] = -2Y$ and $[X, Y] = H$. Therefore there are two roots α and $-\alpha$, where $\alpha(H) = 2$. Thus $\Delta = \Phi^{+} = \{\alpha\}$.

Example 1.2.15. Let $\mathfrak{g}$ is the special Lie algebra $\mathfrak{sl}_3$ and $\mathfrak{h}$ be the subalgebra made of the diagonal matrices with zero trace. A basis of $\mathfrak{h}$ is given by:

$$H_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \text{ and } H_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Consider the linear functionals $(\alpha_i)_{i=1,2}$ defined by:

$$\alpha_i \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} = \lambda_i - \lambda_{i+1}$$

then $\{\alpha_1, \alpha_2\}$ is a basis of $\mathfrak{h}^{\star}$. We have then:
The rank of $\mathfrak{g}$ is 2, $\Delta = \{\alpha_1, \alpha_2\}$ and $\Phi = \{\alpha_1, \alpha_2, \alpha_1 + \alpha_2, -\alpha_1, -\alpha_2, -\alpha_1 - \alpha_2\}$.

Below is the configuration of the roots.

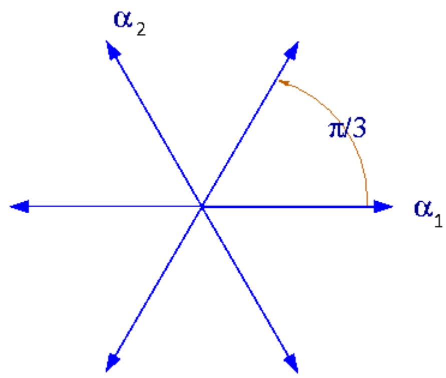


Figure 1.1: Root system of A_2

Definition 1.2.16. The Weyl group of a root system Φ is the group generated by reflections through the hyperplanes orthogonal to the roots, it is a finite reflection group.
The Weyl group of a semisimple Lie algebra, is the Weyl group of the root system of that Lie algebra.

Example 1.2.17. The Weyl group of the simple Lie algebra $\mathfrak{sl}_2$ is isomorphic to $\mathbb{Z}/2\mathbb{Z}$. More generally the Weyl group of $\mathfrak{sl}_n$ is isomorphic to the symmetric group S_n (cf. [7] Appendix C).

Remark 1.2.18.

i. The set of positive roots can be ordered naturally as follows:
$\alpha \leq \beta$ if and only if $\beta - \alpha$ is a non negative linear combination of simple roots.

ii. For each α in Φ, the coroot of α is defined by

$$\alpha^\vee = \frac{2}{(\alpha, \alpha)}\alpha.$$

The set of coroots form another root system $\Phi^\vee$.
Notice that $\alpha^{\vee\vee} = \alpha$, $\Phi^\vee$ is called the dual root system of Φ.

iii. The lattice generated by Φ (resp. $\Phi^\vee$) is called the root lattice (resp. coroot lattice), also Φ and $\Phi^\vee$ define the same Weyl group W (1.2.16) and if Δ is a set of simple roots of Φ then $\Delta^\vee$ is a set of simple roots of $\Phi^\vee$.
Furthermore we have: $s(\alpha)^\vee = s(\alpha^\vee)$ for all s in W.

1.2.4 Dynkin diagrams

Definition 1.2.19. A root system Φ is said to be irreducible if it can not be written as a union of two proper subsets Φ_1 and Φ_2 satisfying $(\alpha, \beta) = 0$ for all α in Φ_1 and β in Φ_2.
To each irreducible root system correspond a graph called a Dynkin diagram (cf. Figure 1.2).
The vertices of a Dynkin diagram correspond to the elements in Δ , the set of simple roots, in the following way:
Each pair of nonorthogonal pair of elements of Δ are connected by:
a. An undirected simple edge if they make an (acute) angle of $\frac{2\pi}{3}$.
b. A directed double edge if they make an angle of $\frac{3\pi}{4}$.
c. A directed triple edge if they make an angle of $\frac{5\pi}{6}$.

The direction is made towards the shortest vector in the pair of vectors in question. Dynkin diagrams are independent of the choice of simple roots.
From the discussion above, we see that the classification of root systems is the same as that of Dynkin diagrams. For instance connected Dynkin diagrams correspond to irreducible root systems.
The graphs below show all the possible irreducible Dynkin diagrams (cf. [7] Chapter II).

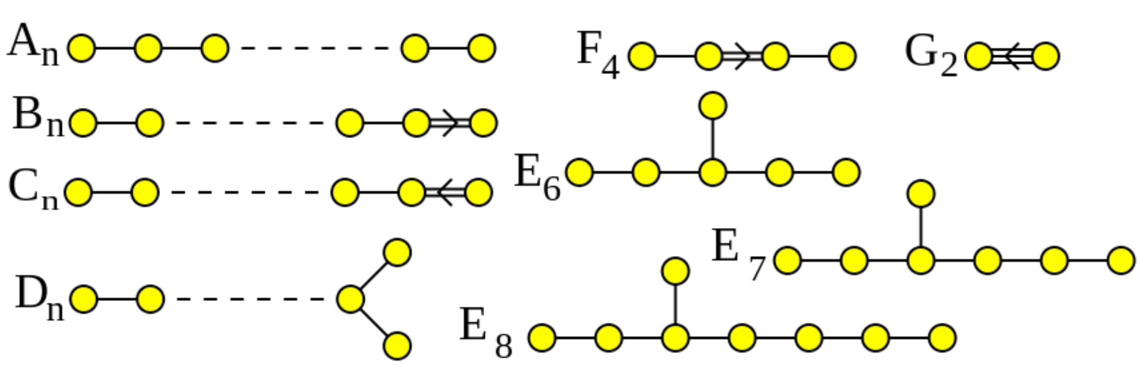


Figure 1.2: Dynkin diagrams of all simple Lie algebras.

1.3 Representations of simple Lie algebras

Here we shed light on irreducible representations of semisimple Lie algebras. We state a few important concepts such as: Highest weight modules. We also introduce the notion of fundamental representations of classical simple Lie algebras.

1.3.1 Weights and weight lattice

Let $\mathfrak{g}$ be a semisimple Lie algebra. Choose a Cartan subalgebra $\mathfrak{h}$ in $\mathfrak{g}$.
The weights of $\mathfrak{g}$ are the linear functionals $\lambda \in \mathfrak{h}^\star$.
Let V be a $\mathfrak{g}$-module, $\alpha \in \mathfrak{h}^\star$, let V_α be defined by:

$$V_\alpha = \{v \in V \mid h \cdot v = \alpha(h)v \text{ for all } h \in \mathfrak{h}\}$$

The dot " $\cdot$ " stands for the action of $\mathfrak{h}$ on V.
An $\alpha \in \mathfrak{h}^\star$ for which V_α is a nonzero vector subspace of V is called a *weight* of V. Furthermore, if $V = \bigoplus_{\alpha \in \mathfrak{h}^\star} V_\alpha$ then V it is called a weight module.
The nonzero vectors of V_α 's are called weight vectors.
Notice that, since V is a finite dimensional space, V has only finitely many weights, say $\alpha_1, \alpha_2, \alpha_3, \cdots \alpha_n$.

Example 1.3.1. Let $\mathfrak{g} = \mathfrak{sl}_3$.
By choosing the Cartan subalgebra $\mathfrak{h}$ to be the (abelian) Lie subalgebra of diagonal matrices with trace zero. Recall that $\mathfrak{h}$ is spanned by:

$$H_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \text{ and } H_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

The weights of $\mathbb{C}^3$ as a $\mathfrak{g}$-module are:
$\alpha_i(H) = h_{ii} - h_{i+1,i+1}$ for $i = 1, 2$ and $H = \begin{pmatrix} h_{11} & 0 & 0 \\ 0 & h_{22} & 0 \\ 0 & 0 & h_{33} \end{pmatrix}$
Note: $h_{11} + h_{22} + h_{33} = 0$
The associated decomposition is then:

$$\mathbb{C}^3 = < e_1 > \oplus < e_2 > \oplus < e_3 >$$

where (e_1, e_2, e_3) is the standard basis of the vector space $\mathbb{C}^3$.
The weight spaces are the one dimensional vector spaces appearing in the decomposition (cf. (`http://math.mit.edu/classes/18.745/Notes/`).

Remark 1.3.2. For a semisimple Lie algebra the weights of the adjoint representation are the roots of the Lie algebra, and the weight space is called root space.

Definition 1.3.3. Let $\mathfrak{g}$ be a simple Lie algebra of rank l and let

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\lambda \in \Phi} \mathfrak{g}_\lambda$$

be the root space decomposition of $\mathfrak{g}$ with respect to $\mathfrak{h}$.
Let $h_1, h_2, \cdots h_l$ be the elements of $\mathfrak{h}$ corresponding to $\frac{2\alpha_i}{<\alpha_i,\alpha_i>}$ of $\mathfrak{h}^\star$ under the bijection of (1.2.13)

we have:

$$\alpha_j(h_i) = \frac{2 < \alpha_i, \alpha_j >}{< \alpha_i, \alpha_i >}$$

is an integer for all $i, j = 1, 2, \cdots l$.
The basis elements $h_i, h_2, \cdots h_l$ are called: *coroots* or *simple coroots*.
An element $\lambda \in \mathfrak{h}^\star$ is called *integral* (resp. *dominant integral*) if $\lambda(h_i) \in \mathbb{Z}$ (resp. $\lambda(h_i) \geq 0$) for all $i = 1, 2, \cdots l$.

Remark 1.3.4. Dominant integral weights can also be described as follows:
Let $\omega_1, \omega_2, \cdots \omega_l$ be the elements of $\mathfrak{h}^\star$ defined by:

$$\omega_i(h_j) = \delta_{ij}, \text{ (Kronecker delta)}$$

Put another way: $\omega_1, \omega_2, \cdots \omega_l$ is the dual basis of the basis $h_1, h_2, \cdots h_l$.
The elements $\{\omega_i\}_{i=1,2,\cdots l}$ are called: *fundamental weights*. An immediate consequence is that any weight $\lambda \in \mathfrak{h}^\star$ is expressed as:

$$\lambda = \lambda(h_1)\omega_1 + \lambda(h_2)\omega_2 + \cdots + \lambda(h_l)\omega_l$$

The set of integral weights (resp. dominant integral weights) is denoted by $P(\mathfrak{g})$ (resp. $P_+(\mathfrak{g})$).
$P(\mathfrak{g})$ is a weight lattice in $\mathfrak{h}^\star$, and $P_+(\mathfrak{g})$ is a cone in $\mathfrak{h}^\star$ (cf. [3]).

Example 1.3.5. Below are the fundamental roots and weights of the simple Lie algebra A_2.

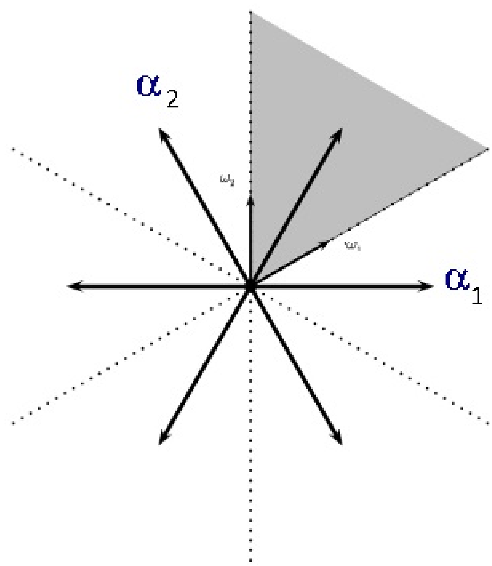


Figure 1.3: Fundamental roots and weights of A_2

Notice that: $\omega_1 + \omega_2 = \alpha_1 + \alpha_2$, ω_1 is orthogonal to α_2 and ω_2 is orthogonal to α_1. The weight $\rho = \omega_1 + \omega_2 = \alpha_1 + \alpha_2$ is a of combinatorial importance, it is used in both the *Weyl's character and dimension formulas* (cf. [3] or [5]).

Remark 1.3.6. We recall that in (1.2.18), the set of simple roots has been ordered. In a similar way, the weights can be ordered as follows:
For $\lambda, \mu \in \mathfrak{h}^\star_\mathbb{R}$, $\lambda \leq \mu$ if and only if $\mu - \lambda$ if a nonnegative linear combination of simple roots. The convex hull of dominant weights of $\mathfrak{h}^\star_\mathbb{R}$ is known as the *Weyl fundamental chamber.*

1.3.2 Highest weight modules

Definition 1.3.7. Let V be a representation of Lie algebra $\mathfrak{g}$.
a). A weight λ of V is called a *highest weight* if for every weight μ of V, $\mu \leq \lambda$.
b). A weight vector v_λ of weight λ is called a highest weight vector if λ is a highest weight of V.

For a complex Lie algebra $\mathfrak{g}$, choose a Cartan subalgebra $\mathfrak{h}$ and a set positive roots Φ^+. One has:

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{n}^+ \oplus \mathfrak{n}^-$$

where

$$\mathfrak{n}^+ = \bigoplus_{\alpha\in\Phi^+} \mathfrak{g}_\alpha \quad \textit{and} \quad \mathfrak{n}^- = \bigoplus_{\alpha\in\Phi^+} \mathfrak{g}_{-\alpha}$$

Definition 1.3.8. A representation V of a Lie algebra $\mathfrak{g}$ is called *highest weight module* if V is generated by a weight vector v such that:

$$\mathfrak{n}^+ v = 0$$

where $\mathfrak{n}^+$ is defined as above.

There is a bijective correspondence between irreducible finite dimensional representation of a simple Lie algebra $\mathfrak{g}$ (up to isomorphism) and the set of dominant integral weights λ. The simple $\mathfrak{g}$-module corresponding to λ is denoted by $L(\lambda)$. Thus we have:

Theorem 1.3.9 (*Highest weight theorem*)**.** *Every finite dimensional irreducible representation of a complex semisimple Lie algebra $\mathfrak{g}$ is of the form $L(\lambda)$ for some dominant integral weight λ.*

Proof. The proof of this theorem as well as the definition of the $\mathfrak{g}$-module $L(\lambda)$ for $\lambda \in \mathfrak{h}^\star$ can be found in any standard book of representation theory (e.g [3]).

□

This result is due to *E. Cartan*, back in 1894.

Remark 1.3.10. The construction of $L(\lambda)$, for any weight λ, uses the theory of *Verma Modules* (cf. [4], [3]).

1.3.3 Exterior and symmetric algebras

We start here with a complex vector space V.

Definition 1.3.11. The tensor algebra of V is defined by:

$$T(V) = \mathbb{C}1 \oplus V \oplus V \otimes V \oplus V \otimes V \otimes V \oplus \cdots$$

Let I be the two sided ideal of $T(V)$ generated by all elements of the form $u \otimes v - v \otimes u$ where u and v are in V. The symmetric algebra $S(V)$ in defined by:

$$S(V) = T(V)/I$$

For each $i = 0, 1, 2, \cdots$, let $T^i(V)$ be the n-fold tensor product of V with itself. Let $S^i(V)$ be the space of all symmetric tensors of order i defined on V. In other words

$$S^i(V) = T^i(V)/(I \cap T^i(V))$$

We have:

$$T(V) = \bigoplus_{i=0}^{\infty} T^i(V) \quad \text{and} \quad S(V) = \bigoplus_{i=0}^{\infty} S^i(V)$$

We note that $S(V)$ is then a graded associative algebra.
If $dim(V) \neq 0$, the dimension of the vector space $S(V)$ if infinite. However, the dimension of each subspace $S^i(V)$ (the i^{th} symmetric power of V) is give by:

$$dim\ S^i(V) = \binom{n+i-1}{i} \tag{1.2}$$

$S(V)$ is also denoted by $S(V)$, and $S^i(V)$ is called the i^{th}- symmetric power of V.
With a similar construction, we define the exterior algebra $\bigwedge(V)$ of an n-dimensional vector space V over the field $\mathbb{C}$ by letting the two sided ideal I' of $T(V)$ in 1.3.11) be generated by the elements elements $v \otimes v$, with $v \in V$.
The exterior algebra of V is defined by

$$\bigwedge(V) = T(V)/I'$$

We also have:

$$\bigwedge(V) = \bigoplus_{i=0}^{n} \wedge^i(V) \quad \text{where} \quad \wedge^i (V) = T^i(V)/(I' \cap T^i(V))$$

The dimension of each subspace $\bigwedge^i(V)$ (i^{th} exterior power of V) and that of the exterior algebra are give by:

$$dim\ \wedge^i (V) = \binom{n}{i} \quad and \quad dim\ \bigwedge(V) = 2^n \tag{1.3}$$

$\bigwedge(V)$ becomes an algebra with the wedge product $a \wedge b$.
If $\{v_1, v_2, \cdots v_n\}$ is a basis of V then a basis for $\bigwedge^k(V)$ is:

$$\{v_{i_1} \wedge v_{i_2} \wedge \cdots \wedge v_{i_k} | \quad 1 \leq i_1 < i_2 \cdots < i_{n-1} \leq n\}$$

For more details about tensors, symmetric and exterior algebras (cf. [7] Appendix A, [3]).
The symmetric and exterior algebras of the defining representation are extremely crucial to

representation theory, in that, if V be the defining representation of a semisimple Lie algebra $\mathfrak{g}$, then both these vector spaces are $\mathfrak{g}$-modules, the action of $\mathfrak{g}$ on each is given by:

$$g(v_{i_1} \wedge v_{i_2} \wedge \cdots v_{i_k}) = gv_{i_1} \wedge v_{i_2} \wedge \cdots \wedge v_{i_k} + v_{i_1} \wedge gv_{i_2} \wedge \cdots \wedge v_{i_k} + \cdots \cdots v_{i_1} \wedge v_{i_2} \wedge \cdots \wedge gv_{i_k}$$

and

$$g(v_{i_1} \cdot v_{i_2} \cdot \cdots v_{i_k}) = gv_{i_1} \cdot v_{i_2} \cdot \cdots \cdot v_{i_k} + v_{i_1} \cdot gv_{i_2} \cdot \cdots \cdot v_{i_k} + \cdots \cdots v_{i_1} \cdot v_{i_2} \cdot \cdots \cdot gv_{i_k}$$

1.3.4 Fundamental representations:

The highest weights representations of a semisimple Lie algebra $\mathfrak{g}$ corresponding to the fundamental weights: $\omega_1, \omega_2, \cdots \omega_l$, are called *fundamental representations.*
It is a fundamental fact in representation theory that if $\mathfrak{g}$ is a semisimple Lie algebra with fundamental weights $\omega_1, \omega_2, \cdots \omega_l$ then:
For any weight: $\lambda = m_1\omega_1 + m_2\omega_2 + \cdots m_l\omega_l$, $L(\lambda)$ is a submodule of :

$$\underbrace{L(\omega_1) \otimes \cdots \otimes L(\omega_1)}_{m_1} \oplus \underbrace{L(\omega_2) \otimes \cdots \otimes L(\omega_2)}_{m_2} \oplus \cdots \oplus \underbrace{L(\omega_l) \otimes \cdots \otimes L(\omega_l)}_{m_l}$$

for some nonnegative integers $m_1, m_2, \cdots m_l$ (cf. [3]).

$L(\omega_1)$ is called: *the defining (or standard) representation.*
The fundamental representations of exceptional simple Lie algebras, which are not essential here, can be found in ([3] chap 3.5).
For classical simple Lie algebras A_l, B_l, C_l and D_l, the table below gives the dimensions of the fundamental representations $L(\omega_1), L(\omega_2), \cdots Ł(\omega_l)$, each corresponding to a node in associated Dynkin diagram (Figure 1.2).

	$L(\omega_1)$	$L(\omega_2)$	$L(\omega_3)$	$\cdots\cdots$	$L(\omega_{l-1})$	$L(\omega_l)$
A_l	$l+1$	$\binom{l+1}{2}$	$\binom{l+1}{3}$	$\cdots\cdots$	$\binom{l+1}{2}$	$l+1$
B_l	$2l+1$	$\binom{2l+1}{2}$	$\binom{2l+1}{3}$	$\cdots\cdots$	$\binom{2l+1}{l-1}$	2^l
C_l	$2l$	$\binom{2l}{2}-1$	$\binom{2l}{3}-2l$	$\cdots\cdots$	$\binom{2l}{l-1}-\binom{2l}{l-3}$	$\binom{2l}{l-1}-\binom{2l}{l-2}$
D_l	$2l$	$\binom{2l}{2}$	$\binom{2l}{3}$	$\cdots\cdots$	$\binom{2l}{l-2}$	2^l-1

Figure 1.4: Dimensions of fundamental representations of A_l, B_l, C_l and D_l.

1.4 Simple modules for $\mathfrak{sl}_2$

Here we focus on the simple Lie algebra $\mathfrak{sl}_2$, its finite dimensional irreducible representations and their characters. We provide an explicit description of these.

1.4.1 Simple modules for $\mathfrak{sl}_2$

We recall here the fixed $\mathfrak{sl}_2$ basis X, H and Y (of 1.2.14) satisfying:

$$[H,X]=2X, \ \ [H,Y]=-2Y \text{ and } \ [X,Y]=H$$

Let V be an irreducible finite dimensional representation of $\mathfrak{sl}_2$, V has the decomposition:

$$V=\bigoplus V_\alpha$$

where each V_α is non trivial and $H(v)=\alpha\cdot v$ for all v in V_α.
Now we need to explicitly describe how X and Y above act on V, or equivalently, on each V_α.
Assume v in a vector in V_α, we get:

$$\begin{aligned} H(X(v)) &= X(H(v))+[H,X](v) \\ &= X(\alpha\cdot v)+2X(v) \\ &= (\alpha+2)X(v) \end{aligned}$$

Therefore if $v \in V_\alpha$ then $X(v) \in V_{\alpha+2}$. Similarly, if $v \in V_\alpha$ then $Y(v) \in V_{\alpha-2}$

The action of H, X and Y which summarize the action of $\mathfrak{sl}_2$ on V show that:

$$V = \bigoplus_{n\in\mathbb{Z}} V_{\alpha_0+2n} \tag{1.4}$$

where α_0 is any complex number such that V_{α_0} is non trivial.

Since $V = \bigoplus\limits_{n\in\mathbb{Z}} V_{\alpha_0+2n}$ is invariant under $\mathfrak{sl}_2$ it must be the whole space V.
$\dim V < \infty$ implies that the set of complex numbers (actually integers) $\{v_{\alpha_0} + 2n\}_n$ is finite.
Let n be the maximum of these integers. The subspaces of (1.4) are:

$$\cdots, V_{n-4},\ V_{n-2},\ V_n$$

Let now v be any nonzero vector in the subspace V_n, we must have:

$$X(v) \in V_{n+2} = (0) \ \text{ thus } \ X(v) = 0$$

Proposition 1.4.1. *The vectors $v, Y(v), Y^2(v), \cdots$ span the space V.*

Proof. (cf. [4], chap 11). □

It is not difficult to show by induction (cf. [4], chap 11), that the action of X on an element of the form $Y^m(v)$ is given by:

$$X(Y^m(v) = m(n-m+1) \cdot Y^{m-1}(v) \tag{1.5}$$

Furthermore since V is finite dimensional, let m be the smallest power of Y such that $Y^m(v) = 0$ We have then:

$$m(n-m+1) \cdot Y^{m-1}(v) = X(Y^m)(v) = 0$$

Thus $n = m - 1$, which shows that n must be a nonnegative integer.
Moreover, if we use the notation V^n in lieu of V, we have: $\dim(V^n) = m = n + 1$.
Finally for each non negative integer n, the eigenvalues of H are $-n, -n+2, \cdots, n-2, n$.
Also, each of the V_α for $\alpha = -n, -n+2, \cdots, n-2, n$, is a one dimensional subspace of V.
As a consequence

$$dim(V) = n + 1 \tag{1.6}$$

Example 1.4.2. Let $V = \mathbb{C}^2$ as vector space over $\mathbb{C}$ with standard basis $x = (1, 0)$ and $y = (0, 1)$.

Consider the vector space $S^2(V)$ with the basis $\{x^2, xy, y^2\}$. The action of H is given by:

$$\begin{aligned} H(x \cdot x) &= 2x \cdot x, \\ H(x \cdot y) &= 0, \\ H(y \cdot y) &= -2y \cdot y \end{aligned}$$

thus the decomposition of V_2 as an $\mathfrak{sl}_2$- representation is:

$$V_2 = \mathbb{C}x^2 \oplus \mathbb{C} \cdot xy \oplus \mathbb{C}y^2$$

The eigenvalues of H are -2, 0 and 2.

This result can be generalized to $S^n(V)$, the n^{th} symmetric tensor of $V = \mathbb{C}^2$.
We take the basis $\{x^n, x^{n-1}y, x^{n-2}y^2, \cdots y^n\}$ as in the case of $n = 2$ above.
The action of H on an element $x^{n-k}y^k$ is:

$$\begin{aligned} H(x^{n-k}y^k) &= (n-k) \cdot H(x) \cdot x^{n-k-1}y^k + k \cdot H(y) \cdot x^{n-k}yk - 1 \\ &= (n-2k) \cdot x^{n-k}y^k \end{aligned}$$

Thus the action of H on $s^n(V)$ has eigenvalues $-n, -n+2, \cdots n-2, n$, the same as its eigenvalues on $V^{(n)}$. since both $S^n(V)$ and $V^{(n)}$ are irreducible, it follows that:

$$V^{(n)} = S^n(V)$$

In other words, Every irreducible $\mathfrak{sl}_2$- representation is of the form $S^n(V)$ where $V \cong \mathbb{C}^2$ (cf. [4], chap 11).

1.4.2 Important properties

A few important results, one of which describes the decomposition of the tensor product of two irreducible $\mathfrak{sl}_2$- representations, are given below: (cf. [4], chap 11).
As above, $V = \mathbb{C}^2$, is the defining representation of $\mathfrak{sl}_2$.

Proposition 1.4.3. *For any nonnegative integers a and b, $a \geq b$:*

$$S^aV \otimes S^bV = S^{a+b}V \oplus S^{a+b-2}V \oplus \cdots \oplus S^{a-b}V$$

Proposition 1.4.4. *For any non negative integer n,*

$$S^n(S^2V) = \bigoplus_{\alpha=0}^{[\frac{n}{2}]} S^{2n-4\alpha}V$$

Example 1.4.5. We have the following decompositions:

$$S^4V \otimes S^3V = S^7V \oplus S^5V \oplus S^3V \oplus S^1V$$

$$S^4(S^2V) = S^8V \oplus S^4V \oplus S^0V$$

We also will use the following property in the proof of our result:

Proposition 1.4.6.

$$\wedge^m(S^nV) \cong S^m(S^{n+1-m}V)$$

(cf. [4] chap 11, Ex 35).

1.4.3 Characters of $\mathfrak{sl}_2$

Let (F_k, ρ) be the irreducible representation of the group SL_2 with dim $F_k = k+1$. We know that every diagonalizable element in SL_2 is conjugate to an element of the form $d(q) = \begin{bmatrix} q & 0 \\ 0 & q^{-1} \end{bmatrix}$ for some $q \in \mathbb{C}^\times$. Hence the character $ch(F_k)$ is uniquely determined by

$$chF_k(d(q)) = q^k + q^{k-2} + \cdots + q^{-k+2} + q^{-k} = [k+1]_q$$

where by definition:

$$[n]_q = \; q^{n-1} + q^{n-3} + \cdots + q^{-n+3} + q^{-n+1}.$$

Furthermore, we define the *q-factorial* as follows:

$$[0]_q = 1, \quad \text{and} \quad [n]_{q!} = \prod_{j=0}^{n} [n-j]_q \quad \text{for } n \geq 1$$

and the *q-binomial coefficient* as

$$\begin{bmatrix} n+m \\ n \end{bmatrix}_q = \frac{[n+m]_{q!}}{[m]_{q!}[n]_{q!}}$$

Theorem 1.4.7. *For all* $q \in \mathbb{C}^\times$

$$chS^m(F_k)(d(q)) = \begin{bmatrix} k+m \\ k \end{bmatrix}_q$$

Where $S^m(F_k)$ *is the* m^{th} *symmetric power of* F_k*. The latter being the* k^{th}*-symmetric power of* $V = \mathbb{C}^2$ *or* $F_k = S^k(\mathbb{C}^2)$*.*

Proof. Details of the proof can be found in numerous sources like ([5], chap 4). □

Corollary 1.4.8. *We do have the following isomorphism:*

$$S^n(S^m V) \cong S^m(S^n V)$$

as $\mathfrak{sl}_2$*- representations. Here again* $V = \mathbb{C}^2$*.*

This corollary is known as *Hermite reciprocity* and will be used in the proof of our result.

Chapter 2

Our result and its proof

In this chapter our result and its proof are exposed.

2.1 Introduction

Key words: Principal embedding, small subalgebra, Cartan-Helgason theorem, Pieri rules, Hermite reciprocity theorem, branching algebra.

2.1.1 Definitions, examples and notations

Definition 2.1.1. Let k be a nonnegative integer. Using the same notation as in (1.4.3), let F_k be the complex irreducible $\mathfrak{sl}_2$- representation of dimension $n = k + 1$. We then have the following Lie algebra homomorphism:

$$\pi : \mathfrak{sl}_2 \to \mathrm{End}(F_k)$$

If a basis is fixed in F_k, $\mathrm{End}(F_k)$ is identified with $\mathfrak{gl}_n$.
For $k \geq 1$ The kernel of π is trivial since $\mathfrak{sl}_2$ is simple. Therefore $\mathfrak{sl}_2 \cong \pi(\mathfrak{sl}_2)$.
Under this isomorphism , $\mathfrak{sl}_2$ intersects the center of $\mathfrak{gl}_n$ trivially, therefore $\mathfrak{sl}_2$ is embedded in $\mathfrak{sl}_n$. It is called the principal $\mathfrak{sl}_2$ subalgebra of $\mathfrak{sl}_n$.

Other embeddings, not needed for our purpose, are possible. For instance, there are (up to conjugation) three nonisomorphic $\mathfrak{sl}_2$ maximal Lie subalgebras embedded in the exceptional Lie algebra E_8, only one of which is principal. The LiE software can be used to quickly check this fact (cf. [10]).

We recall here some standard assumptions and notations used: The ground field is $\mathbb{C}$, the field of complex numbers. Unless explicitly stated, all Lie algebras and representations are assumed to be finite dimensional and over $\mathbb{C}$.

If m is a positive integer and V is a vector space then we write,

$$mV = \underbrace{V \oplus V \oplus \cdots \oplus V}_{m \text{ copies.}}$$

If $m = 0$ then $mV = (0)$.

If G is a group (resp. Lie algebra), which acts on a vector space V, we will denote the subspace of pointwise fixed vectors (i.e. invariant vectors) by V^G.
If G acts on both V_1 and V_2 then G acts on $\mathrm{Hom}(V_1, V_2)$ by

$$g \cdot T := gTg^{-1}$$

The G-invariant vectors in $\mathrm{Hom}(V_1, V_2)$ are exactly the the G-equivariant homomorphisms, denoted $\mathrm{Hom}_G(V_1, V_2)$.
We define $\mathrm{mult}_G(V_1 : V_2)$ by:

$$\mathrm{mult}_G\,(V_1 : V_2) = dim\ \mathrm{Hom}_G(V_1, V_2).$$

If G (resp. $\mathfrak{g}$) is understood, it will be dropped from notation.

Note that in the case that V_1 (resp. V_2) is irreducible and V_2 (resp. V_1) is completely reducible then $\mathrm{mult}(V_1, V_2)$ is the *multiplicity* of V_1 (resp. V_2) in V_2 (resp. V_1).

Fix $\{V_\lambda : \lambda \in \widehat{G}\}$ to be the set of distinct representatives of the equivalence classes of irreducible representations of G , with index set $\widehat{G}$. Then, for an arbitrary completely reducible G-representation, V we have:

$$V \cong \bigoplus_{\lambda \in \widehat{G}} m_\lambda V_\lambda$$

where the nonnegative integers m_λ are the multiplicities, that is $m_\lambda = \mathrm{mult}(V_\lambda : V)$.

If G is a subgroup of a larger group H, and V is an irreducible H-representation then V becomes a G-representation under the restricted action, which we denote by $\mathrm{Res}^H_G V$.
The numbers $\mathrm{mult}_G(V_\lambda : \mathrm{Res}^{\mathrm{H}}_G\, V)$ are sometimes called the *branching multiplicities* (cf. [5]).

2.1.2 Invariant theory

Proposition 2.1.2. *Let V and W be finite dimensional representations of a semi simple Lie algebra $\mathfrak{g}$. Then we have the natural isomorphism:*

$$Hom(V, W) \cong W \otimes V^\star$$

(cf. [5] Appendix B 2.2).

The $\mathfrak{g}$-representation $\mathrm{Hom}(V, W)$ is afforded by the action

$$g \cdot T = gT - Tg$$

for all g in $\mathfrak{g}$ and T in $\mathrm{Hom}(V, W)$.

Corollary 2.1.3. *Let V and W be as in (2.1.2), we have:*

$$Hom_g\,(V, W) \cong [W \otimes V^\star]^g$$

Induced representation

The process of constructing a representation of a Lie group G by starting by a representation of a subgroup H of G is called induction.

Definition 2.1.4. Let V be a representation of a group G and Let H be a subgroup of G. The restricted representation of V to H is the representation obtained by restricting the action of G to H. It is denoted by, $\mathrm{Res}_H^G(V)$.

Now let us reverse this process, in other words, one can construct a representation of G given a representation V of a subgroup H of G.

Definition 2.1.5. Let V be a representation of a subgroup H of a group G. The induced representation $\mathrm{Ind}_H^G(V)$ is defined by:

$$\mathrm{Ind}_H^G(V) = \{f : G \to V | \, f \text{ regular and } \; f(hx) = h \cdot f(x) \; \text{ for all } x \text{ in G and } h \text{ in } H\}$$

where the "$\cdot$" stands for the action of H on V.

Note: The action of G on $\mathrm{Ind}_H^G(V)$ is given by:

$$g(f)(x) = f(xg) \; \text{ for all g in G}$$

It is easy to check that this action in fact makes of $\mathrm{Ind}_H^G(V)$ a G-representation.

In the language of category theory, the functor "Res" is the functor right adjoint to the functor "Ind".

Remark 2.1.6. Induced representations can be constructed using tensor products as follows: A representation V of the subgroup H of a group G is also a module over the group ring $\mathbb{C}[H]$, the induced representation is then:

$$Ind_H^G V \; = \mathbb{C}[G] \otimes_{\mathbb{C}[H]} V$$

Theorem 2.1.7 (*Frobenius reciprocity*)**.** *Let H be a subgroup of G and V and W be representations of G and H respectively. We have*

$$Hom_G(V\ ,\ Ind_H^G W)\ \cong\ Hom_H(Res_H^G V\ ,\ W)$$

Proof. (cf. [4], chap 3). □

Example 2.1.8. $\mathrm{Ind}_H^G \mathbb{1} = \mathbb{C}[G]^H$.

2.1.3 Lie theoretic setup

Now let $\mathfrak{g}$ denote a rank l simple Lie algebra, and fix a Cartan subalgebra $\mathfrak{h}$.
Denote the set of roots (resp. positive roots) determined by $(\mathfrak{g}, \mathfrak{h})$ by Φ and Φ^+ as in (1.3.6).
Let $\omega_1, \omega_2, \cdots \omega_l$ be the fundamental weights of $(\mathfrak{g}, \mathfrak{h})$.
Also, let $P(\mathfrak{g})$ be the lattice of integral weights and $P_+(\mathfrak{g})$ be the cone of dominant weights of $\mathfrak{g}$ (1.3.4). We have:

$$P(\mathfrak{g}) = \sum_{j=1}^{l} \mathbb{Z}\omega_j$$

where $\mathbb{Z}$ is the set of integers and

$$P_+(\mathfrak{g}) = \sum_{j=1}^{l} \mathbb{N}\omega_j$$

where $\mathbb{N}$ is the set of non-negative integers. Set $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}^+$, the semi direct sum of $\mathfrak{h}$ and $\mathfrak{n}^+$ (cf. 1.3.2).
Let G be the unique (up to isomorphism) simply connected algebraic group whose Lie algebra is $\mathfrak{g}$; T_G be the maximal torus in G with Lie algebra $\mathfrak{h}$; and U_K denote the maximal unipotent group with Lie algebra $\mathfrak{n}^+$.
The coordinate ring of the regular functions on G is denoted , as usual, by $\mathbb{C}[G]$.
The group G acts on itself by left and right multiplication. Thus, $\mathbb{C}[G]$ is an infinite dimensional representation of $G \times G$ with respect to the action defined by:

$$[(g, g') \cdot f](x) = f(g^{-1}xg')$$

The algebraic version of the Peter-Weyl is given by the decomposition:

$$\mathbb{C}[G] \cong \bigoplus_{\lambda \in P_+(\mathfrak{g})} L_G(\lambda)^* \otimes L_G(\lambda).$$

where $L_G(\lambda)^*$ stands for the dual (irreducible) representation of the (irreducible) representation $L_G(\lambda)$.

2.1.4 A branching algebra

If we fix a simple Lie subalgebra $\mathfrak{s}$ of $\mathfrak{g}$, there exists a connected, Zariski closed, subgroup S of G with Lie algebra $\mathfrak{s}$. Let T_S (resp. U_S) denote the corresponding maximal torus (resp. unipotent subgroup) in S.

We will restrict the action of $G \times G$ on $\mathbb{C}[G]$ to the subgroup $S \times G$, that is, we restrict the left translation by K to that of S.

Define

$$\mathcal{B}_S^G = \mathbb{C}[G]^{U_S \times U_G}$$

For each of the groups S and G, the maximal torus normalizes the unipotent group.

$T_S \times T_G$ acts on these unipotent invariants. As a consequence, we obtain a gradation by the lattice cone $P_+(\mathfrak{s}) \times P_+(\mathfrak{g})$.

$\mathcal{B}_S^G$ consists of the highest weight vectors for the $S \times G$- action on $\mathbb{C}[G]$.

Let $\mathcal{B}_{S,\mu}^{G,\lambda}$ denote the $T_S \times T_G$-eigenspaces. We have

$$\mathcal{B}_S^G = \bigoplus \mathcal{B}_{S,\mu}^{G,\lambda}$$

where the sum is over $(\lambda, \mu) \in P_+(\mathfrak{s}) \times P_+(\mathfrak{g})$, this is a graded algebra.

Proposition 2.1.9. *Let $\lambda_1, \lambda_2 \in P_+(\mathfrak{g})$ and $\mu_1, \mu_2 \in P_+(\mathfrak{s})$.*
If for $j = 1$ and $j = 2$,

$$mult_S(L_S(\mu_j) : Res_S^G\, L_G(\lambda_j)) > 0$$

then

$$mult_S(L(\mu_1 + \mu_2) : Res_S^G\, L(\lambda_1 + \lambda_2)) > 0.$$

Proof. The dimension of the graded components of the branching algebra are equal to the branching multiplicities. Since G is connected, the branching algebra is a subalgebra of the integral domain $\mathbb{C}[G]$, and therefore has no zero divisors. □

It is worth pointing out that the proposition is particularly useful when we write $\mu_1 = \mu$ and consider the special case $\mu_2 = 0$.

Corollary 2.1.10. *For $\lambda_1, \lambda_2 \in P_+(\mathfrak{g})$ and $\mu \in P_+(\mathfrak{s})$. Assume $L_G(\lambda_2)^S \neq (0)$. Then if $L_S(\mu)$ occurs in $L_G(\lambda_1)$, as an S-representation, then $L_S(\mu)$ occurs in*

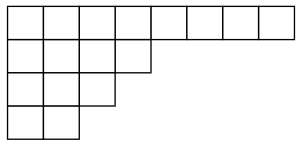
$$L_G(\lambda_1 + \lambda_2),\ L_G(\lambda_1 + 2\lambda_2),\ L_G(\lambda_1 + 3\lambda_2), \cdots, L_G(\lambda_1 + n\lambda_2), \cdots$$

2.1.5 Young diagrams and Pieri rules

Definition 2.1.11. A Young diagram is a finite collection of boxes, arranged in left justified rows, with the row lengths weakly decreasing. Listing the number of boxes in each row gives a partition $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_l)$ of a non-negative integer n, the total number of boxes of the diagram.

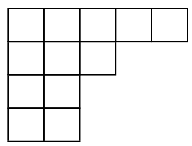
The Young diagram is said to be of shape λ, and it carries the same information as the associated partition.

Example 2.1.12.



(2.1)

and



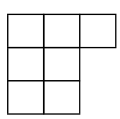
(2.2)

are two Young diagrams of shape $(8, 4, 3, 2)$ and $(5, 3, 2, 2)$, these correspond to two partitions of $n = 17$ and $n = 12$ respectively.

Young diagrams (and Young tableaux) have their original application to representations of the symmetric group (cf. [2] and [4]).

Young diagrams will serve in this work together with the Pieri rules to exhibit a symmetric power $\mathfrak{sp}_n$- representation $S^d(\mathbb{C}^n)$ in any $\mathfrak{sl}_n$- representation, when the action is restricted to $\mathfrak{sp}_n$. Similarly we will exhibit an exterior power $\mathfrak{so}_n$ -representation $\wedge^k(\mathbb{C}^n)$ in any (irreducible) $\mathfrak{sl}_n$- representation, when restricted to the $\mathfrak{sp}_n$ (here n must positive and even).

Example 2.1.13. let F_4^λ be the $\mathfrak{sl}_4$-module associated with the Young diagram below of shape $\lambda = (3, 2, 2, 0)$.



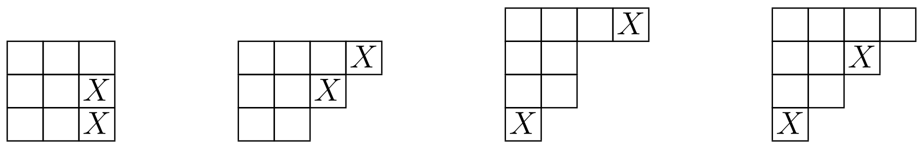
(2.3)

Question: How does the representation $F_4^\lambda \otimes \wedge^2\mathbb{C}^4$ decompose as a sum of $\mathfrak{sl}_4$- modules?

For each $i = 1, 2, 3, 4$, let ϵ_i be the linear functional:

$$\epsilon_i : \begin{bmatrix} d_1 & & & \\ & d_2 & & \\ & & \ddots & \\ & & & d_n \end{bmatrix} \mapsto d_i$$

Since $\mathfrak{sl}_4$ is comprised of 4×4 trace zero matrices, we have then, $\epsilon_1 + \epsilon_2 + \epsilon_3 + \epsilon_4 = 0$.
If ω_1, ω_2, ω_3 are the fundamental weights of $\mathfrak{sl}_4$ then $\lambda = 3\epsilon_1 + 2\epsilon_2 + 2\epsilon_3 = \omega_1 + 2\omega_3$.
Here are the four different Young diagrams obtained by adding two boxes to the initial diagram with no two boxes in the same row.



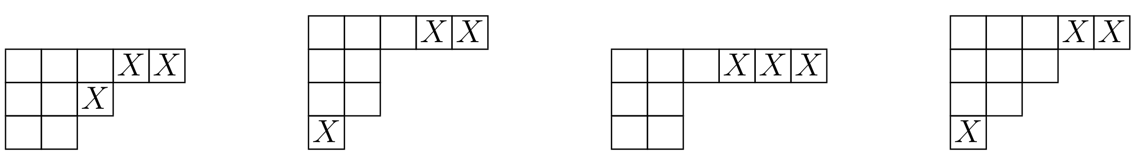
(2.4)

Figure 2.1: Young diagrams corresponding to irreducible constituents in the decomposition of $F_4^\lambda \otimes \wedge^2\mathbb{C}^4$.

Thus the decomposition sought is:

$$F_4^\lambda \otimes \wedge^2\mathbb{C}^4 \cong L(\omega_3) \oplus L(\omega_1 + \omega_2 + 2\omega_3) \oplus L(2\omega_1 + \omega_3) \oplus L(\omega_2 + \omega_3)$$

Example 2.1.14. Considering the irreducible $\mathfrak{sl}_4$-representation F_4^λ of the previous example (2.1.6), the $F_4^\lambda \otimes S^3(\mathbb{C}^4)$ decomposes as a sum of simple $\mathfrak{sl}_4$- modules.
These simple modules correspond to all different Young diagrams obtained by adding three boxes to the initial diagram with no three boxes in the same column.
The six Young tableaux obtained this way are below and they correspond to the irreducible components $L(2\omega_1 + \omega_2 + 2\omega_3)$, $L(3\omega_1 + \omega_3)$, $L(4\omega_1 + 2\omega_3)$, $L(\omega_1 + \omega_2 + \omega_3)$, $L(2\omega_1)$ and $L(\omega_2)$ (form left to right).



(2.5)

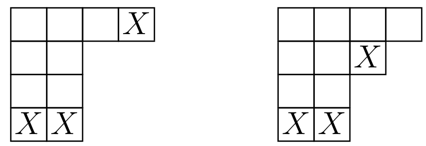


Figure 2.2: Young diagrams corresponding to irreducible constituents in the decomposition of $F_4^\lambda \otimes S^3(\mathbb{C}^4)$.

We have then:

$$F_4^\lambda \otimes S^3(\mathbb{C}^4) \cong L(2\omega_1+\omega_2+2\omega_3) \oplus L(3\omega_1+\omega_3) \oplus L(4\omega_1+2\omega_3) \oplus (\omega_1+\omega_2+\omega_3) \oplus L(2\omega_1) \oplus L(\omega_2)$$

2.1.6 Schur polynomials

Schur polynomials are introduced as well as their relationship with representation theory. These are homogeneous polynomials in n indeterminates with integer coefficients. They form a linear basis of the vector space of symmetric polynomials.
In representation theory, they are the characters of polynomial irreducible representations of $\mathrm{GL}_n(\mathbb{C})$. Schur polynomials in n variables are indexed by partition of the integer n.

Definition 2.1.15. A partition of a positive integer d is a sequence $\lambda = (\lambda_1, \lambda_2, \cdots \lambda_n)$ of a weakly decreasing positive integers such that $\sum_{i=1}^n \lambda_i = d$.
The size of λ is denoted by $|\lambda|$ and is defined to be d.

Associated with the partition $\lambda = (\lambda_1, \lambda_2, \cdots \lambda_n)$ is the (homogeneous) polynomial:

$$a_\lambda(x_1, x_2, \cdots, x_n) = \det \begin{bmatrix} x_1^{\lambda_1+n-1} & x_2^{\lambda_1+n-1} & \dots & x_n^{\lambda_1+n-1} \\ x_1^{\lambda_2+n-2} & x_2^{\lambda_2+n-2} & \dots & x_n^{\lambda_2+n-2} \\ \vdots & \vdots & \ddots & \vdots \\ x_1^{\lambda_n} & x_2^{\lambda_n} & \dots & x_n^{\lambda_n} \end{bmatrix}$$

This polynomial is alternating (antisymmetric), meaning that:

$$a_\lambda(x_{\sigma(1)}, x_{\sigma(2)}, \cdots, x_{\sigma(n)}) = \mathrm{sign}(\sigma)(x_1, x_2, \cdots, x_n)$$

Therefore it is divisible by the polynomial

$$D(x_1, x_2, \cdots, x_n) = \det \begin{bmatrix} x_1^{n-1} & x_2^{n-1} & \dots & x_n^{n-1} \\ x_1^{n-2} & x_2^{n-2} & \dots & x_n^{n-2} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \dots & 1 \end{bmatrix}.$$

Recall that:

$$D(x_1, x_2, \cdots, x_n) = \prod_{1 \le i < j \le n} (x_i - x_j)$$

This alternating polynomial is called the *Vandermonde polynomial* of $(x_1, x_2, \cdots, x_n)$.

Definition 2.1.16. The polynomial

$$s_\lambda(x_1, x_2, \cdots, x_n) = \frac{a_\lambda(x_1, x_2, \cdots, x_n)}{D(x_1, x_2, \cdots, x_n)}$$

is called the *Schur polynomial* associated to the partition λ.

Note: $s_\lambda(x_1, x_2, \cdots, x_n)$ is a symmetric polynomial with integer coefficients. More precisely, it is a homogeneous, symmetric polynomial of degree $|\lambda| = d$.

Schur polynomials and representation theory

There is a strong relationship between Schur polynomials and semistandard Young Tableaux, which we expose here:

Definition 2.1.17. A Young Tableau T of shape $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_l)$ is said to be standard if all the filling are increasing across each row and strictly increasing across each column. A Young Tableau T of shape λ is said to be semistandard if all the filling are weakly increasing across each row and strictly increasing across each column.

Example 2.1.18. The Young tableau T_1 below of shape $\lambda = (3, 2, 2, 1)$ is standard, while the Young tableau T_2 is semistandard.

1	2	3
4	5	
6	7	
8		

1	2	2
5	5	
6	7	
8		

Figure 2.3: From left to right: T_1 standard and T_2 semistandard.

One of the important properties relating Schur polynomials and semistandard tableaux is:

Proposition 2.1.19. *Let λ be a partition of n and let N be a bound on the size of entries in each semistandard tableau T os shape λ. Let $x^T = \prod_{i=1}^{N} x_i^j$, where j is the number of $i's$*

in T.

The Schur polynomial associated with λ is:

$$s_\lambda(x_1, x_2, \cdots, x_n) = \sum_{\textit{semistatard } T} x^T = \sum_{\textit{semistatard } T} x_1^{t_1} x_2^{t_2} \cdots x_n^{t_n}$$

where each t_1 is the number of the number i in T

Example 2.1.20. Let $\lambda = (2, 1)$ and fix a bound on the size entries to be $N = 3$. For this bound N, the list of semi standard tableaux of shape λ are given below:

$$\begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 3 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 3 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 2 & 2 \\ \hline 3 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 2 & 3 \\ \hline 3 \\ \cline{1-1}\end{array}$$

Figure 2.4: List of all semistandard tableaux of shape $\lambda = (2, 1)$ and using the numbers 1, 2 and 3 only.

The corresponding Schur polynomial is :

$$s_{(2,1)}(x_1, x_2, x_3) = x_1^2 x_2 + x_1^2 x_3 + x_1 x_2^2 + x_1 x_2 x_3 + x_1 x_3 x_2 + x_1 x_3^2 + x_2^2 x_3 + x_2 x_3^2$$

Note: This is a homogeneous symmetric polynomial of degree 3 where each term corresponds to one tableau from the list above.

For more details about Schur polynomials (cf. [9]).

2.2 Our original problem

First, let us provide a definition of a semisimple Lie subalgebra of a semisimple Lie algebra being *small*. A concept introduced first by Willenbring and Zuckermman (cf. [11]).

2.2.1 Small subalgebras

Definition 2.2.1. Let $\mathfrak{g}$ be a semisimple Lie algebra.
A semisimple Lie subalgebra $\mathfrak{k}$ of $\mathfrak{g}$ is said to be small in $\mathfrak{g}$ if there exists a positive integer $b(\mathfrak{k}, \mathfrak{g})$, such that for every irreducible finite dimensional $\mathfrak{g}$-module V, there exists an injection of $\mathfrak{k}$-modules $W \hookrightarrow V$, where W is an irreducible $\mathfrak{k}$-module of dimension less than or equal to $b(\mathfrak{k}, \mathfrak{g})$.
In other words, if we let $\{W_i\}_{i=1,2,3,...}$ denote the semigroup of irreducible finite dimensional

representations of $\mathfrak{k}$, if V is an irreducible finite dimensional representation of $\mathfrak{g}$ then upon restriction to $\mathfrak{k}$, V decomposes as:

$$V \cong \bigoplus_{i=1}^{\infty} \underbrace{W_i \oplus W_i \oplus \cdots \oplus W_i}_{m_i(V)}$$

where W_i 's are pairwise inequivalent representations of $\mathfrak{k}$ and $m_i(V) \geq 0$ is the multiplicity of W_i in V for each $i = 1, 2, 3....$ (cf. [11]).

The definition above can be stated as:

Definition 2.2.2. A Lie algebra $\mathfrak{k}$ is said to be small in $\mathfrak{g}$ if:

$$\max \left\{ \min_{m_i(V) \neq 0} (i) \mid V \text{ irreducible finite dimensional representation of } \mathfrak{g} \right\}$$

is finite.

Example 2.2.3. Below are a few examples.

i. Upon restriction to an $\mathfrak{sl}_2$-subalgebra, every irreducible representation of $\mathfrak{sl}_3$ has an irreducible $\mathfrak{sl}_3$-representation of dimension at most 3 (cf. [11]).

ii. Let n be even and $n \geq 2$. Under the standard embedding, the rank $n/2$ symplectic Lie subalgebra $\mathfrak{sp}_n$ is not small in $\mathfrak{sl}_n$ because the symmetric powers of the defining representation of $\mathfrak{sl}_n$ are irreducible as $\mathfrak{sp}_n$- representations.

In this exposition, we consider the Lie subalgebra $\mathfrak{k}$ being the principally embedded $\mathfrak{s}l_2$ in $\mathfrak{s}l_n$ as in (2.1.1). We will determine the sharpest bound $\mathbf{b}(\mathfrak{k}, \mathfrak{g})$ of definition (2.2.1).
The theory describing the multiplicity with which an irreducible representation $(\sigma, \mathfrak{k})$ of $\mathfrak{k}$ occurs in a representation $(\pi, \mathfrak{g})$ for a pair of classical Lie algebras $\mathfrak{k} \subset \mathfrak{g}$ is known as: *Branching rules* (cf. [5]).

2.2.2 Our main result

We will in fact prove that:
For all positive integer $n \geq 3$, the principally embedded Lie subalgebra $\mathfrak{sl}_2$ is small in $\mathfrak{sl}_n$ and a sharp bound of (2.2.1) is:

$$\mathbf{b}(\mathfrak{k}, \mathfrak{g}) = n$$

This result has been proved for $n = 3$ (cf. [5]). Also, this same result implies the following:

Theorem 2.2.4. *Let $\mathfrak{k}$ be the principally embedded $\mathfrak{sl}_2$-subalgebra of $\mathfrak{sl}_n$.*
For any integer $n \geq 3$, and any complex representation V of $\mathfrak{sl}_n$, there exists $0 \leq d \leq n-1$, such that $F_d \hookrightarrow V$ as $\mathfrak{sl}_2$-representations.

2.3 Proof: step one

The first step is to examine the symmetric powers (of all order) of the defining representation $\mathbb{C}^n$ of $\mathfrak{sl}_n$, when the action is restricted to the principally embedded $\mathfrak{sl}_2$.
By setting $d = n-1$ we have $S^d(\mathbb{C}^2) \cong \mathbb{C}^n$.
Obviously the m^{th} symmetric powers of $S^d(\mathbb{C}^2)$ are irreducible representations of $\mathfrak{sl}_n$. When the action of $\mathfrak{sl}_n$ is restricted to $\mathfrak{sl}_2$ we get a decomposition of the form:

$$S^m[S^d(\mathbb{C}^2)] \cong k_0\mathrm{F}_0 \oplus k_1\mathrm{F}_1 \oplus k_2\mathrm{F}_2 \oplus \cdots$$

where $\dim \mathrm{F}_k = k+1$ for all $k = 0, 1, 2, 3\cdots$
and $k_i \geq 0$ for all $i = 0, 1, 2, \cdots$
Define

$$\ell(m, d) := \min\{\dim \mathrm{F}_i \mid k_i \geq 1\}$$

Our *golden* table below, gives the bound $\ell(m, d)$ of for all nonnegative integers $m, d = 0, 1, 2, \cdots, 20$.
As a consequence of *Hermite reciprocity* (1.4.8), we have:

$$\ell(m, d) = \ell(d, m), \text{ for all } m \text{ and } d.$$

Thus all information are encoded in the upper (or lower) triangle of the table. For convenience we will let m denote the order of the row and d that of the column.

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
1	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21
2	1	3	1	3	1	3	1	3	1	3	1	3	1	3	1	3	1	3	1	3	1
3	1	4	3	4	1	4	3	4	1	4	3	4	1	4	3	4	1	4	3	4	1
4	1	5	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
5	1	6	3	4	1	2	3	2	1	2	3	2	1	2	3	2	1	2	1	2	1
6	1	7	1	3	1	3	1	3	1	3	1	3	1	3	1	1	1	1	1	1	1
7	1	8	3	4	1	2	3	2	1	2	3	2	1	2	1	2	1	2	1	2	1
8	1	9	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
9	1	10	3	4	1	2	3	2	1	2	1	2	1	2	1	2	1	2	1	2	1
10	1	11	1	3	1	3	1	3	1	1	1	1	1	1	1	1	1	1	1	1	1
11	1	12	3	4	1	2	3	2	1	2	1	2	1	2	1	2	1	2	1	2	1
12	1	13	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
13	1	14	3	4	1	2	3	2	1	2	1	2	1	2	1	2	1	2	1	2	1
14	1	15	1	3	1	3	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
15	1	16	3	4	1	2	1	2	1	2	1	2	1	2	1	2	1	2	1	2	1
16	1	17	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
17	1	18	3	4	1	2	1	2	1	2	1	2	1	2	1	2	1	2	1	2	1
18	1	19	1	3	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
19	1	20	3	4	1	2	1	2	1	2	1	2	1	2	1	2	1	2	1	2	1
20	1	21	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1

Figure 2.5: Values of $\ell(m,d)$ as $m,d = 0,1,2...20$.

From the table, the following results are immediate:

$$\ell(0,d) = 1$$

for all d since $S^0[S^d(\mathbb{C}^2)] \cong \mathbb{C}$.

$$\ell(1,d) = d+1$$

for all d since $S^1[S^d(\mathbb{C}^2)] \cong \mathbb{C}^{d+1}$.

This last relation shows that the sharp bound $\mathbf{b}(\mathfrak{k},\mathfrak{g})$ of (2.2.2) is at least n.
Now we need to show that $\mathbf{b}(\mathfrak{k},\mathfrak{g})$ is at most n. We investigate case by case:

Case 1: l = 2

For all $d = 0, 1, 2\cdots$,

$$\ell(2,d) = \begin{cases} 1 & if \quad d \text{ even;} \\ 3 & if \quad d \text{ odd.} \end{cases}$$

Proof. $\ell(2,0) = 1$ is a special case of $\ell(d,0)$.
If d is even then $\ell(2,d) = 1$ by taking $\alpha = \frac{d}{2}$ in the proposition (1.4.4).
If d is odd then $\ell(2,d) = 1$ by taking $\alpha = \frac{d+1}{2}$ in the proposition (1.4.4). □

Case 2: l = 3

For all $d = 0, 1, 2\cdots$,

$$\ell(3,d) = \begin{cases} 1 & if \quad d = 0, \\ 3 & if \quad d = 2, \\ 4 & if \quad d \in \{1,3\}. \end{cases}$$

Proof. Taking $d = 3$ in $\ell(2,d)$ together with (1.4.8), we have $(3,2) = \ell(2,3) = 3$
$\ell(3,3) = 4$ is a consequence of the following relation (cf. [4], Chap 11).

$$S^3(S^3V) \cong S^9V \oplus S^5V \oplus S^3V$$

□

Case 3: l = 4

$$\ell(4,d) = \begin{cases} 5 & if \quad d = 1, \\ 1 & if \quad d = 0, 2, 3. \end{cases}$$

Proof. $d = 0$ and $d = 1$ are trivial.
$\ell(4,2) = \ell(2,4) = 1$ (from case 1).
$\ell(4,3) = 1$ because of the decomposition:

$$S^4(S^3V) = S^{12}V \oplus S^8V \oplus S^6V \oplus S^4V \oplus S^0V$$

□

Finally the whole left upper-corner 5×5 subtable of our golden table is fully understood.
In other words $l(l,d)$ is known for $l, d = 0, 1, 2, 3, 4$.
Now $\ell(4,2) = \ell(4,3) = 1$ and $\ell(4,0) = 1$ implies that: $\ell(4,d) = 1$ for all $d = 0, 2, 3, 4, \cdots$

This is because there are invariants in $S^4(S^dV)$ for all $d \geq 2$, these are obtained by taking the sum and/or products of those invariants in $S^4(S^2V)$ and $S^4(S^3V)$.
Using the corollary (2.1.10), together with ((2.3) of case 3), we can provide an explicit upperbound of $\ell(m,d)$ for all $m \in \{0,1,2,3,4\}$ and all $d = 0,1,2,3,\cdots$ as follows.

Proposition 2.3.1. *For all nonnegative integer d*

- $\ell(0,d) = 1 \text{ for all } \quad d = 0,1,2\cdots$
- $\ell(1,d) = d+1 \text{ for all } \quad d = 0,1,2\cdots$
- $\ell(2,d) = \begin{cases} 1 & if \quad d \equiv 0 \text{ or } 2 \bmod 4, \\ 3 & if \quad d \equiv 1 \text{ or } 3 \bmod 4. \end{cases}$
- $\ell(3,d) = \begin{cases} 1 & if \quad d \equiv 0 \bmod 4, \\ 3 & if \quad d \equiv 2 \bmod 4, \\ 4 & if \quad d \equiv 1 \text{ or } 3 \bmod 4. \end{cases}$
- $\ell(4,d) = \begin{cases} 5 & if \quad d = 1, \\ 1 & if \quad d \neq 1. \end{cases}$

By applying again the corollary (2.1.10) and the *Hermite reciprocity* (property (1.4.8)) to the upper left corner five by five subtable, we extend the previous results to all $m, d = 0,1,2\cdots$. Here is how:

Proposition 2.3.2. *For all nonnegative integers m and d*

- *If* $m \equiv 0 \bmod 4,$ *then*

$$\ell(m,d) = 1 \text{ for all } d \neq 1 \text{ and } \ell(m,1) = m+1$$

- *If* $m \equiv 1 \bmod 4,$ *then*

$$\ell(m,d) = \begin{cases} d+1 & if \quad m = 1, \\ m+1 & if \quad d = 1, \\ \leq 4 & \quad otherwise. \end{cases}$$

- *If* $m \equiv 2 \bmod 4,$ *then*

$$\ell(m,d) = \begin{cases} m+1 & if \quad d = 1, \\ \leq 3 & \quad otherwise. \end{cases}$$

- *If* $m \equiv 3 \bmod 4,$ *then*

$$\ell(m,d) = \begin{cases} m+1 & if \quad d = 1, \\ \leq 4 & \quad otherwise. \end{cases}$$

As a consequence of the last two propositions, we have:

Proposition 2.3.3. *For all nonnegative integers m and $d \geq 2$,*

$$\ell(m, d) \leq d + 1$$

Which is to say, For all nonnegative integers m and $n \geq 3$,

$$\ell(m, n-1) = \ell(m, d) \leq n$$

2.4 Proof: step two

In this section we are still considering the principally embedded $\mathfrak{sl}_2$ in $\mathfrak{sl}_n$ for $n \geq 3$
As usual, we identify $P_+(\mathfrak{sl}_2)$ with nonnegative integers and let $\{F_i\}_{i=0,1,2,\cdots}$ be the corresponding irreducible representations such that dim $F_i = i + 1$.
Fix $n \geq 3$, let λ be in $P_+(\mathfrak{sl}_n)$, and let $L(\lambda)$ be the irreducible $\mathfrak{sl}_n$- representation with highest weight λ.

The proof of the two propositions (2.4.8) and (2.4.9) uses the famous *Cartan-Helgason* theorem (cf. [7], chap 8), which we will state here, but first some review.

Definition 2.4.1. Let $\mathfrak{g}$ be a semisimple Lie algebra. An automorphism θ of $\mathfrak{g}$ is called involution on $\mathfrak{g}$ if $\theta^2 = 1_{\mathfrak{g}}$.

Any semisimple Lie algebra has an involution.

Example 2.4.2. we have

i. The identity map is an involution.

ii. For $\mathfrak{g} = \mathfrak{sl}_n$, the map defined by $\theta(X) = X^*$ is an involution. Here X^* stands for the complex conjugation.

Definition 2.4.3. Let θ be an involution on a Lie algebra $\mathfrak{g}$.
Since $\theta^2 = 1_{\mathfrak{g}}$ we have:

$$\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$$

where $\mathfrak{k}$ is Lie subalgebra and $\mathfrak{p}$ is a vector subspace whose Lie subalgebras are all commutative.
It is obvious that:

$$[\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{k};\ [\mathfrak{k}, \mathfrak{p}] \subset \mathfrak{p} \text{ and } [\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k}$$

The pair $(\mathfrak{k}, \mathfrak{p})$ is called a *Cartan pair* and the pair $(\mathfrak{g}, \mathfrak{k})$ is called a *symetric pair* (cf. [7]). One has similar definitions at the level of Lie groups.
Let G be a compact semisimple connected and simply connected Lie group and τ an involution of G.
Define

$$G^\tau = \{g \in G \mid \tau g = g\} \text{ and } P = \{g \in G \mid \tau g = g^{-1}\}$$

Let T be a maximal torus of G invariant under τ, such that $T \cap P$ is a maximal torus in P, and let $S = K \cap T$ where $K = G^\tau$. Also, notice that $S = T^\tau$.
The *Cartan-Helgason* theorem states.

Theorem 2.4.4. *The irreducible representation of G admitting a nonzero vector fixed by K are precisely those with highest weights corresponding to homomorphisms from T to $\mathbb{C}^*$ trivial on S.*

Proof. (cf. [7], chap 8). □

One of the consequences of the theorem above (2.4.4) is the following:

Theorem 2.4.5. *Every symmetric pair (G, K) of algebraic groups is spherical, that is, the affine variety G/K has a multiplicity free coordinate ring. in other words, for any irreducible representation, V, of G, the dimension of the K-invariant subspace, V^K, is at most one dimensional.*

Recall that: $V^K = \{x \in V \mid gx = x \; for \; all \; g \in K\}$
Tow classes of symmetric pairs (SL_n, SO_n) and (SL_n, Sp_n) are needed for the proof of our result.

Proposition 2.4.6. *Let λ be in $P_+(\mathfrak{sl}_n)$.*

$$dim \; L(\lambda)^{SO_n} = 1$$

if and only if the number of boxes in each row of the Young diagram associated with λ is even. In other words:

$$\lambda \in 2\mathbb{N}\omega_1 \oplus 2\mathbb{N}\omega_2 \oplus \cdots \oplus 2\mathbb{N}\omega_{n-1}$$

In a similar way, when considering the symmetric pair (SL_n, Sp_n) and n even.

Proposition 2.4.7. *Let λ be in $P_+(\mathfrak{sl}_n)$.*

$$dim \; L(\lambda)^{Sp_n} = 1$$

if and only if all the number of boxes in each column of the Young diagram associated with λ is even. In other words:

$$\lambda \in \mathbb{N}\omega_2 \oplus \mathbb{N}\omega_4 \oplus \cdots \oplus \mathbb{N}\omega_{n-2}$$

Proof. (cf. [5], chap 12). □

Suppose $n \geq 3$ and let $L(\lambda)$ be an irreducible representation of $\mathfrak{sl}_n$, we will prove the two following propositions.

Proposition 2.4.8. *If n is even then there exists a unique nonnegative integer d such that*

$$\wedge^d(\mathbb{C}^n) \hookrightarrow Res^{\mathfrak{sl}_n}_{\mathfrak{so}_n} L(\lambda)$$

that is to say,

$$mult\,(\wedge^d(\mathbb{C}^n) : Res^{\mathfrak{sl}_n}_{\mathfrak{so}_n} L(\lambda)) = 1$$

for some nonnegative integer d.

Proof. Let $n \geq 3$, n odd.
Given $L(\lambda)$, using Pieri rules (2.1.5) (if needed) with the appropriate number of boxes, $k \geq 1$, will produce one diagram with all rows having an even number of boxes. This diagram correspond to the constituent in the decomposition of $\wedge^k(\mathbb{C}^n) \otimes L(\lambda)$ which has an invariant nonzero vector under the action of SO_n. Therefore,

$$\dim\,[\wedge^k(\mathbb{C}^n) \otimes L(\lambda)^*]^{SO_n} = 1$$

and thus, using the corollary (2.1.3) we get:

$$\dim\,\mathrm{Hom}_{SO_n}\,(\wedge^k(\mathbb{C}^n), L(\lambda)) = 1$$

Finally, an SL_n- representation of the form $\wedge^k(\mathbb{C}^n)$ occurs in the decomposition of any given $\mathfrak{sl}_n$ (resp. SL_n) irreducible representation $L(\lambda)$ when the action is restricted to $\mathfrak{so}_n$ (resp. SO_n). □

Proposition 2.4.9. *If n is even then there exists a unique nonnegative integer d such that*

$$S^d(\mathbb{C}^n) \hookrightarrow Res^{\mathfrak{sl}_n}_{\mathfrak{sp}_n} L(\lambda)$$

that is to say,

$$mult\,(S^d(\mathbb{C}^n) : Res^{\mathfrak{sl}_n}_{\mathfrak{sp}_n} L(\lambda)) = 1$$

for some nonnegative integer d.

Proof. First, it is worth to mention that the irreducible $\mathfrak{sl}_n$- representations are the same as the irreducible $\mathfrak{gl}_n$- representations, where the center of $\mathfrak{gl}_n$ acts trivially. Let $n \geq 3$, n even Again, once $L(\lambda)$ is given, using Pieri rules of (2.1.5) (if needed) with the appropriate number of boxes, $d \geq 1$ we will produce one diagram with all columns have an even number of boxes. This diagram correspond to the constituent in the decomposition of $S^d(\mathbb{C}^n) \otimes L(\lambda)$ which has an a nonzero invariant vector under the action of Sp_n. and this implies that:

$$\dim [S^d(\mathbb{C}^n) \otimes L(\lambda)^*]^{Sp_n} = 1$$

and therefore using (2.1.3) we get:

$$\dim \mathrm{Hom}_{Sp_n} (S^d(\mathbb{C}^n), L(\lambda)) = 1$$

therefore an $S^d(\mathbb{C}^n)$ occurs in the decomposition of any $\mathfrak{sl}_n$ (resp. SL_n) irreducible representation $L(\lambda)$ when the action is restricted to $\mathfrak{sp}_n$ (resp. Sp_n). □

Remark 2.4.10. It is important to note that in both (2.4.8) and (2.4.9) the corresponding nonnegative integers k and d associated with $\wedge^k(\mathbb{C}^n)$ and $S^d(\mathbb{C}^n)$ are unique.

2.5 Proof: step three

In this last part of the proof, we consider an integer $n \geq 3$ and we will, as we should, use propositions (2.4.9 and 2.4.8) to discuss two separate cases as follows.

Case 1: n even

By considering the double embedding:

$$\mathfrak{sl}_2 \hookrightarrow \mathfrak{sp}_n \hookrightarrow \mathfrak{sl}_n.$$

Let $L(\lambda)$ be an irreducible $\mathfrak{sl}_n$-representation. By reducing the action to $\mathfrak{sp}_n$ and using (2.4.9), we know that there exits $d = 0, 1, 2 \cdots$ such that the irreducible $\mathfrak{sp}_\mathfrak{n}$-representation $S^d(\mathbb{C}^n)$ occurs in $\mathrm{Res}^{\mathfrak{sl}_\mathfrak{n}}_{\mathfrak{sp}_n} L(\lambda)$.
For this same $S^d(\mathbb{C}^n)$, when the action is restricted to the principally embedded $\mathfrak{sl}_2$.
Our previous result (2.3.3) implies the existence of an $\mathfrak{sl}_2$ representation F_k occurring in $S^d(\mathbb{C}^n)$ where $k \leq n$.

Case 2: n odd

By considering the double embedding:

$$\mathfrak{sl}_2 \hookrightarrow \mathfrak{so}_n \hookrightarrow \mathfrak{sl}_n$$

Let $L(\lambda)$ be an irreducible $\mathfrak{sl}_n$-representation.
From (2.4.8), there exists a $k = 0, 1, 2 \cdots, n.$ such that $\wedge^k(\mathbb{C}^n)$ occurs in $L(\lambda)$ as an irreducible $\mathfrak{so}_\mathfrak{n}$-representation.
For that same k and from proposition (1.4.6) we have:

$$\wedge^k[S^n(\mathbb{C}^2)] \cong S^k[S^{n+1-k}(\mathbb{C}^2)]$$

Here again, from (2.3.3), there exits a $k' \in \{0, 1, 2 \cdots, n-k\}$ such that the $\mathfrak{sl}_2$- representation $\mathrm{F}_{k'}$ occurs in $S^k[S^{n+1-k}(\mathbb{C}^2)]$. Therefore that same $\mathrm{F}_{k'}$ is a constituent of $\wedge^k[S^n(\mathbb{C}^2)$ as representations of $\mathfrak{sl}_2$.
If $k = 0$ then $k' = 0 \leq n$.
If $k \geq 1$, we need to justify why $n + 1 - k \geq 2$. Here is why!
We have $k \leq \frac{n-1}{2} \leq n - 1$ thus $n - k + 1 \geq 2$.
From the previous case (n even) we know $k' \leq n - k + 1$, therefore $k' \leq n$ (since $k \geq 1$).
This concludes our proof.

2.5.1 More readings from our golden table

Although a sharp bound for the lowest dimension of $\mathfrak{sl}_2$- type occurring in the $\mathfrak{sl}_2$ representation $S^m(S^d(\mathbb{C}^2))$ has been proven to be $\ell(m, d) = d + 1$. For more subtle reading, let us consider the golden table again (see below).

Proposition 2.5.1. *For all m odd and d odd we have:*

$$\ell(m, d) \geq 1$$

Proof. The $\mathfrak{sl}_2$-representation $S^m(S^d(\mathbb{C}^2))$ has weights:

$$m_d d + m_{d-2}(d-2) + \cdots m_{-d}(-d)$$

with $m_d + m_{d-2} + \cdots m_{-d} = m$ thus the trivial representation does not occur since it has zero weight. □

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
1	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21
2	1	3	1	3	1	3	1	3	1	3	1	3	1	3	1	3	1	3	1	3	1
3	1	4	3	4	1	4	3	4	1	4	3	4	1	4	3	4	1	4	3	4	1
4	1	5	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
5	1	6	3	4	1	2	3	2	1	2	3	2	1	2	3	2	1	2	1	2	1
6	1	7	1	3	1	3	1	3	1	3	1	3	1	3	1	1	1	1	1	1	1
7	1	8	3	4	1	2	3	2	1	2	3	2	1	2	1	2	1	2	1	2	1
8	1	9	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
9	1	10	3	4	1	2	3	2	1	2	1	2	1	2	1	2	1	2	1	2	1
10	1	11	1	3	1	3	1	3	1	1	1	1	1	1	1	1	1	1	1	1	1
11	1	12	3	4	1	2	3	2	1	2	1	2	1	2	1	2	1	2	1	2	1
12	1	13	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
13	1	14	3	4	1	2	3	2	1	2	1	2	1	2	1	2	1	2	1	2	1
14	1	15	1	3	1	3	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
15	1	16	3	4	1	2	1	2	1	2	1	2	1	2	1	2	1	2	1	2	1
16	1	17	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
17	1	18	3	4	1	2	1	2	1	2	1	2	1	2	1	2	1	2	1	2	1
18	1	19	1	3	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
19	1	20	3	4	1	2	1	2	1	2	1	2	1	2	1	2	1	2	1	2	1
20	1	21	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1

Figure 2.6: Values of $\ell(m,d)$ for $m,d=0,1,2...20$.

From the table above we also have the following lemmas corresponding to $m=5,6$ and 7

Lemma 2.5.2.

$$\ell(5,d)=\begin{cases} 2 & if \quad d \text{ odd and } d\geq 5,\\ 3 & if \quad d\in\{2,6,10,14\},\\ 1 & if \quad d \text{ even and } d\notin\{2,6,10,14\},\\ 4 & if \quad d=3,\\ 1 & if \quad d=1.\end{cases}$$

Proof. (cf. http://arxiv.org/abs/1603.04935). □

Lemma 2.5.3.

$$\ell(6,d)=\begin{cases} 1 & \text{for all} \quad d\notin\{1,3,,5,7,9,11,13\},\\ 3 \quad if & d\in\{3,,5,7,9,11,13\},\\ 7 \quad if & d=1.\end{cases}$$

Proof. (cf. http://arxiv.org/abs/1603.04935). □

Lemma 2.5.4.

$$\ell(7,d) = \begin{cases} 2 & if \quad d \quad odd\ and\ d \notin \{1,3\}, \\ 4 & if \quad d = 3, \\ 3 & if \quad d \in \{2,6,10\}, \\ 8 & if \quad d = 1, \\ 1 & if \quad d\ is\ even\ and\ \ d \notin \{2,6,10\}. \end{cases}$$

Proof. (cf. http://arxiv.org/abs/1603.04935). □

Because of the symmetry of our table, we will assume here that $d \geq m$.
First, notice the four matrix:

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 1 & 2 \\ 1 & 1 & 1 & 1 \\ 1 & 2 & 1 & 2 \end{bmatrix}$$

corresponding to the rows 8 through 11 and columns 8 through 11, combined with the corollary (2.1.10), will repeat itself (mod 4) to the right, downward and/or both, indefinitely many times.

Remark 2.5.5. For all $m \geq 8$ and all $d \geq 8$

$$\ell(m,d) \in \{1,2\}$$

even more precisely,
For all $m \geq 8$ and for all $d \geq m$

$$\ell(m,d) = \begin{cases} 2 \text{ for all } & m \text{ and } d \text{ odd }, \\ 1 & \quad\text{otherwise.} \end{cases}$$

It is also worth to note the same four by four matrix of 1's and 2's corresponding to the the rows 4 through 7 and columns 16 through 19. This matrix will, as well, repeat itself (mod 4) to the right, downward and/or both indefinitely many times.

2.6 Motivation and new directions

2.6.1 Motivation

One of the sources of motivation of our work is the generalization of the results of BGG's paper (cf. [1]).

Another motivation stems from a study of generalized Harish-Chandra modules (cf. [8]) as follows:
Let $\mathfrak{k}$ denote a reductive subalgebra of a semisimple Lie algebra $\mathfrak{g}$. A $(\mathfrak{g}, \mathfrak{k})$-module M is a $\mathfrak{g}$-module such that $\mathfrak{k}$ acts locally finitely. That is, for all $v \in M$, $\dim \mathcal{U}(\mathfrak{k})v < \infty$ ($\mathcal{U}(\mathfrak{k})$ being the universal enveloping algebra of $\mathfrak{k}$).
We say that the $(\mathfrak{g}, \mathfrak{k})$-module is *admissible* if every irreducible finite dimensional representation of $\mathfrak{k}$ occurs with finite multiplicity in M. We then have the following proposition:

Proposition 2.6.1. *Given an infinite dimensional, admissible* $(\mathfrak{g}, \mathfrak{k})$*-module* M*. For every semisimple small subalgebra* $\mathfrak{k}' \subset \mathfrak{k}$*, if* $\mathfrak{k}'$ *is small in* $\mathfrak{k}$*, then* M *can not be an admissible* $(\mathfrak{g}, \mathfrak{k}')$*-module.*

Proof. (cf. [11]) □

An important special case is worth mentioning:
When the action of $\mathfrak{sl}_n$ on the adjoint representation is restricted to $\mathfrak{sl}_2$, one obtains that the special unitary group $\mathrm{SU}(n)$ has the same homology as the product of spheres of dimension $3, 5, 7, \cdots 2n-1$ (cf. `http://www.ams.org/mathscinet/pdf/974333.pdf`).

2.6.2 New directions

- The result stated here is classified in the area of branching, our hope is to extend it to a large family of pairs $(\mathfrak{g}, \mathfrak{k})$.
- In our case of the pair $(\mathfrak{sl}_n, \mathfrak{sl}_2)$, although a the sharp bound $\mathbf{b}(\mathfrak{k}, \mathfrak{g})$ of (2.2.1) has been determined to be n, the multiplicities of the smallest occurring $\mathfrak{sl}_2$-type remains an open question, which we hope to solve in the future.

Chapter 3

Computational data

3.1 Maple Data

We have used both Maple and LiE software to produce data used to conjecture our result.

Data using Maple:

```
> restart: # Clear the Maple RAM
with(LinearAlgebra): # load LinearAlgebra package
> # We express sl(2) characters on the diagonal matrices of SL(2,C)
> Matrix(2,2, [[q,0],[0,1/q]]);
```

$$\begin{bmatrix} \mathrm{q} & 0 \\ 0 & 1/\mathrm{q} \end{bmatrix}$$

```
# Exterior powers (i.e. alternating tensors)
# Compute the sl(2) character: \wedge^k(F_n), where F_n = S^n(C^2)
# Here k and n are nonnegative integers
wkn := proc(k,n)
local i, pd: pd := 1:
for i from 0 to n do
pd := pd*(1+q^(n-2*i)*t)
od:
sort(expand(coeff(expand(pd), t, k)));
end:
> # Example: What is the sl(2) character of \wedge^2 F_3 ?
wkn(2, 3);
```

$$q^4 + q^2 + \frac{1}{q^2} + \frac{1}{q^4} + 2$$

```
> # Symmetric powers (i.e. symmetric tensors)
# Compute the sl(2) character: S^d(F_n)
# Here d and n are nonnegative integers
sdn := proc(d,n)
local i, pd: pd := 1:
for i from 0 to n do
pd := pd/(1-q^(n-2*i)*t)
od: pd := series(pd, t=0, d+1):
sort(expand(coeff(pd, t, d)));
end:
> # Example: What is the sl(2) character of S^2(F_3) ?
sdn(2, 3);
```

$$q^6 + q^4 + 2q^2 + \frac{2}{q^2} + \frac{1}{q^4} + \frac{1}{q^6} + 2$$

```
> # Example: What is the sl(2) character of S^2 F_2 ?
ch := sdn(2, 2);
```

$$ch := q^4 + q^2 + \frac{1}{q^2} + \frac{1}{q^4} + 2$$

```
# Decompose an sl(2) character into irreps.
# f is a character
# X^d denotes the irrep. F_d
exf := proc(f)
local F,i,sm: F:=expand((q-1/q)*f):
if f<>0 then
sm := 0:
for i from 0 to degree(f)+1 do
sm := sm + coeff(F, q, i+1)*X^i
od: sm;
else 0 fi;
end:
> # Example: decompose ch into irreps.
exf(ch);
```

$$1 + X^4$$

```
# We see X^0 (an sl(2) invariant), and
# We see X^4, the 5-dim. sl(2)-irrep.
> # What is the decomposition of \wedge^4 of the septic (i.e. F_7) ?
exf( wkn(4,7) );
```

$$1 + 2X^4 + 2X^8 + X^{10} + X^{12} + X^{16}$$

```
> # We see irreps. of dimension 1, 5, 9, 11, 13, and 17,
# with the irreps. of dimension 5 and 9 occuring twice.

> # The order to which a polynomial vanishes at the origin.
# f is a polynomail in X
ord := proc(f)
if f=0 then 0 else
if subs(X=0, f)<>0 then 0 else 1+ord(simplify(f/X)) fi:
fi;
end:
> ord( 3*X^5 + 4*X^10 + X^100 );
                  5
> ord( X^2);
                  2
> ord( X);
                  1
> ord( 1);
         0
> # What is the minimal sl(2) type occuring in \wedge^3(F_7) ?
ord( exf( wkn(3,7) ) );
                  3
> # What is the full decomposition of \wedge^3(F_7) ?
exf( wkn(3,7) );
```

$$X^3 + X^5 + X^7 + X^9 + X^{11} + X^{15}$$

```
# What is the character of \wedge^3(F_7) ?
wkn(3,7);
```

$$q^{15} + q^{13} + 2q^{11} + 3q^9 + 4q^7 + 5q^5 + 6q^3 + 6q + \frac{6}{q} + \frac{6}{q^3} + \frac{5}{q^5} + \frac{4}{q^7} + \frac{3}{q^9} + \frac{2}{q^{11}} + \frac{1}{q^{13}} + \frac{1}{q^{15}}$$

```
# Golden table
Matrix(10,10, (i,j) -> 1+ord( exf(sdn(i-1,j-1) ) ) );
```

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 1 & 3 & 1 & 3 & 1 & 3 & 1 & 3 & 1 & 3 \\ 1 & 4 & 3 & 4 & 1 & 4 & 3 & 4 & 1 & 4 \\ 1 & 5 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 6 & 3 & 4 & 1 & 2 & 3 & 2 & 1 & 2 \\ 1 & 7 & 1 & 3 & 1 & 3 & 1 & 3 & 1 & 3 \\ 1 & 8 & 3 & 4 & 1 & 2 & 3 & 2 & 1 & 2 \\ 1 & 9 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 10 & 3 & 4 & 1 & 2 & 3 & 2 & 1 & 2 \end{bmatrix}$$

```
> Schur polynomials
# The character of an irrep. of GL(n,C) indexed by a partition
\la,
# can be expressed in terms of a ratio of alternating
polynomials,
# which is called a "Schur polynomail.
# Here: la is a partition. (i.e. weakly degreesing sequence of
# nonnegative integers.
# The numerator of the Schur polynomial
s_num := proc(la)
local n: n := nops(la):
Determinant(Matrix( n,n, (i,j) -> x[i]^(n-j+la[j])));
end:
# The Schur polynomial
schur := proc(la)
sort(simplify(
s_num(la)/s_num([seq(0,i=1..nops(la))])));
end:
> # Character of the defining rep. of GL(4,C), denoted
V=L([1,0,0,0])
schur( [1,0,0,0] );
```

$$x_1 + x_2 + x_3 + x_4$$

```
> # Character of the symmetric square of V.
schur( [2,0,0,0] );
```

$$x_1^2 + x_1x_2 + x_1x_3 + x_1x_4 + x_2^2 + x_2x_3 + x_2x_4 + x_3^2 + x_3x_4 + x_4^2$$

```
> # Character of the alternating square of V.
schur( [1,1,0,0] );
```

$$x_1^3x_2^2x_3+x_1^3x_2^2x_4+x_1^3x_2x_3^2+2x_1^3x_2x_3x_4+x_1^3x_2x_4^2+x_1^3x_3^2x_4+x_1^3x_3x_4^2+x_1^2x_2^3x_3+x_1^2x_2^3x_4+2x_1^2x_2^2x_3^2+4x_1^2x_2^2x_3x_4+2x_1^2x_2^2x_4^2+x_1^2x_2x_3^3+4x_1^2x_2x_3^2x_4+4x_1^2x_2x_3x_4^2+x_1^2x_2x_4^3+x_1^2x_3^3x_4+2x_1^2x_3^2x_4^2+x_1^2x_3x_4^3+x_1x_2^3x_3^2+2x_1x_2^3x_3x_4+x_1x_2^3x_4^2+x_1x_2^2x_3^3+4x_1x_2^2x_3^2x_4+4x_1x_2^2x_3x_4^2+x_1x_2^2x_4^3+2x_1x_2x_3^3x_4+4x_1x_2x_3^2x_4^2+2x_1x_2x_3x_4^3+x_1x_3^3x_4^2+x_1x_3^2x_4^3+x_2^3x_3^2x_4+x_2^3x_3x_4^2+x_2^2x_3^3x_4+2x_2^2x_3^2x_4^2+x_2^2x_3x_4^3+x_2x_3^3x_4^2+x_2x_3^2x_4^3$$

```
> # We restrict the Schur polynomials to the principal SL(2) torus:
# As before, la is a partition
sq := proc(la)
local ply,n,i: n := nops(la): ply := schur(la):
for i from 0 to n-1 do
ply := subs( x[i+1]=q^(n-1-2*i), ply)
od: ply;
end:
> # sl(2) character of rho
ch_rho := sq([3,2,1,0]);
```

$$ch_{rho} := 8q^4 + 5q^6 + 10q^2 + \frac{10}{q^2} + \frac{3}{q^8} + \frac{5}{q^6} + 10 + \frac{8}{q^4} + 3q^8 + \frac{1}{q^{10}} + q^{10}$$

```
> # Decompose into irreducibles.
exf( ch_rho );
```

$$2X^2 + 3X^4 + 2X^6 + 2X^8 + X^{10}$$

```
> # The order of zero is the minimal sl(2) type:
ord( exf( ch_rho ) );
  2
> # We decompose the restricted Schur polynomials,
# and find the minimal sl(2) type.
```

```
sl2 := la -> ord(exf(sq(la))):
> # Example: rho
sl2( [3,2,1,0] );
 2
> # How many sl4 irreps. have a given dimension
# of a lowest sl(2)-irrep.
# Bound the fundamental coefficients by N.
N := 1:
counts_sl4 := [0,0,0,0,0,0,0,0,0,0]:
for i from 0 to N do
for j from 0 to N do
for k from 0 to N do
ans := 1+sl2( [i+j+k, i+j, i, 0] ):
counts_sl4[ans] := counts_sl4[ans]+1
od od od:
counts_sl4;
[2, 2, 2, 2, 0, 0, 0, 0, 0, 0]
> sl2( [0,0,0,0] );
0
> sl2( [1,1,0,0] );
0
> sl2( [2,1,0,0] );
1
> sl2( [2,2,1,0] );
1
> sl2( [3,2,1,0] );
2
> sl2( [2,1,1,0] );
2
> sl2( [1,0,0,0] );
3
> sl2( [1,1,1,0] );
3
```

```
> N := 2:
counts_sl4 := [0,0,0,0,0,0,0,0,0,0]:
for i from 0 to N do
for j from 0 to N do
for k from 0 to N do
ans := 1+sl2( [i+j+k, i+j, i, 0] ):
counts_sl4[ans] := counts_sl4[ans]+1
od od od:
counts_sl4;
[7, 10, 8, 2, 0, 0, 0, 0, 0, 0]

> N := 3:
counts_sl4 := [0,0,0,0,0,0,0,0,0,0]:
for i from 0 to N do
for j from 0 to N do
for k from 0 to N do
ans := 1+sl2( [i+j+k, i+j, i, 0] ):
counts_sl4[ans] := counts_sl4[ans]+1
od od od:
counts_sl4;
[22, 28, 10, 4, 0, 0, 0, 0, 0, 0]
> N := 4:
counts_sl4 := [0,0,0,0,0,0,0,0,0,0]:
for i from 0 to N do
for j from 0 to N do
for k from 0 to N do
ans := 1+sl2( [i+j+k, i+j, i, 0] ):
counts_sl4[ans] := counts_sl4[ans]+1
od od od:
counts_sl4;
[51, 56, 14, 4, 0, 0, 0, 0, 0, 0]
> N := 5:
counts_sl4 := [0,0,0,0,0,0,0,0,0,0]:
for i from 0 to N do
```

```
for j from 0 to N do
for k from 0 to N do
ans := 1+sl2( [i+j+k, i+j, i, 0] ):
counts_sl4[ans] := counts_sl4[ans]+1
od od od:
counts_sl4;
[90, 102, 18, 6, 0, 0, 0, 0, 0, 0]
> N := 6:
counts_sl4 := [0,0,0,0,0,0,0,0,0,0]:
for i from 0 to N do
for j from 0 to N do
for k from 0 to N do
ans := 1+sl2( [i+j+k, i+j, i, 0] ):
counts_sl4[ans] := counts_sl4[ans]+1
od od od:
counts_sl4;
[149, 162, 26, 6, 0, 0, 0, 0, 0, 0]
> N := 7:
counts_sl4 := [0,0,0,0,0,0,0,0,0,0]:
for i from 0 to N do
for j from 0 to N do
for k from 0 to N do
ans := 1+sl2( [i+j+k, i+j, i, 0] ):
counts_sl4[ans] := counts_sl4[ans]+1
od od od:
counts_sl4;
[228, 248, 28, 8, 0, 0, 0, 0, 0, 0]
> N := 8:
counts_sl4 := [0,0,0,0,0,0,0,0,0,0]:
for i from 0 to N do
for j from 0 to N do
for k from 0 to N do
ans := 1+sl2( [i+j+k, i+j, i, 0] ):
counts_sl4[ans] := counts_sl4[ans]+1:
```

```
od od od:
counts_sl4;
[337, 352, 32, 8, 0, 0, 0, 0, 0, 0]
> N := 9:
counts_sl4 := [0,0,0,0,0,0,0,0,0,0]:
for i from 0 to N do
for j from 0 to N do
for k from 0 to N do
ans := 1+sl2( [i+j+k, i+j, i, 0] ):
counts_sl4[ans] := counts_sl4[ans]+1:
od od od:
counts_sl4;
[464, 490, 36, 10, 0, 0, 0, 0, 0, 0]
> # Of the 1000 irreps. with fundamental coordinates bounded by 9,
# 464 contain the trivial representation
# 490 contain the 2-dim representation (i.e. defining)
# 36 contain the 3-dim representation (i.e. quadratic)
# 10 contain the 4-dim representation (i.e. cubic)

> # Compute the multiplicity of the lowest sl(2)-type
# FUTURE WORK!
low_mult := proc(la)
local lt, ans:
ans := exf(sq(la)):
ans := coeff(ans, X, ord(ans) );
end:
```

Bibliography

[1] I. N. Bernšteĭn, I. M. Gel′fand, and S. I. Gel′fand. Models of representations of compact Lie groups. *Funkcional. Anal. i Priložen.*, 9(4):61–62, 1975.

[2] Daniel Bump. *Lie groups*, volume 225 of *Graduate Texts in Mathematics*. Springer, New York, second edition, 2013.

[3] Roger Carter, Graeme Segal, and Ian Macdonald. *Lectures on Lie groups and Lie algebras*, volume 32 of *London Mathematical Society Student Texts*. Cambridge University Press, Cambridge, 1995. With a foreword by Martin Taylor.

[4] William Fulton and Joe Harris. *Representation theory*, volume 129 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1991. A first course, Readings in Mathematics.

[5] Roe Goodman and Nolan R. Wallach. *Symmetry, representations, and invariants*, volume 255 of *Graduate Texts in Mathematics*. Springer, Dordrecht, 2009.

[6] James E. Humphreys. *Introduction to Lie algebras and representation theory*. Springer-Verlag, New York-Berlin, 1972. Graduate Texts in Mathematics, Vol. 9.

[7] Anthony W. Knapp. *Lie groups beyond an introduction*, volume 140 of *Progress in Mathematics*. Birkhäuser Boston, Inc., Boston, MA, second edition, 2002.

[8] Ivan Penkov and Gregg Zuckerman. Generalized Harish-Chandra modules: a new direction in the structure theory of representations. *Acta Appl. Math.*, 81(1-3):311–326, 2004.

[9] Claudio Procesi. *A primer of invariant theory*, volume 1 of *Brandeis Lecture Notes*. Brandeis University, Waltham, MA, 1982. Notes by Giandomenico Boffi.

[10] M. A. A. van Leeuwen. LiE, a software package for Lie group computations. *Euromath Bull.*, 1(2):83–94, 1994.

[11] Jeb F. Willenbring and Gregg J. Zuckerman. Small semisimple subalgebras of semisimple Lie algebras. In *Harmonic analysis, group representations, automorphic forms and invariant theory*, volume 12 of *Lect. Notes Ser. Inst. Math. Sci. Natl. Univ. Singap.*, pages 403–429. World Sci. Publ., Hackensack, NJ, 2007.