Linear Numerical Stability and Implicit Numerical Damping

Learning Outcomes

Following this lecture, students will be able to:

- Describe the differences between absolutely unstable, absolutely stable, and conditionally stable finite difference approximations.
- Describe the general procedure by which the stability of a given spatial and temporal finite difference approximation combination can be determined.
- Understand the impacts of implicit numerical damping and computational instability on a model solution, including the wavelength dependence of these impacts for the schemes in which such dependence exists.

A Primer on Waves, Damping, and Dispersion

This and the next lecture introduce the concepts of **implicit numerical damping** and **numerical dispersion**, each of which result from the model's chosen finite difference methods. They have similar-looking yet distinct impacts on the modeled representation of wave-like features, however. Let's establish a framework through which we can better understand these concepts.

Consider a block that is eleven units tall. Let's assume that a westerly wind of 10 m s^{-1} advects the block to the east. In the real world, this eleven-unit-tall block will move eastward at 10 m s^{-1}.

Things are a bit different in the modeled world, however. A model views this block as the sum of blocks of all different sizes, similar to how Fourier series allow us to write any continuous function as the sum of waves with varying frequencies or wavelengths. For the sake of illustration, let us assume that a model views this eleven-unit block as the sum of three blocks that are seven units, three units, and one unit tall. Together, these three blocks sum to eleven, so at the outset nothing is different from the real-world representation.

The chosen finite difference methods are approximations. Though the specific details vary between differencing methods, the two most common manifestations of these approximations are implicit numerical damping (which causes features to lose amplitude) and numerical dispersion (which causes features to move at different phase speeds and group velocities). A finite difference method can be associated with one, both, or neither of these properties.

Both implicit numerical damping and numerical dispersion depend on wavelength, such that parts of our eleven-unit block example above will be impacted by implicit numerical damping and numerical dispersion to varying degrees.

Under *implicit numerical damping* alone, the seven-, three-, and one-unit blocks will move eastward at 10 m s^{-1} (i.e., their phase speed is unaffected) but their sizes are reduced, perhaps to 6.5, 2.5, and 0.5 units, respectively. Under *numerical dispersion* alone, the seven-, three-, and one-unit blocks will remain that size (i.e., their amplitude is unaffected) but their propagation speed is reduced, perhaps to 9.75, 9, and 7 m s^{-1}, respectively. **Both give the appearance of an eleven-unit block that loses amplitude!** However, this appearance is deceiving since only implicit numerical damping causes the features to lose amplitude.

Envisioning model fields as not having a single wavelength but being comprised of the sum of a bunch of different wavelength features, each of which have different amplitudes, is key to understanding linear numerical stability, implicit numerical damping, and numerical dispersion. Keep this hypothetical example in mind along with the concepts from our "Thinking in Waves" lecture as we dive into these concepts starting...now!

Introduction to Linear Stability

Numerical stability is defined by how the model solution evolves with time: does it grow exponentially time, leading to floating point overflow – one or more model variables becoming too large for the computer to represent – and the model crashing? In general, we assess stability in the context of identifying the conditions under which this occurs.

In its most general form, the *CFL criterion* is a stability criterion for a linear advection term: under what conditions (U, Δt, and Δx) does the model solution become unstable? In this lecture, we will formally derive the CFL criterion for (linear) advection terms for several temporal and spatial finite differencing schemes. Note, however, that while *all* terms in the primitive equations contribute to numerical stability, the advection terms are the most problematic and thus form the basis of our investigation here.

As in the atmosphere for vertical parcel displacements, there are three types of numerical stability, listed from least to most common:

- **Absolutely unstable**: the model will always crash no matter what values are chosen for the dependent parameters (e.g., U, Δt, and Δx).
- **Absolutely stable**: the model will never crash no matter what values are chosen for the dependent parameters.
- **Conditionally stable**: the model solution will remain stable so long as the chosen values for the dependent parameters adhere to an appropriate stability criterion.

Consider a one-dimensional advection equation for a generic variable h that is advected by a mean velocity U:

$$\left.\frac{\partial h}{\partial t}\right|_j^\tau = -U\left.\frac{\partial h}{\partial x}\right|_j^\tau$$

Here, subscripts indicate that the terms are evaluated at a point *j* along the *x*-axis, while superscripts indicate that the terms are evaluated at a time step τ.

We wish to specify harmonic, or wave-like, solutions for *h* of the form:

$$h = \hat{h}e^{i(kx-\omega t)}$$

Here, $\hat{h}$ is amplitude, *k* is a zonal wavenumber equal to $2\pi/L$, *L* is wavelength, and ω is frequency (s^{-1}) and equal to Uk. In the above, the exponential function specifies wave-like structure through Euler's formula, where $e^{i\theta} = cos\,\theta + i\,sin\,\theta$ (with a real component equal to $cos\,\theta$). We assume that the frequency ω has both real and imaginary components, such that $\omega = \omega_R + i\omega_I$. Though ω_I itself is real-valued, the leading *i* makes it imaginary.

We express *h* in terms of a single wave for simplicity, though we will explore the wavenumber dependence (i.e., multiple waves comprising the total solution) in the linear stability criteria we will soon derive. This will show that some differencing schemes are computationally stable for some wavelengths and unstable for others. As we will see in our next lecture, this is also important for numerical dispersion.

If we substitute the definition for ω into the definition for *h*, we obtain:

$$h = \hat{h}e^{i(kx-\omega t)} = \hat{h}e^{i(kx-(\omega_R+i\omega_I)t)} = \hat{h}e^{\omega_I t}e^{i(kx-\omega_R t)}$$

It is clear that the amplitude of *h* is no longer equal to a constant value $\hat{h}$ but is now a function of time through $e^{\omega_I t}$, such that ω_I determines whether the amplitude of *h* **grows exponentially** ($|e^{\omega_I t}| > 1$, such that $\omega_I > 0$), **remains constant** ($|e^{\omega_I t}| = 1$, such that $\omega_I = 0$), or **dampens** with time ($|e^{\omega_I t}| < 1$, such that $\omega_I < 0$).

Thus, to determine linear numerical stability, we need to determine the conditions under which $|e^{\omega_I t}| \leq 1$ (for stability; values < 1 indicate **implicit numerical damping**) and $|e^{\omega_I t}| > 1$ (for unstable solutions). This process is known as *linear stability analysis* or, in some references, *von Neumann stability analysis*, and is but one stability assessment method.

Linear Stability of Forward-in-Time, Backward-in-Space Finite Differences

Let us examine the stability of the forward-in-time, backward-in-space combination of finite difference schemes. Though this is a combination of differencing schemes that is associated with particularly large truncation error, it also provides for a direct evaluation of numerical stability. In this case, the one-dimensional advection equation becomes:

$$\frac{h_j^{\tau+1} - h_j^{\tau}}{\Delta t} = -U\frac{h_j^{\tau} - h_{j-1}^{\tau}}{\Delta x}$$

Or, equivalently:

$$h_j^{\tau+1} - h_j^{\tau} = -\frac{U\Delta t}{\Delta x}\left(h_j^{\tau} - h_{j-1}^{\tau}\right)$$

Note that $x = j\Delta x$, such that the location is equal to the grid point j multiplied by the grid spacing Δx, and $t = \tau\Delta t$, such that the time is equal to the time index τ multiplied by the time step Δt. The Δx in the above is that on the Earth (Δx_e), which is often smaller than that on the model grid (Δx_g).

The wave-like solution for h can thus be rewritten in terms of j and τ:

$$h = \hat{h}e^{\omega_I t}e^{i(kx-\omega_R t)} = \hat{h}e^{\omega_I \tau\Delta t}e^{i(kj\Delta x-\omega_R \tau\Delta t)}$$

If we substitute this solution into the finite difference form of the 1-D equation above, we obtain:

$$\hat{h}e^{\omega_I(\tau+1)\Delta t}e^{i(kj\Delta x-\omega_R(\tau+1)\Delta t)} - \hat{h}e^{\omega_I \tau\Delta t}e^{i(kj\Delta x-\omega_R \tau\Delta t)}$$
$$= -\frac{U\Delta t}{\Delta x}\left(\hat{h}e^{\omega_I \tau\Delta t}e^{i(kj\Delta x-\omega_R \tau\Delta t)} - \hat{h}e^{\omega_I \tau\Delta t}e^{i(k(j-1)\Delta x-\omega_R \tau\Delta t)}\right)$$

Divide by a common factor of $\hat{h}e^{\omega_I \tau\Delta t}e^{i(kj\Delta x-\omega_R \tau\Delta t)}$ to obtain:

$$e^{\omega_I\Delta t}e^{-i\omega_R\Delta t} - 1 = -\frac{U\Delta t}{\Delta x}\left(1 - e^{-ik\Delta x}\right)$$

Note the difference in this equation from that which is equation (3.38) in the course text; here, there is a leading negative sign in the last exponential, which is correct, whereas the course text lacks such a leading negative despite obtaining the correct solution at the end.

The exponentials involving i can be rewritten using Euler's formula, where $e^{i\theta} = cos\,\theta + i\,sin\,\theta$ and $e^{-i\theta} = cos\,\theta - i\,sin\,\theta$. Doing so, we obtain:

$$e^{\omega_I\Delta t}(cos(\omega_R\Delta t) - i\,sin(\omega_R\Delta t)) - 1 = -\frac{U\Delta t}{\Delta x}\left(1 - (cos(k\Delta x) - i\,sin(k\Delta x))\right)$$

If we separate this equation into its real (top) and imaginary (bottom) parts, we obtain:

$$e^{\omega_I\Delta t}\,cos(\omega_R\Delta t) - 1 = -\frac{U\Delta t}{\Delta x}(1 - cos(k\Delta x))$$

$$-ie^{\omega_I\Delta t}\,sin(\omega_R\Delta t) = -i\frac{U\Delta t}{\Delta x}sin(k\Delta x)$$

Or, written equivalently,

$$e^{\omega_I \Delta t} cos(\omega_R \Delta t) = 1 - \frac{U\Delta t}{\Delta x}(1 - cos(k\Delta x))$$

$$e^{\omega_I \Delta t} sin(\omega_R \Delta t) = \frac{U\Delta t}{\Delta x} sin(k\Delta x)$$

We wish to combine these equations so that we can obtain an equation for $e^{\omega_I \Delta t}$. To do so, we want to eliminate ω_R, which is associated with wave propagation and dispersion rather than implicit numerical damping. This can be done by squaring each equation and adding them together, since $cos^2 \theta + sin^2 \theta = 1$. Doing so, we obtain:

$$e^{\omega_I \Delta t} e^{\omega_I \Delta t} = \left(1 - \frac{U\Delta t}{\Delta x}(1 - cos(k\Delta x))\right)^2 + \left(\frac{U\Delta t}{\Delta x} sin(k\Delta x)\right)^2$$

Taking the square root of both sides of this equation, we obtain:

$$\left|e^{\omega_I \Delta t}\right| = \sqrt{\left(1 - \frac{U\Delta t}{\Delta x}(1 - cos(k\Delta x))\right)^2 + \left(\frac{U\Delta t}{\Delta x} sin(k\Delta x)\right)^2}$$

Note that we have taken the absolute value of the left-hand side to keep only the positive root. We are less interested in the sign of $e^{\omega_I \Delta t}$ as we are in whether its magnitude is greater than 1.

We can expand everything under the radical as follows:

$$\left(\frac{U\Delta t}{\Delta x} sin(k\Delta x)\right)^2 = \frac{U^2(\Delta t)^2}{(\Delta x)^2} sin^2(k\Delta x)$$

$$\left(1 - \frac{U\Delta t}{\Delta x}(1 - \cos(k\Delta x))\right)^2 = 1 - 2\frac{U\Delta t}{\Delta x}(1 - \cos(k\Delta x)) + \frac{U^2(\Delta t)^2}{(\Delta x)^2}(1 - \cos(k\Delta x))^2$$

$$= 1 - 2\frac{U\Delta t}{\Delta x} + 2\frac{U\Delta t}{\Delta x}\cos(k\Delta x) + \frac{U^2(\Delta t)^2}{(\Delta x)^2}\left(1 - 2\cos(k\Delta x) + \cos^2(k\Delta x)\right)$$

Adding these two equations, substituting for the resulting $\frac{U^2(\Delta t)^2}{(\Delta x)^2}(sin^2(k\Delta x) + cos^2(k\Delta x))$ term, and combining like terms, we obtain:

$$\left|e^{\omega_I \Delta t}\right| = \sqrt{1 - 2\frac{U\Delta t}{\Delta x} + 2\frac{U\Delta t}{\Delta x}\cos(k\Delta x) + 2\frac{U^2(\Delta t)^2}{(\Delta x)^2} - 2\frac{U^2(\Delta t)^2}{(\Delta x)^2}\cos(k\Delta x)}$$

$$= \sqrt{1 + 2\frac{U\Delta t}{\Delta x}\left(\cos(k\Delta x) - 1 + \frac{U\Delta t}{\Delta x} - \frac{U\Delta t}{\Delta x}\cos(k\Delta x)\right)}$$

For $(a+b)(c+d) = ab + ad + bc + bd$, if $a = \cos(k\Delta x)$, $b = -1$, $c = 1$, and $d = -\frac{U\Delta t}{\Delta x}$, the terms in the parentheses underneath the radical can be simplified:

$$|e^{\omega_I \Delta t}| = \sqrt{1 + 2\frac{U\Delta t}{\Delta x}\left[(cos(k\Delta x) - 1)\left(1 - \frac{U\Delta t}{\Delta x}\right)\right]}$$

Recall that the value of $e^{\omega_I t}$ determines whether the amplitude of *h* grows, decays, or remains constant in time. We previously defined $t = \tau\Delta t$, such that $e^{\omega_I t} = e^{\omega_I \tau \Delta t} = (e^{\omega_I \Delta t})^{\tau}$. Thus, the value of $|e^{\omega_I \Delta t}|$ determines how the amplitude of *h* will change *over one time step*, which is then raised to the power of τ (i.e., this amplitude change also grows exponentially over time).

The stability criterion above is a function of both the Courant number and of $k\Delta x$, which for $k = 2\pi/L$ becomes $2\pi(\Delta x/L)$, and is thus a function of the ratio of the horizontal grid spacing to the wavelength.

Let us consider a simple case: $\frac{U\Delta t}{\Delta x} = 1$. In that case, everything under the radical collapses to 1, such that $|e^{\omega_I \Delta t}| = 1$. This is **numerically stable**, with *no change in amplitude* with time.

What about when $\frac{U\Delta t}{\Delta x} \neq 1$? Note that the largest-possible value of Δx is $L/2$, defining a grid of three points to represent the $2\Delta x$ wave. There, $k\Delta x = \pi$, with $\cos(\pi) = -1$. The smallest-possible value of Δx is approximately zero, defining a grid of an infinite number of points to resolve all waves. As $k\Delta x$ approaches zero, $\cos(k\Delta x)$ approaches 1. Thus, $\cos(k\Delta x)$ has allowable values between -1 and 1, such that $\cos(k\Delta x) – 1$ has allowable values between -2 and 0; in other words, it is always negative.

For $\frac{U\Delta t}{\Delta x} > 1$, $\left(1 - \frac{U\Delta t}{\Delta x}\right) < 0$. Thus, for $\cos(k\Delta x) – 1 < 0$, the number under the radical in the equation for $|e^{\omega_I \Delta t}|$ above is always greater than 1. Consequently, $|e^{\omega_I \Delta t}| > 1$, which defines a **numerically unstable** solution with *exponential amplitude growth* over time.

Conversely, for $\frac{U\Delta t}{\Delta x} < 1$, $\left(1 - \frac{U\Delta t}{\Delta x}\right) > 0$. Thus, for $\cos(k\Delta x) – 1 < 0$, the number under the radical in the equation for $|e^{\omega_I \Delta t}|$ above is always less than 1. Consequently, $|e^{\omega_I \Delta t}| < 1$, which defines a **numerically stable** solution with (implicit) *exponential amplitude damping* over time.

Thus, for the backward-in-space, forward-in-time differencing scheme, the stability criterion is given by the generic form of the CFL criterion:

$$\frac{U\Delta t}{\Delta x} \leq 1$$

The precise degree to which the wave's amplitude grows or dampens over time is *a function of its wavelength* given the $\cos(k\Delta x)$ term that appears in the equation for $|e^{\omega_I \Delta t}|$above. Recall: $k =$

$2\pi/L$, such that $k\Delta x$ is equal to $2\pi\Delta x/L$. Thus, a wave of wavelength $2\Delta x$ will have $k\Delta x = \pi$ (with $\cos \pi = -1$) while a wave of wavelength $20\Delta x$ will have $k\Delta x = \pi/10$ (with $\cos \pi/10 = 0.951$). The precise degree to which the wave's amplitude grows or dampens over time is also a function of the Courant number itself, given the relationship to $\frac{U\Delta t}{\Delta x}$ in the expression for $|e^{\omega_I \Delta t}|$.

These dependencies are illustrated in Fig. 1, plotting the damping magnitude per time step for ten selected waves of wavelengths $2\Delta x$ to $50\Delta x$ over a range of stable Courant numbers. *For this combination of differencing schemes*, shorter-wavelength features are damped more per time step than are longer-wavelength features. The damping magnitude also varies with the Courant number, with the maximum damping occurring at a Courant number of 0.5.

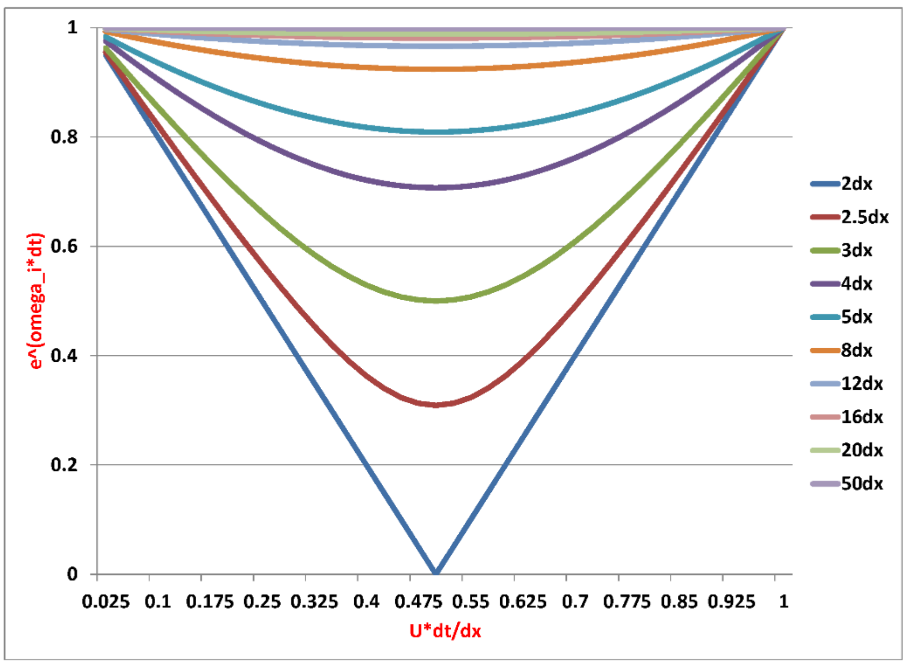

Figure 1. The value of $|e^{\omega_I \Delta t}|$ as a function of Courant number (numerically stable values only) for waves of wavelength between $2\Delta x$ and $50\Delta x$ for the forward-in-time, backward-in-space finite differencing scheme. Please refer to the text for further details.

Linear Stability for Other Spatial and Temporal Differencing Schemes

The numerical stability of any combination of spatial and temporal differencing schemes can be assessed using the process outlined above. The course text describes this in some detail for the centered-in-time, centered-in-space scheme and states only the end results for the forward-in-time, second-order-accurate centered-in-space and centered-in-time, fourth-order-accurate centered-in-space differencing schemes. Here, we consider only basic insight for each; please refer to the course text for more details.

(1) Forward-in-time, second-order centered-in-space

$$\left|e^{\omega_I \Delta t}\right| = \sqrt{1 + \left(\frac{U\Delta t}{\Delta x}\right)^2 sin^2(k\Delta x)}$$

Both $\left(\frac{U\Delta t}{\Delta x}\right)^2$ and $sin^2(k\Delta x)$ are positive-definite, such that the value under the radical is greater than 1 for all values of $k\Delta x$ and $\frac{U\Delta t}{\Delta x}$. Consequently, *no matter the time step,* this combination of differencing schemes is **numerically unstable**. As a result, this scheme is only used to advance the model for the first model time step when a centered-in-time scheme is otherwise used, with the amplitude growth between the first and second time steps being acceptably small for this single model advance.

(2) Centered-in-time (leapfrog), second-order centered-in-space

$$\frac{U\Delta t}{\Delta x} sin(k\Delta x) \leq 1$$

As was stated above, the allowable values of Δx range from $L/2$ to ~0, such that the allowable values of $k\Delta x$ range from π to ~0. The sin function in both cases evaluates to 0. Between 0 and π, the maximum value of $\sin(k\Delta x)$ is 1, which occurs when $k\Delta x = \pi/2$ (for $\Delta x = L/4$). Thus, the sin function will always be between 0 and 1. This allows us to state the stability criterion as:

$$\frac{U\Delta t}{\Delta x} \leq 1$$

(3) Centered-in-time (leapfrog), fourth-order-accurate centered-in-space

$$\frac{U\Delta t}{\Delta x} \leq 0.73$$

It can be shown that $|e^{\omega_I \Delta t}| = 1$ for all stable values of the Courant number for both (2) and (3). In other words, those schemes are either *numerically stable without damping* or they are numerically unstable. **This is true of all centered even-order differencing schemes!** In contrast, all odd-order time and space differencing schemes are associated with implicit damping for stable values of the Courant number, with the specific damping magnitude dependent on the wavelength and the Courant number.

Skamarock et al. (2019; section 3.3.1) lists the Courant number that cannot be exceeded to maintain computational stability for twelve combinations of spatial and temporal differencing schemes. This table, first published in Wicker and Skamarock (2002), is reproduced below, where an X indicates that the differencing-scheme combination is always numerically unstable:

	3rd Order	**4th Order**	**5th Order**	**6th Order**
Leapfrog	X	0.72	X	0.62
Runge-Kutta 2	0.88	X	0.30	X
Runge-Kutta 3	1.61	1.26	1.42	1.08

These values do not tell us anything about how implicit numerical damping magnitudes vary as a function of wavelength (apart from knowing that even-ordered schemes do not dampen so long as the stability criterion is met), nor do they tell us anything about how rapidly a wavelength would grow if it is numerically unstable.

The default choices for WRF-ARW – Runge-Kutta 3 in time, 5th order in space – strike an effective balance between accuracy (higher-order in time and space) and computational efficiency (high Courant number) as compared to other available differencing schemes. Based on this, the general guidance for the model time step Δt in WRF-ARW is $6*\Delta x$, where Δx is input in km and the resulting Δt is in s.

As noted earlier, the *U* in the Courant number is not determined by the meteorology. Rather, it is instead determined by rapidly moving sound waves if these waves are not addressed in another fashion (e.g., semi-implicit or split-explicit temporal differencing). For split-explicit models, the shorter time step used to address sound waves is usually 3-4 times shorter than that used by the rest of the model since the speed of sound is usually 3-4 times faster than the meteorologically dependent *U*.

Further, vertical advection terms in the primitive equations also pose a constraint on numerical stability, where $U \rightarrow W$ and $\Delta x \rightarrow \Delta z$. Though both *W* and Δz are typically smaller than *U* and Δx, Δz is typically non-uniform over the model domain, with smaller values near the surface and tropopause and larger values in the middle troposphere. Fortunately, *W* is typically large where Δz is typically large, with the inverse being true as well. However, where *W* is large when Δz is small, such as may be observed with intense vertical circulations within the boundary layer or thunderstorms, the vertical advection term may limit stability more than horizontal advection terms. In practice, the CFL criterion is most frequently violated for vertical advection in WRF-ARW simulations.

Physical and Mathematical Interpretations of Linear Stability

At its essence, a stability criterion based on the Courant number indicates that there is a limit to the ratio between the maximum distance that can be traveled in one timestep and the grid spacing. This limit varies as a function of the chosen spatial and temporal differencing schemes used. For the general CFL condition, the maximum distance that can be traveled in one timestep cannot be larger than the grid spacing. For less restrictive differencing schemes, a greater maximum distance relative to the grid spacing can be traveled in one timestep; the opposite is true for more restrictive differencing schemes.

What does this mean, however? Mathematically, the CFL condition can be phrased in terms of the numerical and physical domains of dependence. The former is defined by the finite differencing scheme and contains the grid points that contribute to the approximated solution. The latter is defined by the meteorology and contains the region that contributes to the physical solution. The CFL condition thus states that the chosen differencing scheme remains numerically stable so long as the numerical domain does not exceed the physical domain's bounds.