Synoptic Meteorology II: Q-Vectors

Readings: Section 2.3 of *Midlatitude Synoptic Meteorology*.

Motivation

The quasi-geostrophic omega equation is an excellent diagnostic tool and has served as the one of the foundations for introductory synoptic-scale weather analysis for over 50 years. However, it is not infallible. Rather, it has two primary shortcomings:

- The two primary forcing terms in the omega equation, the differential geostrophic-vorticity advection and Laplacian of the potential-temperature advection terms, often have different signs from one another. Without computing each term's actual magnitude, it is impossible to assess which one is larger (and is the primary control on synoptic-scale vertical motions).

- The terms in the omega equation are sensitive to the reference frame – stationary or moving with the flow – in which they are computed. In other words, different results are obtained if the reference frame is changed, even if the meteorological features are the same! This is not a primary concern for us, however, as we are primarily considering stationary reference frames (e.g., as depicted on standard weather charts).

These shortcomings motivate a desire to obtain a new equation for synoptic-scale vertical motions that does not have these problems. As we will show, the equation that we will derive can be used not just to diagnose synoptic-scale vertical motions, but it can also be used to diagnose synoptic-scale frontogenesis! This equation, the Q-vector form of the quasi-geostrophic omega equation, is derived below.

Obtaining The Q-Vector Form of the Quasi-Geostrophic Omega Equation

To obtain the Q-vector form of the quasi-geostrophic omega equation, we will start with the quasi-geostrophic horizontal momentum and thermodynamic equations. For simplicity, we will assume that the Coriolis parameter f is constant, i.e., $f = f_0$, such that all terms involving β, the meridional variability in f, are zero. We will also neglect diabatic heating.

Thus, our basic equation set is given by:

$$\frac{\partial u_g}{\partial t} + \vec{\boldsymbol{v}}_g \cdot \nabla u_g - f_0 v_{ag} = 0 \qquad (1)$$

$$\frac{\partial v_g}{\partial t} + \vec{\boldsymbol{v}}_g \cdot \nabla v_g + f_0 u_{ag} = 0 \qquad (2)$$

$$\frac{\partial T}{\partial t} + \vec{\boldsymbol{v}}_g \cdot \nabla T - S_p \omega = 0 \tag{3}$$

First, find $p*\partial/\partial p$ of (1):

$$p\frac{\partial}{\partial p}\left(\frac{\partial u_g}{\partial t}\right) + p\frac{\partial}{\partial p}\left(u_g \frac{\partial u_g}{\partial x}\right) + p\frac{\partial}{\partial p}\left(v_g \frac{\partial u_g}{\partial y}\right) = f_0 p \frac{\partial v_{ag}}{\partial p} \tag{4}$$

Next, find $R/f_0*\partial/\partial y$ of (3):

$$\frac{R}{f_0}\frac{\partial}{\partial y}\left(\frac{\partial T}{\partial t}\right) + \frac{R}{f_0}\frac{\partial}{\partial y}\left(u_g \frac{\partial T}{\partial x}\right) + \frac{R}{f_0}\frac{\partial}{\partial y}\left(v_g \frac{\partial T}{\partial y}\right) = \frac{RS_p}{f_0}\frac{\partial \omega}{\partial y} \tag{5}$$

We need to use the chain rule on the second and third partial derivatives on the left-hand sides of (4) and (5). Where applicable, we also need to commute the order of the partial derivatives; note that this applies to all terms on the left-hand sides of (4) and (5). After doing so, compute (4) – (5) and group like terms where possible to obtain:

$$\begin{aligned} \frac{RS_p}{f_0}\frac{\partial \omega}{\partial y} - f_0 p \frac{\partial v_{ag}}{\partial p} &= -\left(\frac{\partial}{\partial t} + u_g \frac{\partial}{\partial x} + v_g \frac{\partial}{\partial y}\right)\left(p\frac{\partial u_g}{\partial p} - \frac{R}{f_0}\frac{\partial T}{\partial y}\right) \\ &-p\left[\frac{\partial u_g}{\partial p}\frac{\partial u_g}{\partial x} + \frac{\partial v_g}{\partial p}\frac{\partial u_g}{\partial y}\right] + \frac{R}{f_0}\left[\frac{\partial u_g}{\partial y}\frac{\partial T}{\partial x} + \frac{\partial v_g}{\partial y}\frac{\partial T}{\partial y}\right] \end{aligned} \tag{6}$$

For further simplification, recall the thermal wind relationship:

$$p\frac{\partial v_g}{\partial p} = -\frac{R}{f_0}\frac{\partial T}{\partial x} \tag{7a}$$

$$p\frac{\partial u_g}{\partial p} = \frac{R}{f_0}\frac{\partial T}{\partial y} \tag{7b}$$

Note that in (7), we have substituted f_0 for f given that we are currently assuming that f = constant. By substituting (7b) into (6), the first set of terms on the right-hand side of (6) goes away. Likewise, we can apply both (7a) and (7b) to rewrite the second set of terms on the right-hand side of (6) in terms of R/f_0. Doing so, we obtain:

$$\frac{RS_p}{f_0}\frac{\partial \omega}{\partial y} - f_0 p \frac{\partial v_{ag}}{\partial p} = -\frac{R}{f_0}\frac{\partial u_g}{\partial x}\frac{\partial T}{\partial y} + \frac{R}{f_0}\frac{\partial v_g}{\partial y}\frac{\partial T}{\partial y} + \frac{R}{f_0}\frac{\partial u_g}{\partial y}\frac{\partial T}{\partial x} + \frac{R}{f_0}\frac{\partial u_g}{\partial y}\frac{\partial T}{\partial x} \tag{8}$$

Recall that, by definition, for f = constant, the divergence of the geostrophic wind is zero. This is equivalent to stating that:

$$\frac{\partial u_g}{\partial x} = -\frac{\partial v_g}{\partial y} \tag{9}$$

If we substitute (9) into the first term on the right-hand side of (8), we find that it is equal to the second term on the right-hand side of (8). Likewise, it is apparent that the last two terms on the right-hand side of (8) are equivalent. Thus,

$$\frac{RS_p}{f_0}\frac{\partial \omega}{\partial y} - f_0 p \frac{\partial v_{ag}}{\partial p} = 2\frac{R}{f_0}\left(\frac{\partial u_g}{\partial y}\frac{\partial T}{\partial x} + \frac{\partial v_g}{\partial y}\frac{\partial T}{\partial y}\right) \tag{10}$$

Finally, noting that $S_p = \sigma p/R$, if we substitute for S_p, divide (10) by p, and then multiply by f_0, we obtain:

$$\sigma\frac{\partial \omega}{\partial y} - f_0^2\frac{\partial v_{ag}}{\partial p} = 2\frac{R}{p}\left(\frac{\partial u_g}{\partial y}\frac{\partial T}{\partial x} + \frac{\partial v_g}{\partial y}\frac{\partial T}{\partial y}\right) = 2\frac{R}{p}\left(\frac{\partial \vec{\boldsymbol{v}}_g}{\partial y} \cdot \nabla T\right) \tag{11}$$

In (11), note that:

$$Q_2 = -\frac{R}{p}\left(\frac{\partial \vec{\boldsymbol{v}}_g}{\partial y} \cdot \nabla T\right) \tag{12}$$

Such that:

$$\sigma\frac{\partial \omega}{\partial y} - f_0^2\frac{\partial v_{ag}}{\partial p} = -2Q_2 \tag{13}$$

Now, we wish to find $p*\partial/\partial p$ of (2):

$$p\frac{\partial}{\partial p}\left(\frac{\partial v_g}{\partial t}\right) + p\frac{\partial}{\partial p}\left(u_g\frac{\partial v_g}{\partial x}\right) + p\frac{\partial}{\partial p}\left(v_g\frac{\partial v_g}{\partial y}\right) = -f_0 p\frac{\partial u_{ag}}{\partial p} \tag{14}$$

Next, find $R/f_0*\partial/\partial x$ of (3):

$$\frac{R}{f_0}\frac{\partial}{\partial x}\left(\frac{\partial T}{\partial t}\right) + \frac{R}{f_0}\frac{\partial}{\partial x}\left(u_g\frac{\partial T}{\partial x}\right) + \frac{R}{f_0}\frac{\partial}{\partial x}\left(v_g\frac{\partial T}{\partial y}\right) = \frac{RS_p}{f_0}\frac{\partial \omega}{\partial x} \tag{15}$$

Apply the chain rule, commute the order of the partial derivatives, compute (14) + (15), and group like terms where possible to obtain:

$$\frac{RS_p}{f_0}\frac{\partial\omega}{\partial x} - f_0 p\frac{\partial u_{ag}}{\partial p} = \left(\frac{\partial}{\partial t} + u_g\frac{\partial}{\partial x} + v_g\frac{\partial}{\partial y}\right)\left(p\frac{\partial v_g}{\partial p} + \frac{R}{f_0}\frac{\partial T}{\partial x}\right)$$
$$+p\left[\frac{\partial u_g}{\partial p}\frac{\partial v_g}{\partial x} + \frac{\partial v_g}{\partial p}\frac{v_g}{\partial y}\right] + \frac{R}{f_0}\left[\frac{\partial u_g}{\partial x}\frac{\partial T}{\partial x} + \frac{\partial v_g}{\partial x}\frac{\partial T}{\partial y}\right] \quad (16)$$

Applying (7a) causes the first set of terms on the right-hand side of (16) to go away. Applying (7a) and (7b) allows us to rewrite the second set of terms on the right-hand side of (16) in terms of R/f_0. Doing so, we obtain:

$$\frac{RS_p}{f_0}\frac{\partial\omega}{\partial x} - f_0 p\frac{\partial u_{ag}}{\partial p} = -\frac{R}{f_0}\frac{\partial v_g}{\partial y}\frac{\partial T}{\partial x} + \frac{R}{f_0}\frac{\partial u_g}{\partial x}\frac{\partial T}{\partial x} + \frac{R}{f_0}\frac{\partial v_g}{\partial x}\frac{\partial T}{\partial y} + \frac{R}{f_0}\frac{\partial v_g}{\partial x}\frac{\partial T}{\partial y} \quad (17)$$

Applying (9) causes the first term on the right-hand side of (17) to equal the second term on the right-hand side of (17). We also find that the last two terms on the right-hand side of (17) are equal. Simplifying (17) with this, substituting for S_p, dividing by p, and multiplying by f_0, we obtain:

$$\sigma\frac{\partial\omega}{\partial x} - f_0^2\frac{\partial u_{ag}}{\partial p} = 2\frac{R}{p}\left(\frac{\partial u_g}{\partial x}\frac{\partial T}{\partial x} + \frac{\partial v_g}{\partial x}\frac{\partial T}{\partial y}\right) = 2\frac{R}{p}\left(\frac{\partial\vec{\boldsymbol{v}}_g}{\partial x}\cdot\nabla T\right) \quad (18)$$

In (18), note that:

$$Q_1 = -\frac{R}{p}\left(\frac{\partial\vec{\boldsymbol{v}}_g}{\partial x}\cdot\nabla T\right) \quad (19)$$

Such that:

$$\sigma\frac{\partial\omega}{\partial x} - f_0^2\frac{\partial u_{ag}}{\partial p} = -2Q_1 \quad (20)$$

(13) and (20) contain forcing terms related to both the vertical motion ω and ageostrophic wind $\mathbf{v}_{ag}$. We wish to eliminate the latter. We do so by computing $\partial/\partial x$ of (20) + $\partial/\partial y$ of (13). If we do so, commuting the derivatives on the ageostrophic wind terms in so doing, we obtain:

$$\sigma\frac{\partial^2\omega}{\partial x^2} + \sigma\frac{\partial^2\omega}{\partial y^2} - f_0^2\frac{\partial}{\partial p}\left(\frac{\partial u_{ag}}{\partial x} + \frac{\partial v_{ag}}{\partial y}\right) = -2\left(\frac{\partial Q_1}{\partial x} + \frac{\partial Q_2}{\partial y}\right) \quad (21)$$

The last term on the left-hand side of (21), the divergence of the ageostrophic wind, can be written in terms of ω by making use of the continuity equation applicable in the quasi-geostrophic system,

$$\nabla\cdot\vec{\boldsymbol{v}}_{ag} + \frac{\partial\omega}{\partial p} = 0 \quad (22)$$

Substituting (22) into (21) and simplifying the equation, we obtain:

$$\sigma \nabla^2 \omega + f_0^2 \frac{\partial^2 \omega}{\partial p^2} = -2\nabla \cdot \boldsymbol{Q} \tag{23}$$

Where **Q** = (Q_1, Q_2), with Q_1 and Q_2 as defined in (19) and (12), respectively.

Basic Interpretation and Application of the Q-Vector Equation

(23) gives the *Q-vector form of the quasi-geostrophic omega equation*. The quasi-geostrophic vertical motion is due entirely to **Q-vector divergence** (where $\nabla \cdot (\quad)$ is the divergence operator), which we can readily compute or estimate. Unlike with the regular form of the omega equation, there are not multiple forcing terms that may conflict with one another, a significant advantage! Likewise, there are no partial derivatives with respect to pressure, meaning that we only need data on one isobaric level to diagnose the vertical motion – another advantage!

As with the regular form of the omega equation, we apply this equation for the *diagnosis* of middle tropospheric vertical motions. Note, however, that we still do not actually 'solve' for the vertical motion, given the second-order partial derivative operators on the left-hand side of (23) that require iterative methods to solve.

The basic interpretation of (23) is straightforward. Recalling that $\nabla^2\omega \propto -\omega$ (because the second partial derivative of a local maximum is negative and that of a local minimum is positive), $\omega \propto \nabla \cdot \boldsymbol{Q}$. Thus,

- Synoptic-scale ascent ($\omega < 0$) is found where there is Q-vector ***convergence*** ($\nabla \cdot \boldsymbol{Q} < 0$).
- Synoptic-scale descent ($\omega > 0$) is found where there is Q-vector ***divergence*** ($\nabla \cdot \boldsymbol{Q} > 0$).

Without actually *computing* **Q**, its components (namely, the horizontal partial derivatives of $\mathbf{v}_g$ and *T*), and its divergence, how can we *estimate* **Q** and its divergence from a weather map? We use a minor coordinate transformation into a coordinate system akin to a *natural coordinate system* to aid in estimating **Q** and its divergence.

Let us define the *x*-axis to be along, or parallel to, an isotherm, **with warm air to the right of the positive *x*-axis**. The *y*-axis is defined perpendicular to the *x*-axis. An idealized schematic of this is provided in Fig. 1 below. This is akin to placing the *x*-axis along the direction in which the thermal wind blows, albeit using temperature on one isobaric level rather than the layer-mean temperature.

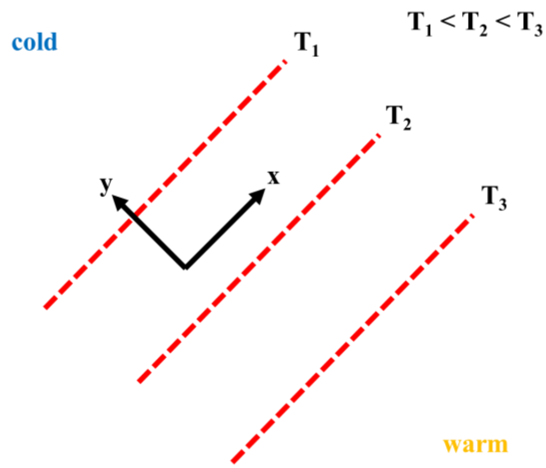

Figure 1. Idealized depiction of the coordinate transformation described in the text above.

In this coordinate system, $\partial T/\partial x$, or the change in temperature along the isotherm, is inherently zero. Thus, the $\partial T/\partial x$ terms in (19) and (12) are 0. As a result, **Q** becomes:

$$\boldsymbol{Q} = -\frac{R}{p}\frac{\partial T}{\partial y}\left(\frac{\partial v_g}{\partial x}\boldsymbol{i} + \frac{\partial v_g}{\partial y}\boldsymbol{j}\right) \tag{24}$$

We can use (9) to rewrite the second term of (24), such that:

$$\boldsymbol{Q} = -\frac{R}{p}\frac{\partial T}{\partial y}\left(\frac{\partial v_g}{\partial x}\boldsymbol{i} - \frac{\partial u_g}{\partial x}\boldsymbol{j}\right) \tag{25}$$

And, by vector identity, the term in the parentheses of (25) can be rewritten, such that:

$$\boldsymbol{Q} = \frac{R}{p}\frac{\partial T}{\partial y}\left(\boldsymbol{k} \times \frac{\partial \vec{\boldsymbol{v}}_g}{\partial x}\right) \tag{26}$$

Thus, to evaluate **Q**, we first want to find the *vector change* in $\mathbf{v}_g$ *along the isotherm*. The **k** x () operator, geometrically, signifies a 90° clockwise rotation of the just-determined change vector. The length, or magnitude, of **Q** is determined by multiplying it by $\frac{R}{p}\frac{\partial T}{\partial y}$, although this may not be necessary for a qualitative evaluation (since these terms are all positive given how we have defined the *y*-axis). If we do this at several locations on a weather map, we can estimate the divergence of **Q** and, thus, estimate at least the sign of the vertical motion.

Let us now consider three examples. As we do so, we will demonstrate how the same answer is given by the Q-vector analysis as would be obtained from the regular form of the omega equation.

Example 1: Idealized Trough/Ridge Pattern

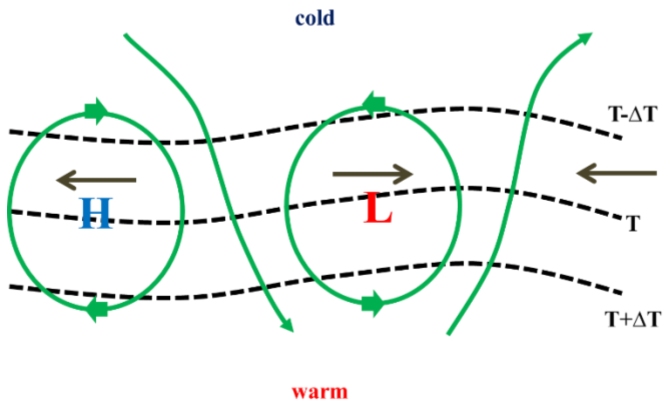

Figure 2. **Q** vectors (grey arrows) for an idealized trough/ridge pattern. Isotherms are depicted in black dashed lines, with cold air to the north, and streamlines depicting the geostrophic flow are depicted in solid green lines.

Near the region of high heights, the geostrophic wind is northerly along the positive *x*-axis (to the east) and southerly along the negative *x*-axis (to the west). Near the region of low heights, the geostrophic wind is southerly along the positive *x*-axis and northerly along the negative *x*-axis.

In each case, we want to subtract the geostrophic wind vector along the negative *x*-axis from the geostrophic wind vector along the positive *x*-axis. To do so, recall principles of vector subtraction. To subtract two vectors, take the vector being subtracted, flip it 180°, and add it to the first vector. Vectors are added by placing the origin of the second vector at the tip/end of the first vector, then by drawing a new vector from the origin of the first vector to the tip/end of the second vector. This is depicted in Fig. 3 below for both situations described above.

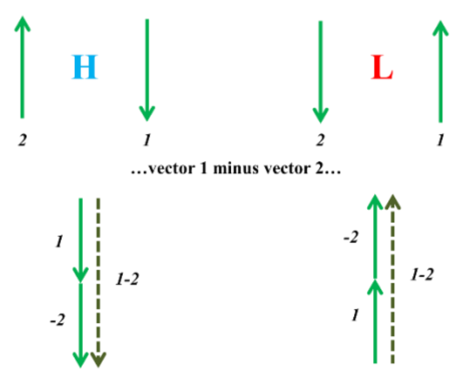

Figure 3. Illustration of the vector subtraction operations described in the text above.

After subtracting the vectors, applying the **k** x () operator necessitates rotating the new vector 90° to the right. This results in a vector pointing from east to west with the areas of high heights and a vector pointing from west to east with the area of low heights. The precise length of each vector can be determined by multiplying the vector by the magnitude of the temperature gradient and the leading factor of *R*/*p*, if desired.

The pattern of vertical motion can thus be determined by evaluating the **Q**-vector divergence. To the west of the area of lower heights, there is **Q**-vector divergence, implying descent. To the east of the area of lower heights, there is **Q**-vector convergence. This implies ascent.

For a sanity check, let us compare this evaluation with that which can be obtained from the omega equation. To the west of the area of lower heights, there is cold geostrophic temperature advection, typically associated with descent. Conversely, to the east of the area of lower heights, there is warm geostrophic temperature advection, typically associated with ascent. Both are consistent with our **Q**-vector interpretation.

Since the magnitude of the geostrophic relative vorticity ζ_g is maximized at the base of troughs and apex of ridges, we can infer cyclonic geostrophic relative vorticity advection to the east of the area of lower heights and anticyclonic geostrophic relative vorticity advection to the west the area of lower heights. If we presume that the geostrophic relative vorticity advection is relatively small in the lower troposphere, this implies cyclonic geostrophic relative vorticity advection increasing with height to the east of the area of lower heights and anticyclonic geostrophic relative vorticity advection increasing with height to the west the area of lower heights. This implies ascent east and descent west of the area of lower heights, again consistent with our **Q**-vector interpretation.

Example 2: Idealized Trough/Ridge Pattern with No Temperature Advection

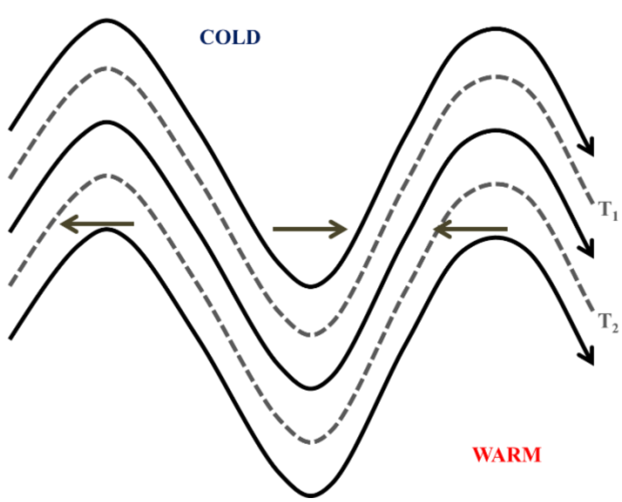

Figure 4. **Q** vectors (grey arrows) for an idealized trough/ridge pattern. Isotherms are depicted in grey dashed lines, with cold air generally to the north, and streamlines depicting the geostrophic flow are depicted in solid black lines.

In many ways, this example is similar to the previous example. However, in this case, the isotherms are parallel to the isohypses and geostrophic wind. In the following, as before, “east” refers to the positive *x*-axis along an isotherm while “west” refers to the negative *x*-axis along an isotherm.

In the base of the trough, the geostrophic wind is from the southwest to the east and from the northwest to the west. Subtracting the latter vector from the former results in a vector pointing from south to north. Applying the **k** x () operator rotates this vector 90° to the right, such that the **Q**-vector points from west to east. In the apex of the ridges, the geostrophic wind is from the northwest to the east and from the southwest to the west. Subtracting the latter vector from the former results in a vector pointing from north to south. Applying the **k** x () operator rotates this vector 90° to the right, such that the **Q**-vector points from east to west. The precise magnitude of each **Q**-vector can be obtained by multiplying each by the magnitude of the temperature gradient and the leading factor of R/p.

Thus, in Fig. 4, **Q**-vectors converge to the east of the trough and diverge to the west of the trough. As before, this signifies ascent and descent, respectively.

Let us again interpret the scenario depicted in Fig. 4 in terms of the omega equation. With no geostrophic temperature advection, forcing is exclusively due to differential geostrophic vorticity advection. The pattern of differential geostrophic vorticity advection is identical to that described in our first example for the same physical reasons: cyclonic geostrophic relative vorticity advection increasing with height to the east of the trough and anticyclonic geostrophic relative vorticity advection increasing with height to the west. As before, this implies ascent and descent, respectively, consistent with the **Q**-vector interpretation.

Example 3: Confluent Flow in the Entrance Region of a Jet Streak

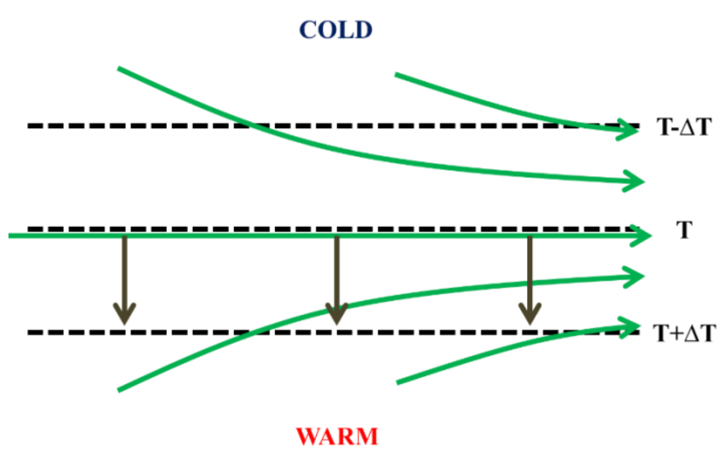

Figure 5. **Q**-vectors (grey arrows) for confluent flow (inferred from the green streamlines obtained from the geostrophic flow) associated with a westerly jet streak. Isotherms are depicted by the dashed black lines with cold air to the north.

In the confluent flow scenario depicted above, the geostrophic wind accelerates (becomes larger) to the east. Thus, along an isotherm, the magnitude of the geostrophic wind – primarily westerly – is always larger to the east. Vector subtraction results in a relatively short vector pointing from west to east. Applying the **k** x () operator to this vector rotates it to the right by 90°, such that it points from north to south, as depicted in Fig. 5 above.

As depicted in Fig. 5, there is no explicit **Q**-vector convergence or divergence. Thus, what do the **Q**-vectors look like further to the north and south? Visually, we can see that along the isotherms, the streamlines are not much different in direction or tightness. This implies no meaningful change in the direction or magnitude of the geostrophic wind along these isotherms, further implying that the magnitude of the **Q**-vectors there is relatively small.

Thus, **Q**-vectors diverge to the north in the colder air and converge to the south in the warmer air. Thus, we see ascent to the south and descent to the north. This is consistent with the four-quadrant jet model: ascent in the right jet entrance region and descent in the left jet entrance region.

Again, we wish to confirm our evaluation using the omega equation. There is warm geostrophic temperature advection and implied ascent to the south and cold geostrophic temperature advection and implied descent to the north. The interpretation in terms of differential geostrophic vorticity advection is more nuanced. The streamlines imply the presence of a trough to the southwest and a ridge to the northwest. This implies cyclonic geostrophic vorticity advection on the south side of the jet streak and anticyclonic geostrophic vorticity advection on the north side of the jet streak. If we again presume that geostrophic vorticity advection is weak near the surface, this pattern implies ascent to the south and descent to the north. Once again, this is consistent with our interpretation from the **Q**-vector analysis.

More Examples

We will work through more examples of how the **Q**-vector form of the omega equation may be applied to diagnose synoptic-scale vertical motion in class (using the Real-Time QG Diagnostics page linked on the course website) and in lab. However, please do make use of the Real-Time QG Diagnostics website outside of class to aid your own study and interpretation of these concepts!

Relationship to the Horizontal Temperature Gradient

To this point, we have discussed how the **Q**-vector may be computed and/or estimated. Likewise, we have shown how its divergence can be used to infer the direction and magnitude of the synoptic-scale vertical motion. However, we have yet to describe the physical meaning of the components of the **Q**-vector. That is the focus of this section.

Q_1 and Q_2 are given by (19) and (12), respectively. Both appear to represent temperature advection by the horizontal shear of the geostrophic wind. More generally, Q_1 and Q_2 could be interpreted as being related to the evolution of the horizontal temperature gradient. However, it is fair to ask if these interpretations are the best possible interpretations for the **Q**-vector.

To do so, let us return to the form of the quasi-geostrophic thermodynamic equation posed in (3). Let's make this equation even simpler: let us state that the flow is purely geostrophic such that the vertical velocity term $S_p\omega$ vanishes. This allows us to write:

$$\frac{\partial T}{\partial t} + u_g \frac{\partial T}{\partial x} + v_g \frac{\partial T}{\partial y} = 0 \tag{27}$$

Note that we have expanded the advection term in (3) into its components in writing (27). We first wish to find ∂/∂x of (27). In so doing, we must make use of the product rule when taking the partial derivatives of the second and third terms on the left-hand side of (27). Doing so, we obtain:

$$\frac{\partial}{\partial x}\left(\frac{\partial T}{\partial t}\right) + \frac{\partial u_g}{\partial x}\frac{\partial T}{\partial x} + u_g \frac{\partial}{\partial x}\left(\frac{\partial T}{\partial x}\right) + \frac{\partial v_g}{\partial x}\frac{\partial T}{\partial y} + v_g \frac{\partial}{\partial x}\left(\frac{\partial T}{\partial y}\right) = 0 \tag{28}$$

Next, we wish to commute the order of the partial derivatives in the first and last terms on the left-hand side of (28). Doing so and grouping like terms, we obtain:

$$\left(\frac{\partial}{\partial t} + \vec{\boldsymbol{v}}_g \cdot \nabla\right)\frac{\partial T}{\partial x} + \frac{\partial u_g}{\partial x}\frac{\partial T}{\partial x} + \frac{\partial v_g}{\partial x}\frac{\partial T}{\partial y} = 0 \tag{29}$$

The last two terms on the right-hand side of (29) can be written in vector form, resulting in:

$$\left(\frac{\partial}{\partial t} + \vec{\boldsymbol{v}}_g \cdot \nabla\right)\frac{\partial T}{\partial x} + \frac{\partial \vec{\boldsymbol{v}}_g}{\partial x} \cdot \nabla T = 0 \tag{30}$$

However, we know that this term is related to Q_1 – in fact, from (19), we know that it is equivalent to $-Q_1 p/R$. Moving this term to the right-hand side of (30), we obtain:

$$\left(\frac{\partial}{\partial t} + \vec{\boldsymbol{v}}_g \cdot \nabla\right)\frac{\partial T}{\partial x} = \frac{D_g}{Dt}\left(\frac{\partial T}{\partial x}\right) = \frac{Q_1 p}{R} \tag{31}$$

We now wish to find ∂/∂y of (27). In so doing, we must again make use of the product rule when taking the partial derivatives of the second and third terms on the left-hand side of (27). Doing so, we obtain:

$$\frac{\partial}{\partial y}\left(\frac{\partial T}{\partial t}\right) + \frac{\partial u_g}{\partial y}\frac{\partial T}{\partial x} + u_g \frac{\partial}{\partial y}\left(\frac{\partial T}{\partial x}\right) + \frac{\partial v_g}{\partial y}\frac{\partial T}{\partial y} + v_g \frac{\partial}{\partial y}\left(\frac{\partial T}{\partial y}\right) = 0 \tag{32}$$

Next, we wish to commute the order of the partial derivatives in the first and third terms on the left-hand side of (32). Doing so and grouping like terms, we obtain:

$$\left(\frac{\partial}{\partial t} + \vec{\boldsymbol{v}}_g \cdot \nabla\right)\frac{\partial T}{\partial y} + \frac{\partial u_g}{\partial y}\frac{\partial T}{\partial x} + \frac{\partial v_g}{\partial y}\frac{\partial T}{\partial y} = 0 \tag{33}$$

The last two terms on the right-hand side of (33) can be written in vector form, resulting in:

$$\left(\frac{\partial}{\partial t} + \vec{\boldsymbol{v}}_g \cdot \nabla\right)\frac{\partial T}{\partial y} + \frac{\partial \vec{\boldsymbol{v}}_g}{\partial y} \cdot \nabla T = 0 \tag{34}$$

However, we know that this term is related to Q_2 – in fact, from (12), we know that it is equivalent to $-Q_2 p/R$. Moving this term to the right-hand side of (34), we obtain:

$$\left(\frac{\partial}{\partial t} + \vec{\boldsymbol{v}}_g \cdot \nabla\right)\frac{\partial T}{\partial y} = \frac{D_g}{Dt}\left(\frac{\partial T}{\partial y}\right) = \frac{Q_2 p}{R} \tag{35}$$

Combining (31) and (35), we obtain:

$$\frac{D_g}{Dt}\left(\frac{R}{p}\nabla T\right) = Q_1\hat{\boldsymbol{\imath}} + Q_2\hat{\boldsymbol{\jmath}} \tag{36}$$

(36) demonstrates that the **Q**-vector is directly proportional to the *rate of change of the horizontal temperature gradient* following the geostrophic motion! Note that (36) describes changes in the *vectorized* horizontal temperature gradient. Next, we develop an equation that relates the **Q** vector to changes in the *magnitude* of the horizontal temperature gradient – i.e., frontogenesis!

Relationship to Frontogenesis

Return to (36), the relationship between the **Q**-vector and the change of the horizontal temperature gradient following the geostrophic motion. Recall that we can analyze fronts in terms of horizontal temperature gradients: cold fronts are typically found on the leading edge of advancing cold air and warm fronts are typically found on the trailing edge of retreating cold air. Across the front, there is a large horizontal temperature gradient. How the magnitude of this gradient changes with time gives a measure of how the front's intensity changes with time!

We can develop a relationship for the rate of change of the magnitude of the horizontal temperature gradient following the motion, with the generic form of this relationship given by:

$$\frac{D_g}{Dt}(\|\nabla T\|) = \frac{D_g}{Dt}\left(\sqrt{\left(\frac{\partial T}{\partial x}\right)^2 + \left(\frac{\partial T}{\partial y}\right)^2}\right) \tag{37}$$

We desire to expand the right-hand side of (37). In doing so, we make use of the following general relationship:

$$\frac{D_g}{Dt}(f^n) = nf^{n-1}\frac{D_g}{Dt}(f) \tag{38}$$

In (38), *f* is some arbitrary function and *n* is some arbitrary exponent or power. Making use of (38) as we expand the right-hand side of (37), we obtain:

$$\begin{aligned}\frac{D_g}{Dt}(\|\nabla T\|) &= \frac{1}{2}\left(\left(\frac{\partial T}{\partial x}\right)^2 + \left(\frac{\partial T}{\partial x}\right)^2\right)^{-1/2}\frac{D_g}{Dt}\left(\left(\frac{\partial T}{\partial x}\right)^2 + \left(\frac{\partial T}{\partial x}\right)^2\right) \\ &= \frac{1}{2\|\nabla T\|}\left[2\frac{\partial T}{\partial x}\frac{D_g}{Dt}\left(\frac{\partial T}{\partial x}\right) + 2\frac{\partial T}{\partial y}\frac{D_g}{Dt}\left(\frac{\partial T}{\partial y}\right)\right] \\ &= \frac{1}{\|\nabla T\|}\left[\frac{\partial T}{\partial x}\frac{D_g}{Dt}\left(\frac{\partial T}{\partial x}\right) + \frac{\partial T}{\partial y}\frac{D_g}{Dt}\left(\frac{\partial T}{\partial y}\right)\right]\end{aligned} \tag{39}$$

If we substitute the definition of **Q**, as given by (31) and (35) in this instance, (39) becomes:

$$\frac{D_g}{Dt}(\|\nabla T\|) = \frac{1}{\|\nabla T\|}\left[\frac{\partial T}{\partial x}\frac{p}{R}Q_1 + \frac{\partial T}{\partial y}\frac{p}{R}Q_2\right] \tag{40}$$

Or, in vector notation,

$$\frac{D_g}{Dt}(\|\nabla T\|) = \frac{1}{\|\nabla T\|}\frac{p}{R}\left(\nabla T \cdot \vec{\boldsymbol{Q}}\right) \tag{41}$$

What does (41) signify? This equation signifies that the change in the *magnitude* of the horizontal temperature gradient following the geostrophic flow is directly proportional to the orientations of the horizontal temperature gradient (∇T) and **Q**-vectors with respect to each another.

This latter remark arises from the definition of the dot product contained within (41). To facilitate interpretation of (41), it is helpful to recall the properties of the dot product:

- For any two vectors **A** and **B**, if **A** is *perpendicular* to **B**, their dot product is zero.
- If **A** points in *the same direction* as **B**, their dot product is positive.
- If **A** points in *the opposite direction* as **B**, their dot product is negative.

As applied to (41), the change in the magnitude of the horizontal temperature gradient following the geostrophic flow is zero if the horizontal temperature gradient (always from cold toward warm air) and **Q**-vectors are perpendicular to each another. If the horizontal temperature gradient and **Q**-vectors point in the same direction, the change in the magnitude of the horizontal temperature gradient is positive. Conversely, if the horizontal temperature gradient and **Q**-vectors point in the opposite direction, the change in the magnitude of the horizontal temperature gradient is negative.

Let us apply this concept to Example 3 earlier in this lecture. The horizontal temperature gradient in that example points from north to south, from cold toward warm air. Likewise, in that example, we demonstrated that the **Q**-vectors point from north to south. Thus, in this case, the horizontal temperature gradient and **Q**-vectors point in the same direction. This means that the magnitude of the horizontal temperature gradient will grow larger with time, a frontogenetic situation.

We can confirm this by considering how the geostrophic wind, given by the streamlines in Fig. 5, flows with respect to the isotherms. To the northwest, the geostrophic wind flows from cold toward warm air. Conversely, to the southwest, the geostrophic wind flows from warm toward cold air. This pattern of geostrophic temperature advection increases the horizontal temperature gradient's magnitude to the west, as we deduced above.

This exercise can be repeated for diffluent flow in the exit region of a jet streak. In that case, the **Q**-vectors and horizontal temperature gradient point in opposite directions, causing the magnitude of the horizontal temperature gradient to become smaller with time, a frontolytic situation. To gain experience with these concepts, I encourage you to work through this exercise on your own.

Thus, in the quasi-geostrophic system, the development and decay of fronts can be evaluated by considering how the **Q**-vectors are oriented with respect to the isotherms! The dual utility of the **Q**-vector – to evaluate synoptic-scale vertical motion (and its associated ageostrophic flow) and frontal development and decay – illustrates a powerful advantage of the **Q**-vector formulation over the omega equation! It highlights how ascent, clouds, and precipitation are often found in regions of frontogenesis, given the strong relationship between vertical motion and frontogenesis manifest by the **Q**-vector.

It should be noted, however, that the above framework only considers how *geostrophic processes* change the horizontal temperature gradient's magnitude. Ageostrophic processes can also change this magnitude but are not accounted for in the Q-vector–based frontogenesis formulation.

Why Does the Q-Vector Relate to Frontogenesis?

In the preceding section, we documented the mathematical relationship between the **Q**-vector and frontogenesis, then illustrated this relationship with some hypothetical examples. However, these do not entirely describe *why* the **Q**-vector relates to frontogenesis. To do so, let's look more closely at the $(\nabla T \cdot \vec{\boldsymbol{Q}})$ term in parentheses on the right-hand side in (41).

$\vec{\boldsymbol{Q}}$ is defined by (19) and (12), with its full-component forms given by (18) and (11), respectively. Use these to find the dot product from (41):

$$(\nabla T \cdot \vec{\boldsymbol{Q}}) = \left(\frac{\partial T}{\partial x}Q_1 + \frac{\partial T}{\partial y}Q_2\right) = \left(\frac{\partial T}{\partial x}\left(-\frac{R}{p}\left(\frac{\partial u_g}{\partial x}\frac{\partial T}{\partial x} + \frac{\partial v_g}{\partial x}\frac{\partial T}{\partial y}\right)\right) + \frac{\partial T}{\partial y}\left(-\frac{R}{p}\left(\frac{\partial u_g}{\partial y}\frac{\partial T}{\partial x} + \frac{\partial v_g}{\partial y}\frac{\partial T}{\partial y}\right)\right)\right)$$

Rearranging, we obtain:

$$(\nabla T \cdot \vec{\boldsymbol{Q}}) = -\frac{R}{p}\left(\frac{\partial u_g}{\partial x}\left(\frac{\partial T}{\partial x}\right)^2 + \frac{\partial v_g}{\partial y}\left(\frac{\partial T}{\partial y}\right)^2 + \frac{\partial T}{\partial x}\frac{\partial T}{\partial y}\left(\frac{\partial v_g}{\partial x} + \frac{\partial u_g}{\partial y}\right)\right)$$

Frontogenesis requires that the right-hand side of this equation be positive. Since *R* and *p* are both positive-definite, the leading negative sign on it means that each right-hand side forcing term must be negative for frontogenesis to occur. Let us consider each of these terms in order:

- $\frac{\partial u_g}{\partial x}\left(\frac{\partial T}{\partial x}\right)^2$: The square of the partial derivative of temperature in the *x*-direction is positive-definite, such that the partial derivative of the zonal component of the geostrophic wind in the *x*-direction must be negative. This occurs when u_g becomes less westerly/positive along the positive *x*-axis; i.e., a fast westerly flow slowing toward the east. This represents *speed convergence* along the *x*-axis.
- $\frac{\partial v_g}{\partial y}\left(\frac{\partial T}{\partial y}\right)^2$: The square of the partial derivative of temperature in the *y*-direction is positive-definite, such that the partial derivative of the meridional component of the geostrophic wind in the *y*-direction must be negative. This occurs when v_g becomes less southerly/positive along the positive y-axis; i.e., a fast southerly flow slowing toward the north. This represents *speed convergence* along the *y*-axis.
- $\frac{\partial T}{\partial x}\frac{\partial T}{\partial y}\left(\frac{\partial v_g}{\partial x} + \frac{\partial u_g}{\partial y}\right)$: The term inside the parentheses represents *shearing deformation* of the geostrophic wind. Consider the case where this is negative and the two partial derivatives with respect to temperature are positive (i.e., temperature increasing toward the north and east). Negative shearing deformation has its axis of dilatation extending from northwest to southeast and axis of contraction extending from northeast to southwest. The axis of contraction is thus parallel to the isotherms, indicating that contraction will bring the isotherms closer together.

From this, we can see how (41) represents the combined effects of divergence and deformation on frontogenesis – much as did equation (19) of the Frontogenesis lecture notes!

Q-Vectors and the Destruction and Restoration of Geostrophic Balance

Again, Q_1 and Q_2 are given by (19) and (12), respectively. The forcings upon Q_1 and Q_2 are entirely geostrophic in nature, whether directly ($\mathbf{v}_g$) or indirectly (T, related to geopotential height via the hydrostatic equation). Thus, *purely geostrophic flow is responsible for departures from geostrophy* (i.e., for ageostrophic flow)! The resultant ageostrophic circulation, which is related to Q_1 and Q_2 by (11) and (18), works to restore geostrophic and thermal wind balance.

We can further demonstrate the concept of geostrophic flow being responsible for departures from geostrophy in the context of the thermal wind relationship. First, we wish to use the thermal wind relationship posed in (7) to re-write (31) and (35) in terms of the vertical shear of the geostrophic wind. Substituting (7b) into (35) and (7a) into (31), we obtain:

$$\frac{D_g}{Dt}\left(f_0\frac{\partial u_g}{\partial p}\right) = Q_2 \tag{42a}$$

$$\frac{D_g}{Dt}\left(f_0\frac{\partial v_g}{\partial p}\right) = -Q_1 \tag{42b}$$

For comparison, return to (1) and (2). As we did with the simplified form of the quasi-geostrophic thermodynamic equation, consider the case where the flow is purely geostrophic. This allows us to state:

$$\frac{\partial u_g}{\partial t} + \vec{\boldsymbol{v}}_g \cdot \nabla u_g = 0 \tag{43a}$$

$$\frac{\partial v_g}{\partial t} + \vec{\boldsymbol{v}}_g \cdot \nabla v_g = 0 \tag{43b}$$

By finding ∂/∂p of (43a) and (43b), we can develop alternate expressions for the left-hand sides of (42a) and (42b). We make use of the product rule and commute the order of partial derivatives in obtaining these expressions, as we did with (28)-(30) when operating on the quasi-geostrophic thermodynamic equation. In mathematical form,

$$\begin{aligned}\frac{\partial}{\partial p}\left(\frac{\partial u_g}{\partial t} + \vec{\mathbf{v}}_g \cdot \nabla u_g\right) &= \frac{\partial}{\partial p}\left(\frac{\partial u_g}{\partial t}\right) + \frac{\partial u_g}{\partial p}\frac{\partial u_g}{\partial x} + u_g\frac{\partial}{\partial p}\left(\frac{\partial u_g}{\partial x}\right) + \frac{\partial v_g}{\partial p}\frac{\partial u_g}{\partial y} + v_g\frac{\partial}{\partial p}\left(\frac{\partial u_g}{\partial y}\right) \\ &= \frac{D_g}{Dt}\left(\frac{\partial u_g}{\partial p}\right) + \frac{\partial u_g}{\partial p}\frac{\partial u_g}{\partial x} + \frac{\partial v_g}{\partial p}\frac{\partial u_g}{\partial y}\end{aligned} \tag{44a}$$

$$\frac{\partial}{\partial p}\left(\frac{\partial v_g}{\partial t}+\vec{\mathbf{v}}_g\cdot\nabla v_g\right)=\frac{\partial}{\partial p}\left(\frac{\partial v_g}{\partial t}\right)+\frac{\partial u_g}{\partial p}\frac{\partial v_g}{\partial x}+u_g\frac{\partial}{\partial p}\left(\frac{\partial v_g}{\partial x}\right)+\frac{\partial v_g}{\partial p}\frac{\partial v_g}{\partial y}+v_g\frac{\partial}{\partial p}\left(\frac{\partial v_g}{\partial y}\right)$$
$$=\frac{D_g}{Dt}\left(\frac{\partial v_g}{\partial p}\right)+\frac{\partial u_g}{\partial p}\frac{\partial v_g}{\partial x}+\frac{\partial v_g}{\partial p}\frac{\partial v_g}{\partial y} \tag{44b}$$

If we apply the thermal wind relationship (7) to re-write the second and third terms of (44a) and (44b), we obtain:

$$\frac{D_g}{Dt}\left(\frac{\partial u_g}{\partial p}\right)=\frac{R}{f_0 p}\left(\frac{\partial T}{\partial x}\frac{\partial u_g}{\partial y}-\frac{\partial T}{\partial y}\frac{\partial u_g}{\partial x}\right) \tag{45a}$$
$$\frac{D_g}{Dt}\left(\frac{\partial v_g}{\partial p}\right)=\frac{R}{f_0 p}\left(-\frac{\partial T}{\partial y}\frac{\partial v_g}{\partial x}+\frac{\partial T}{\partial x}\frac{\partial v_g}{\partial y}\right) \tag{45b}$$

If we apply (9) to the second term on the right-hand sides of (45a) and (45b), we obtain:

$$\frac{D_g}{Dt}\left(\frac{\partial u_g}{\partial p}\right)=\frac{R}{f_0 p}\left(\frac{\partial T}{\partial x}\frac{\partial u_g}{\partial y}+\frac{\partial T}{\partial y}\frac{\partial v_g}{\partial y}\right) \tag{46a}$$
$$\frac{D_g}{Dt}\left(\frac{\partial v_g}{\partial p}\right)=-\frac{R}{f_0 p}\left(\frac{\partial T}{\partial y}\frac{\partial v_g}{\partial x}+\frac{\partial T}{\partial x}\frac{\partial u_g}{\partial x}\right) \tag{46b}$$

Substituting from (12) and (19), the definitions of Q_1 and Q_2, we obtain:

$$\frac{D_g}{Dt}\left(f_0\frac{\partial u_g}{\partial p}\right)=-Q_2 \tag{47a}$$
$$\frac{D_g}{Dt}\left(f_0\frac{\partial v_g}{\partial p}\right)=Q_1 \tag{47b}$$

Compare (47) to (42). The left-hand sides of both equations are equivalent. However, while the right-hand sides of both equations are of equivalent magnitude, they are of opposite sign! This means that the forcing from the temperature gradient and vertical wind shear are not in balance.

If thermal wind balance is maintained, (42) and (47) do not have opposite signs. Instead, because a sign discrepancy exists, geostrophic forcing manifest through the **Q**-vector destroys thermal wind balance. The ageostrophic circulation and accompanying vertical motion manifest through the **Q**-vector, as described above, works to offset this sign discrepancy and restore thermal wind balance.