<u>Synoptic Meteorology II: Potential-Vorticity Inversion and Anomaly Structure</u>

Readings: Sections 4.2 and 4.4 of *Midlatitude Synoptic Meteorology*.

(<u>Note</u>: moving forward, references to "potential vorticity" mean isentropic potential vorticity.)

Potential-Vorticity Inversion

Introduction

One of the most important facets of potential vorticity is that if we know the three-dimensional potential-vorticity distribution, we can obtain information about the field that define it – namely, the kinematic (through the absolute vorticity) and thermodynamic (through the static stability) fields. We can see this solely by looking at the definition of the isentropic potential vorticity:

$$P = -g\eta \frac{\partial \theta}{\partial p} \tag{1}$$

The absolute vorticity depends on the horizontal wind. Under the assumption of hydrostatic balance, the vertical change in potential temperature is related to the geopotential height field (and thus the geostrophic wind) through the hypsometric equation. Thus, if we know the three-dimensional distribution of P, we should be able to extract out information about u, v, θ, Φ, and so on. This is what is known as the *invertibility principle*.

To help illustrate this point, let us consider something familiar to us, the relationship between the geostrophic relative vorticity and geopotential height, i.e.,

$$\zeta_g = \frac{1}{f_0}\nabla^2\Phi \tag{2}$$

We typically observe the geopotential height, from which we can compute the geostrophic relative vorticity using (2). However, we can also use the geostrophic relative vorticity to solve (2) for the geopotential height. Fundamentally, (2) is a second-order partial differential equation – specifically, a Poisson equation – that can be solved for the geopotential height. Thus, if we know the distribution of the distribution of the geostrophic relative-vorticity field, we can *quantitatively* obtain the geopotential height field.

Principles of Potential-Vorticity Inversion

The invertibility principle is a robust way for us to be able to identify the wind, geopotential height, and thermal fields associated with a given potential vorticity anomaly. While *qualitative* estimates for these fields can be obtained more simply than by using the invertibility principle, it nevertheless has great utility in obtaining *quantitative* measures of these fields for specific features.

The right side of (1) contains two unknowns, absolute vorticity (which can be directly written in terms of the horizontal wind) and static stability. Before we can use the potential-vorticity distribution to obtain the absolute vorticity or static stability, we must relate one to the other so that the right side of (1) contains only a single unknown. This requires invoking one or more balance relationships. For example, hydrostatic balance can be used to relate the potential-temperature distribution to the geopotential height, and geostrophic balance can be used to relate the geopotential height to the horizontal wind, such that the horizontal wind is the only unknown on the right side of (1). Once the horizontal wind is known, the balance condition or conditions can be reversed to obtain the other unknown quantities. More commonly, however, we use a more complex balance relationship (such as a non-linear balance, which is beyond the scope of this course) for this purpose.

In addition to one or more balance relationships, we also need *boundary conditions* to invert the potential vorticity. Why? Note the presence of partial derivatives with respect to pressure in the definitions of both the isentropic and Ertel potential vorticity. Thus, mathematically, to obtain the three-dimensional distribution of our fundamental variables (u, v, θ) from the three-dimensional potential-vorticity distribution, we must specify what those variables look like along the upper and lower boundaries of the three-dimensional volume.

Application of Potential-Vorticity Inversion Principles

We can let gradient-wind and hydrostatic balances serve as the balance relationships for inverting the isentropic potential vorticity. With some manipulation, applying these balance relationships permits us to develop a thermal wind-like relationship between the thermal and wind fields, from which we can then obtain an equation relating the horizontal wind speed V to the isentropic potential vorticity distribution. That equation is subsequently simplified by making approximations consistent with the study of *relatively weak*, *synoptic-scale* isentropic potential vorticity anomalies. The resulting equation, given appropriate boundary conditions for V, can be solved to obtain the wind speed from the isentropic potential vorticity distribution. The balance relationships can then be reversed to obtain the thermal fields from the wind speed, thereby completing the inversion process.

In the context of the invertibility principle, we can state that a given potential-vorticity anomaly *is associated with* its accompanying wind field. As the wind field is in balance with the thermal field due to the chosen balance relationships, we can state that a given potential-vorticity anomaly is also *associated with* its thermal structure. These associations are sometimes referred to as *inductions* – i.e., a potential-vorticity anomaly *induces* its wind and thermal fields – though the associative terminology arguably better reflects the relationship between these quantities.

Later in this lecture, we will examine the vertical extent of and physical constraints upon these associations. Before doing so, however, we first examine the basic structure of the wind and thermal fields associated with weak synoptic-scale upper-tropospheric potential vorticity and lower-tropospheric potential temperature anomalies.

The Structure of Upper-Tropospheric Potential-Vorticity Anomalies

To begin, let us return to the basic definition of the isentropic potential vorticity presented in (1). We previously defined *positive potential vorticity anomalies* as localized maxima of isentropic potential vorticity ($P > 0$) and *negative potential vorticity anomalies* as localized minima of isentropic potential vorticity ($P < 0$).

By examining (1), we can describe the basic structure of upper-tropospheric positive and negative potential-vorticity anomalies:

- A **positive potential-vorticity anomaly** is associated with locally large *cyclonic absolute vorticity* ($\eta > 0$) **and** *enhanced static stability* with tightly packed isentropes in the vertical ($-\partial\theta/\partial p >> 0$).

- A **negative potential-vorticity anomaly** is associated with locally large *anticyclonic absolute vorticity* ($\eta < 0$) **and** *reduced static stability* with weakly packed isentropes in the vertical ($-\partial\theta/\partial p > 0$).

If we apply appropriate balance relationships and boundary conditions, the specific distributions of absolute vorticity and static stability associated with a given potential-vorticity anomaly may be obtained, as described above. The following illustrations are derived using hydrostatic and gradient-wind balance as the balance relationships.

The structure of a positive potential-vorticity anomaly is depicted in Fig. 1. A positive potential-vorticity anomaly is associated with a lowered dynamic tropopause. Accompanying this is upper-tropospheric cyclonic rotation and large static stability. Isentropes are tightly packed in the vertical through the anomaly, resulting in locally high potential temperature above and locally low potential temperature below the anomaly. Cyclonic rotation and large static stability each weaken with increasing vertical distance away from the anomaly.

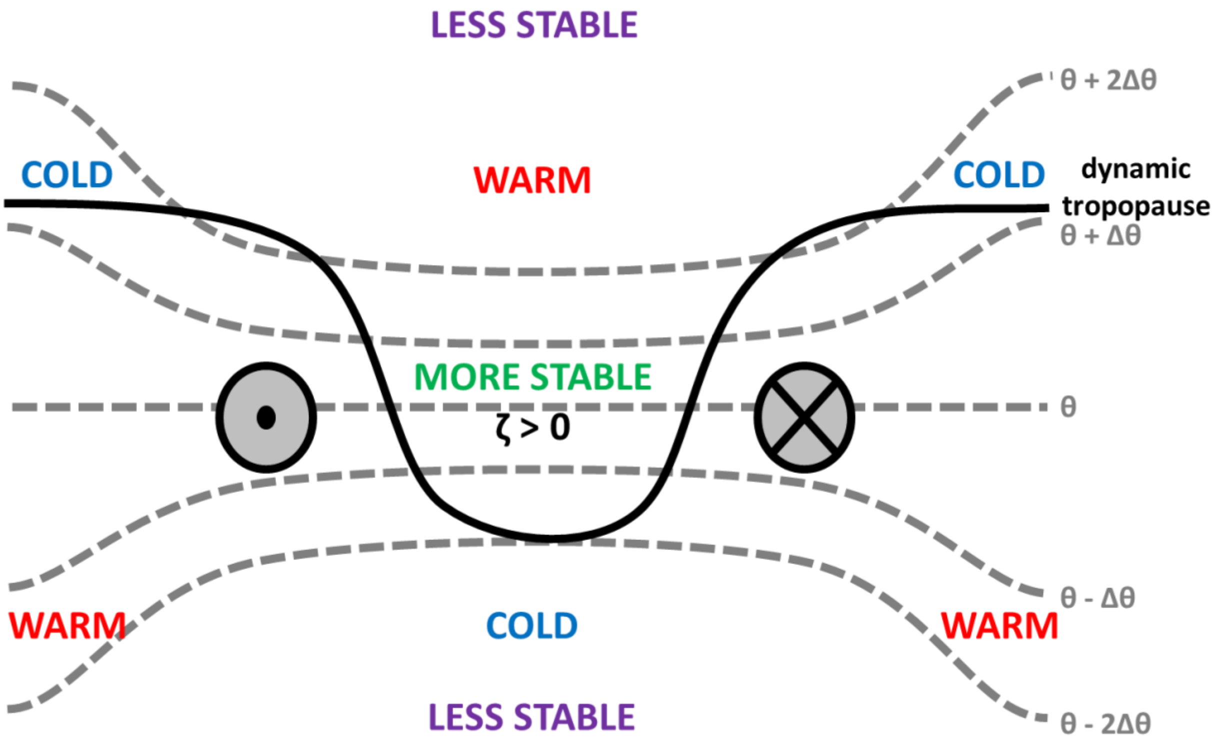

Figure 1. The structure of an idealized upper tropospheric positive potential vorticity anomaly. The dynamic tropopause is depicted by the solid black line while isentropes are depicted by the dashed grey lines. The circle with a dot in its center denotes flow out of the page while the circle with an X in its center denotes flow into the page.

The structure of a negative potential-vorticity anomaly is depicted in Fig. 2. A negative potential-vorticity anomaly is associated with an elevated dynamic tropopause. Accompanying this is upper-tropospheric anticyclonic rotation and weak static stability. Isentropes are loosely packed in the vertical through the anomaly, resulting in locally low potential temperature above and locally high potential temperature below the anomaly. Anticyclonic rotation and small static stability each weaken with increasing vertical distance away from the anomaly.

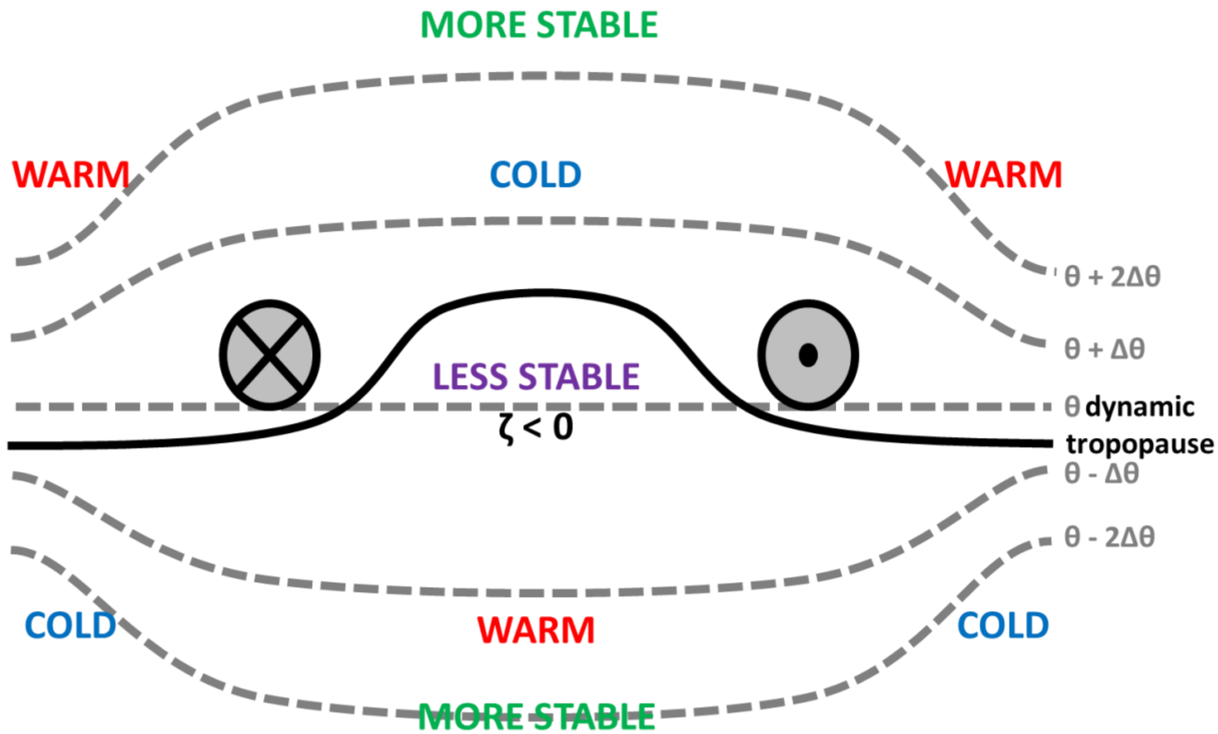

Figure 2. The structure of an idealized upper tropospheric negative potential vorticity anomaly. The dynamic tropopause is depicted by the solid black line while isentropes are depicted by the dashed grey lines. The circle with a dot in its center denotes flow out of the page while the circle with an X in its center denotes flow into the page.

The Structure of Surface Potential-Temperature Anomalies

To find the full three-dimensional structure of the wind and thermal fields associated with a given potential-vorticity anomaly, we need to know the three-dimensional potential-vorticity distribution. This only poses an issue at the upper and lower boundaries, where we need to specify solutions for the wind and thermal fields to complete our analysis. These prescribed solutions are known as *boundary conditions*.

These boundary conditions vary between the upper and lower boundaries. The upper boundary is typically placed in the stratosphere (e.g., p = 60 hPa), where synoptic-scale meteorological variability is negligible. The lower boundary is given by the surface, where we know that synoptic-scale meteorological variability is far from negligible. Here, the boundary condition on this lower boundary is given by the surface potential temperature. Thus, it stands to follow that surface potential-temperature anomalies may have similar structure to potential-vorticity anomalies found along the dynamic tropopause.

As before, the specific distributions of absolute vorticity and static stability with a given surface potential-temperature anomaly may be obtained by specifying appropriate balance relationships and boundary conditions. The illustrations below are again obtained using balance relationships of hydrostatic and gradient-wind balance.

First, let us introduce the concept of "thermal vorticity." In defining the thermal vorticity, we wish to recall the relationship between the geopotential height and geostrophic relative vorticity:

$$\zeta_g = \frac{1}{f_0}\nabla^2\Phi \tag{4}$$

If we take the partial derivative of (4) with respect to p and commute the order of the partial derivatives on the right side, we obtain:

$$\frac{\partial \zeta_g}{\partial p} = \frac{1}{f_0}\nabla^2\frac{\partial \Phi}{\partial p} \tag{5}$$

Substituting the ideal gas law and Poisson's equation, the hydrostatic equation may be written as:

$$\frac{\partial \Phi}{\partial p} = -h\theta \qquad \text{where } h = \frac{R}{p_0}\left(\frac{p_0}{p}\right)^{\frac{c_v}{c_p}} \tag{6}$$

Substituting (6) into (5) and rearranging, we obtain:

$$f_0\frac{\partial \zeta_g}{\partial p} = -h\nabla^2\theta \tag{7}$$

Equation (7) is known as the *thermal vorticity relationship*. Assuming geostrophic balance, the vertical derivative of the geostrophic relative vorticity ζ_g with respect to p is a function of the Laplacian of the potential temperature θ.

Since h is positive, the proportionality implied by the Laplacian on the right side of (7) means that $\partial\zeta_g/\partial p$ is a local maximum where θ is a local maximum and that $\partial\zeta_g/\partial p$ is a local minimum where θ is a local minimum. If we assume that ζ_g at some middle to upper tropospheric level away from the surface is zero, $\partial\zeta_g$ is negative for cyclonic geostrophic relative vorticity at the surface while $\partial\zeta_g$ is positive for anticyclonic geostrophic relative vorticity at the surface. Therefore, since $\partial p < 0$, θ is a local maximum where there is cyclonic geostrophic relative vorticity at the surface and θ is a local minimum where there is anticyclonic geostrophic relative vorticity at the surface.

We first examine the structure of a positive surface potential-temperature anomaly, as depicted in Fig. 3. As described above, a positive surface potential-temperature anomaly is associated with surface cyclonic rotation. Since potential temperature generally increases with height, a positive surface potential-temperature anomaly implies a downward bowing of near-surface isentropes. This leads to relatively large static stability at and just above the surface and relatively weak static stability at higher altitudes. The combination of cyclonic rotation and relatively strong static stability at the surface suggests that positive surface potential-temperature anomalies are conceptually equivalent to positive potential-vorticity anomalies.

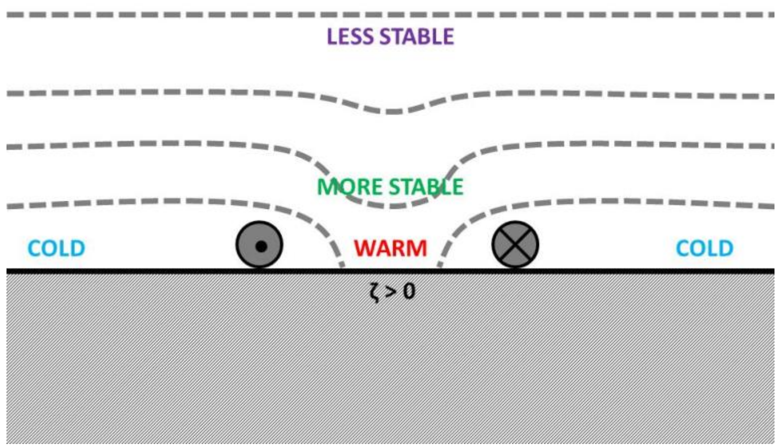

Figure 3. The structure of an idealized surface positive potential-temperature anomaly. The surface is depicted by the solid black line while isentropes are depicted by the dashed grey lines. The circle with a dot in its center denotes flow out of the page while the circle with an X in its center denotes flow into the page.

Next, we examine the structure of a negative surface potential-temperature anomaly, as depicted in Fig. 4. As described above, a negative surface potential-temperature anomaly is associated with anticyclonic surface rotation. Since potential temperature generally increases with height, a negative surface potential-temperature anomaly implies upward-lifted near-surface isentropes. This leads to relatively small static stability at and just above the surface and relatively strong static stability at higher altitudes. The combination of anticyclonic rotation and relatively weak static stability at the surface thus suggests that negative surface potential-temperature anomalies are conceptually equivalent to negative potential-vorticity anomalies.

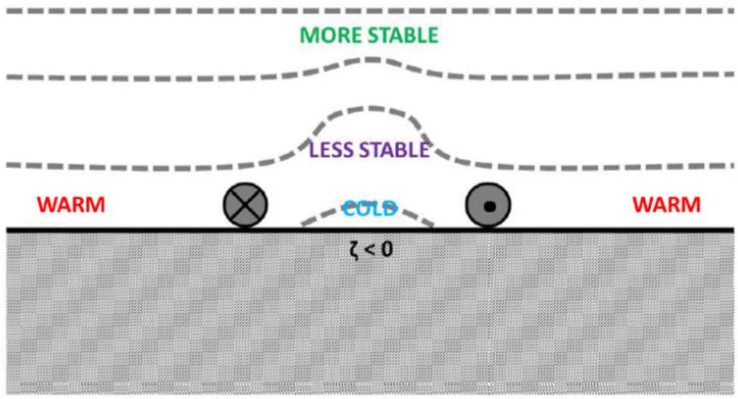

Figure 4. The structure of an idealized surface negative potential-temperature anomaly. The surface is depicted by the solid black line while isentropes are depicted by the dashed grey lines. The circle with a dot in its center denotes flow out of the page while the circle with an X in its center denotes flow into the page.

The Vertical Sphere of Influence of Potential-Vorticity Anomalies

The positive and negative potential-vorticity anomalies described above are synoptic-scale anomalies, with horizontal length scales on the order of 1,000 km or larger. However, it is worthwhile to also consider the vertical length scale of such anomalies; i.e., over how deep of a vertical layer are the wind and thermal fields associated with positive and negative potential-vorticity anomalies found?

It can be shown that the vertical depth $\Delta\theta$ over which the wind and thermal fields are induced by a given potential vorticity anomaly can be expressed as:

$$\Delta\theta = \frac{f_0 L}{\sqrt{\sigma^*_{ref}\, g K_{ref}}} \tag{8}$$

In (8), L is the horizontal length scale, $f_0 = 1 \times 10^{-4}$ s^{-1}, $g = 9.81$ m s^{-2}, σ^*_{ref} is an *inverse* measure of the reference-state static stability, and K_{ref} is related to the reference-state density.

The key take-home messages from (8) are:

- A larger-scale potential-vorticity anomaly (large L) will influence the wind and thermal fields over a greater vertical depth.

- Potential-vorticity anomalies embedded within environments of larger static stability (and thus small σ^*_{ref}) will influence the wind and thermal fields over a greater vertical depth.

Plugging in representative values for each of the terms of (8), we find that $\Delta\theta \approx$ 55-60 K. This is typically as large as the depth of the troposphere.

Thus, we state that a weak, synoptic-scale potential-vorticity anomaly is associated with unique wind and thermal fields over a deep vertical depth. An upper-tropospheric potential-vorticity anomaly can feasibly influence wind and thermal fields down to the surface, while a surface potential-temperature anomaly can influence wind and thermal fields up to the tropopause! This is colloquially known as the ***action at a distance principle***; i.e., a potential-vorticity anomaly is not confined to a single atmospheric level but is associated with unique or distinct wind and thermal anomalies over a large vertical depth! These anomalies are strongest at the anomaly's altitude and gradually decay above and below it, but do not return to their background values until a large vertical distance away from the anomaly's center.

We can also repeat the above scaling exercise to get an estimate of the relative *strength* of the wind (and thermal) fields associated with a potential-vorticity anomaly. Doing so, the following relationship is obtained:

$$U = \sigma^*_{ref} PL \tag{9}$$

In (9), P represents the magnitude of the potential-vorticity anomaly and U represents the intensity of the associated wind field (which, through the associated balance relationships, also represents the intensity of the associated thermal field). The key take-home messages from (9) are:

- A stronger potential-vorticity anomaly (either positive or negative; large magnitude to P) is associated with larger wind and thermal anomalies.

- A larger-scale potential-vorticity anomaly (large L) is associated with larger wind and thermal anomalies.

- Potential-vorticity anomalies embedded within environments of larger static stability (and thus small σ^*_{ref}) are associated with weaker wind and thermal anomalies.

- The strength of the wind and thermal fields induced by a given potential-vorticity anomaly are inversely proportional to the reference-state (or environmental) static stability.

- Thus, weak environmental static stability implies shallow yet strong induced wind and thermal fields, whereas stronger environmental static stability implies deep yet weak induced wind and thermal fields.

Finally, although the above discussion is explicitly valid only for upper-tropospheric potential-vorticity anomalies, the same insights hold for surface potential-temperature anomalies as well. Thus, the vertical scale of a surface potential-temperature anomaly is related to its horizontal scale and static stability, whereas the intensity of its associated wind and thermal anomalies is related to its horizontal scale, the magnitude of the surface potential-temperature anomaly, and static stability. As with upper-tropospheric potential-vorticity anomalies, synoptic-scale surface potential-temperature anomalies are associated with distinct or unique wind and thermal fields over a large vertical depth that spans nearly the entire troposphere!