"Whispering" waves and Bate's ridges in numerical experiments

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Abstract: The whispering-gallery effect can be simulated using wave front constructions on rays. Results show that sources placed close to the boundary of an elliptic domain will exhibit the whispering-gallery effect. It illustrates that the energy of wave fronts close to boundaries remains dense. The wave fronts form patterns of repeated ridges which are related to the experimental observations of sharp ridges by Bate. The energy of the wave fronts does not remain dense near the boundary for all convex domains. This fact is illustrated using the stadium billiard, where the Bate's ridges spread when leaving the curved area.
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1. Introduction

Robert E. Apfel wrote a charming little article called "'Whispering' waves in a wineglass" in 1985.[1] It is not only a delightful read with a humorous and refreshingly personal style, but it also is instructive and thought-provoking.

In this paper Apfel shows how widespread dinner-party entertainment contains still open physical questions. Who hasn't tried to make a wine glass ring, by dipping the finger into the liquid and rubbing the rim? Especially those of us who would be inclined to read academic acoustics journals?

Apfel uses this scenario to do three things. First to illustrate a phenomenon, the capillary waves close to the vibrating rim within the wine glass, experimentally. Secondly he describes a related phenomenon theoretically. This is the rather often-mentioned whispering-gallery modes connected with the work of Rayleigh.[2–4] He then points out an analogy between the observed phenomenon and the theoretical one.

Ultimately he poses this and a few more questions left to be answered. Is the similarity of the theoretical model of whispering-gallery waves and creeping capillary waves in a cylindrical container (a less inspired simplification for a wine glass) more than an observed analogy?

This article won't be the place to settle this question, but rather I would like to follow the spirit of Apfel's own paper and elaborate on the theory of whispering galleries, taking the opportunity to revisit some rather old published experiments on them and check how those experiments compare to virtual simulations.

2. Listening to whispering galleries

In current literature the theory of whispering galleries is invariably linked with Rayleigh who wrote about them repeatedly. In his classic text *Theory of Sound*[4] he gave a geometric explanation of the phenomenon. He later followed with articles filling in the wave part, using Bessel functions. This may not be so surprising as Rayleigh was one of the main movers in the establishment of the theory of Bessel functions in the 19th century.[5] In this light Rayleigh's comment in the paper introducing the Bessel solution make a lot of sense. He writes:[2] "I have often wished to illustrate the matter further on distinctively wave principles, but only recently have recognized that most of what I sought lay as it were under my nose." It was the Bessel function, which he knew well to be the solution of wave phenomena in the plane for cylindrical symmetries.

Rayleigh's argument of his 1910 paper goes like this.[2] Bessel functions of high order drop off exponentially approaching the origin, but are oscillatory (for high-frequency arguments) close to the first radial zero, which would be the boundary. This is analogous to the observation that whispers remain near the wall. The essence of the argument why one can hear the whispers in the gallery so clearly is the same in both cases. The energy stays confined to a region close to the wall instead of spreading with distance like the circumference of a circle, in the plane case, or the surface of a sphere in the three-dimensional world. But the connection between the geometric picture and the Bessel argument was not rigorous. Rather Rayleigh used the same device as Apfel: analogy.

The analogy to other physical situation, as put forward by Apfel, also has a long tradition. Rayleigh very early on already suggested that the whispering-gallery effect may exist for seismic phenomena and water waves[4] and Raman and Sutherland[6] in their experimental work gave the analogy to optics.

It is maybe not widely known that the topic wasn't really settled with Rayleigh's pair of papers[2,3] on the wave formulation of the whispering gallery. In fact a number of experimental works were performed later and questioned some of Rayleigh's propositions. Raman and Sutherland[6] performed experiments on the phenomenon, established the existence of the optical whispering galleries, and related new measurements to Rayleigh's model. But this was 2 years too late for Rayleigh to observe and comment on. A 1938 article by Bate[7] discusses the state of affairs and offers his own additional observations. Some of these are quite interesting and I hope to discuss them here.

3. What are Bate's ridges?

The advantage of the Bessel picture is the added information of the wave patterns along the circumference as well as more detailed prediction with respect to the radial distribution of the energy. Raman and Sutherland[6] looked at both of these questions with the experimental eye, showing the interplay between theory and experiment so nicely exhibited in Apfel's paper.[1] Rayleigh predicted that the first zero of the high-order Bessel function would coincide with the wall, hence giving one radial peak with a rapid drop-off of energy approaching the center of curvature. Raman and Sutherland found in a simple optical arrangement that this only held for small curvatures and that additional intensity ripples appeared with increased curvature. Also they found that here is a qualitative difference between the intensity of the wave along circumferential paths if the source is symmetric and directed.

Bate[7] followed with a further set of experiments. Using lycopodium powder and electric sparks he finds that "The powder is ranged into short ridges at right angles to the wall. [..] The distance between the ridges or striae diminishes with intensity."

This is the observation I would like to pick up on. Nowadays most of my experiments are in silico, using computer simulations to shed light on aspects of a phenomenon. To do this I would like to take a number of steps back to Rayleigh's original geometric way of thinking about the whispering modes as rays and add the simple additional structure of position of wave fronts under isotropic propagation. Hence we will take the rays and keep track where on a ray a disturbance is at a given time. This we will then use to visualize the situation.

This method is very simple, yet the geometric figures that arise are very complicated and hence may explain why hand-traced figures are hard to come by—I haven't found any yet. In the case of wave fronts in the plane on a domain with a circular boundary this has already been explained elsewhere.[8] Here I would like to discuss the aspects that are relevant to Bate's observation of the whispering-gallery effect.

It's been known for awhile that whispering rays exist also on elliptical galleries of which the circle is but the special case, where the two elliptic foci coincide.[9,10] The ellipse can be seen in the left part of Fig. 1. In Apfel's sense we can think of this as an elliptical vessel filled with a liquid—my choice would be orange juice—and observe the first front of low amplitude disturbances.

Rays of equal density shoot off that point and reflect off the walls according to the standard geometric rules of reflection. A wave front is drawn as the connection of all points of all
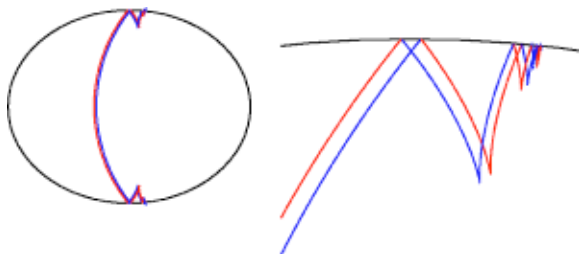
Fig. 1. A wave front started 1% from the right larger radius of the ellipse after it traveled to the shorter radius. Wave fronts that underwent an even number of reflections are shown in red. Those with an odd number of reflection are blue. Left: Full view. Right: Detailed view.

rays that have the same path length to the source having gone through any number of reflections, as required. The algorithm follows these steps:

Algorithm 1:
Step 1: Choose initial wave front point q_0.
Step 2: Calculate discrete set of rays by angle from this point.
Step 3: Calculate oriented intersection points p_0 of all rays with the domain boundary. Set ray order to 0.
Loop (Steps 4–5): Increase wave front distance l.
Step 4: Calculate the wave front points q_n of all rays from l.
Step 5: If q_n lies outside the domain Ω, increase ray order n, calculate intersection point p_n from p_{n-1} using the equal angle reflection law with the boundary.

Initial wave fronts are always assumed to be points. All simulations used 2000 rays with uniform initial angular density. For all figures for the elliptic domain the point source of the disturbance is placed on the longer elliptic axis 1% of the axis away from the rightmost point of the ellipse. The ratio of the long axis to the short axis is 1:0.8. For the stadium the point is given in terms of percentage of the radius of the circle away from the boundary. The ratio of the length of the straight segment to the diameter of the semicircles is 1.25:1.

Following this procedure and looking at the wave front when it has reached the shorter axis of the ellipse we get Fig. 1. The right part of the figure shows a zoom so we can better see the detail of the wave fronts. The first, red, curve is the circular wave front propagated out from the source without reflection (reflection order 0). It is closely chased by a blue front which has reflected off the wall. Of course the distance between them is 2% of the longer axis of the ellipse.

Observe that close to the wall the angle between these first two fronts and the wall is not a right one, contrary to Bate's description of his experiments. However, looking at successive ridges the angles of the wave front with the wall get closer and closer to a right angle. It could be that the observed right angles are in fact Mach stems, which arise from reflections of weak shocks in a flow, as an anonymous reviewer kindly pointed out to me.

We do, however, see Bate's description of the distance between the ridges. They do get closer as can be seen quite clearly in Fig. 1. He also links the distance to intensity.

Luckily some information about intensity is encoded geometrically. The $1/r$ law for plane waves (or the $1/r^2$ law in our three-dimensional world) is just the lengthening of the wave front with increased radius from a point source. We can visualize this by taking a finite subset of uniform rays emerging from the source. Looking at the distance between two rays supporting a piece of wave front, we get a measure of the intensity of the wave front in that area.

The zoomed part of Fig. 1 is now also depicted in the left part of Fig. 2. We observe that the density of rays is rather high for the late ridges very close to the wall and becomes more and more spaced out for ridges reaching further away from the wall. However, it is not clear how to reconcile this picture with Bate's description.
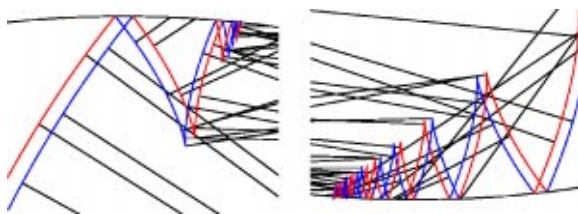
Fig. 2. The density of the rays supporting a wave front indicates the amount of amplitude loss through geometric widening of the wave front. The rays close to the wall remain dense after propagation whereas away from the wall they widen and hence the amplitude drops. Left: Quarter rotation along the ellipse. Right: Three-quarter rotation along the ellipse.

But what is the behavior of wave fronts when the propagation goes on? To study this, we let the wave travel on until the late ridges reach the bottom point close to the short axis of the ellipse. What we get is shown on the right side of Fig. 2. Notice that the densities of the late ridges remain high. This is of course not surprising and was already predicted just using rays by Rayleigh.[4] The reason is that rays with a small grazing (grazing is measured relative to boundary tangent) angle to the reflecting wall keep on reflecting between the wall and a smaller confocal ellipse. This confocal ellipse is closer to the boundary the smaller the angle.[9,10]

Imagine you get an oddly shaped glass container. Will you still see a whispering-gallery effect? This is a question already of concern to Rayleigh,[4] who conjectured very general curved surfaces will generate the effect. This problem is in a strict sense still open and is in a slightly different and more precise form known as the Birkhoff-Poritsky conjecture[11] which claims that only ellipses have this property. In special cases we know how the whispering-gallery effect breaks down.

A particularly easy example is the stadium-shaped domain.[9,12] Only very specific and few paths will be able to generate whispering modes. We can see what happens to a source close to the boundary in a stadium. In Fig. 3 the source was again placed 1% of the circular radius away from the rightmost boundary. When the wave front reaches the point where the circular arc becomes a straight line, the wave front looks very much like the one seen in Fig. 1. In Fig. 3 we see what happens after the wave front traversed the region with straight boundaries. The once small ridges close to the wall have significantly spread out and hence the intensity of those wave front regions has dropped much. Thus the whispering-gallery effect disappears if the glass vessel is stadium shaped. It disappears faster if the straight segment is longer.[9] This is true even though isolated orbits persist close to the wall[12] but neighboring orbits diverge rapidly and the energy in the region dissipates quickly. Hence orbit arguments may not be sufficient to estimate if energy remains dense close to gallery walls.
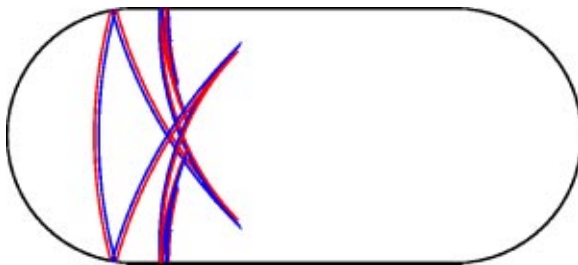

Fig. 3. In a stadium-shaped vessel, the initial whispering fronts after passing through a region without curvature become stretched.

4. Conclusions

This paper was written with the inspiration of Robert Apfel's whimsical article.[1] He advocates finding and seeking ways to illustrate the essence of real phenomenon. I have tried to follow this lead using simple virtual simulations to illustrate essential features of the whispering-gallery effect. Whispering-gallery effects are usually either studied in terms of closed orbits in the vicinity of the boundary using ray arguments or in terms of modes that are supported close to the boundary using spectral arguments. By adding the geometric picture of the wave front we can see aspects of the localized behavior of the waves near the boundary and get local estimates of the energy. We can also see when an isolated orbit is not enough to describe if energy remains confined close to the wall.

Acknowledgments

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