1. Suppose that $(\Omega, \mathcal{F}, \Pr)$ is a probability space. Prove that $\Pr(\emptyset) = 0$.

2. Suppose that $(\Omega, \mathcal{F}, \Pr)$ is a probability space. Prove that $\Pr$ is finitely additive.

3. Suppose that $(\Omega, \mathcal{F}, \Pr)$ is a probability space, and $A \in \mathcal{F}$. Prove that $\Pr(A^c) = 1 - \Pr(A)$.

4. Suppose that $(\Omega, \mathcal{F})$ is a measurable space, and $Q : \mathcal{F} \to [0, 1]$ satisfies

   - $Q(\Omega) = 1$;
   - Q is finitely additive;
   - For each sequence of events $\{A_k\}_{k=1}^{\infty}$ with $A_{n+1} \subset A_n$ for each positive integer n and

     $$\cap_{n=1}^{\infty} A_k = \emptyset$$

     we know $Q(A_n) \to 0$ as $n \to \infty$.

   Prove that Q is a probability measure.

5. Prove that all distribution functions are non-decreasing, are right-continuous with left-hand limits, converge to 1 at infinity and 0 at negative infinity.

6. Suppose that $\Omega = [0, 1]$, $\mathcal{F}$ is the set of all Lebesgue measurable subsets of Ω and $\Pr(F)$ is the integral of $f(x) = x$ with respect to Lebesgue measure over F if F does not contain $1/3$, and is the integral of $f(x) = x$ with respect to Lebesgue measure over F plus $1/2$ if F contains $1/3$. Let $X : \Omega \to (-\infty, \infty)$ defined by $X(\omega) = \omega^2$.

   Verify that X is a real valued random variable, find and graph its distribution function, and compute its mean, variance and characteristic function.

7. Let U be a random variable such that

   $$\Pr(\omega : U(\omega) \in (a, b]) = b - a$$

   for $0 \le a \le b \le 1$. Let $F : (-\infty, \infty) \to [0, 1]$ be a distribution function. Define $G : (0, 1) \to (-\infty, \infty)$ by $G(y) = \sup\{x : F(x) \le y\}$. Show that $G(U)$ is a random variable and that its distribution function is F. Illustrate this theorem with the distribution function in the preceding problem. This problem shows that each distribution function on the real line is the distribution function of a real valued random variable so long as the uniform distribution function is the distribution function of a real valued random variable.

8. (From Chung's book) Suppose that X is a real valued random variable whose distribution function F is continuous. What is the distribution function of $F(X)$? What happens if F is not continuous?

9. Suppose that X_n, $n = 0, 1, 2, \ldots$ is an infinite sequence of independent, identically distributed generalized random variables on $(\Omega, \mathcal{F}, \Pr)$. Let C be a measurable set in their range such that $\Pr(X_1 \in C) > 0$. Let

   $$\Omega' := \{\omega \in \Omega : X_n(\omega) \in C \text{ for some } n > 0\}.$$

   Define $T : \Omega \to [0, \infty)$ by

   $$T(\omega) = \begin{cases} 0 & \text{if} \quad \omega \in (\Omega')^c \\ \min\{n : X_n(\omega) \in C\} & \text{if} \quad \omega \in \Omega' \end{cases}$$

   Show that $\Omega' \in \mathcal{F}$, that $\Pr(\Omega') = 1$ and find a formula for $\Pr(\omega : T(\omega) = n)$ for each positive integer n. What does the value of T tell us?

10. (continuation) Define

    $$Y(\omega) = X_{T(\omega)}(\omega).$$

    Show that Y is a generalized random variable on $(\Omega, \mathcal{F})$, and show that for each measurable set F,

    $$\Pr(Y \in F) = \Pr(X_1 \in F | X_1 \in C).$$

11. (continuation) Use the preceding to show that if U_n, $n = -1, 0, 1, 2, \ldots$ are independent, identically distributed uniform random variables on $[-1, 1]$, and we put $X_n = (U_{2n-1}, U_{2n})$ for $n = 0, 1, 2, \ldots$ and $C = \{(x, y) \in R^2 : x^2 + y^2 \leq 1\}$, then Y is uniformly distributed on C.

    Note: The suite of problems 7 and 9, 10 and 11 form the backbone for a large part of the theory of simulation of sequences of psuedo-random numbers with specified distributions. 7 goes by the name of the inverse transform method, and 10 is called the rejection method. See S. Ross's book on simulation methods for more examples and details.

12. (From Chung) Suppose that $\mathrm{E}[X^2] = 1$ and $\mathrm{E}[|X|] \geq a > 0$. Show that $\Pr(|X| \geq ta) \geq (1-t)^2 a^2$ for $t \in [0, 1]$.

13. Suppose that $\Pr(X \geq 0) = 1$. Show that $\mathrm{E}[X] < \infty$ if and only if

$$\sum_{n=1}^{\infty} \Pr(X \geq n) < \infty.$$

14. We say that a random variable X has a Poisson distribution with rate a if for some $a > 0$,

$$\Pr(X = x) = \frac{a^x}{x!} \exp(-a)$$

    for each non-negative integer x, and

$$\Pr(X = x) = 0$$

    otherwise. Suppose now that $(X_k)_{k=1}^{\infty}$ is a sequence of mutually independent Poisson random variables, and X_k has rate a_k. Put $S_n = X_1 + \ldots + X_n$. Show that S_n is a Poisson random variable with rate $a_1 + \cdots + a_n$. Under what additional assumption on the a_k will

$$\Pr\left(\lim_{n\to\infty} \frac{S_n}{\mathrm{E}[S_n]} = 1\right) = 1?$$

15. (continuation) Under what condition on the a_k is there is a random variable X such that

$$\Pr(\lim_{n\to\infty} S_n = X) = 1?$$

    Determine the distribution of X.

16. Suppose that X and Y are independent random variables. Show that

$$\Pr(X > 0, X + Y > 0) + \Pr(Y > 0, X + Y > 0) = \Pr(X > 0)\Pr(Y > 0) + \Pr(X + Y > 0).$$

17. Show that if X and Y are independent normal random variables with zero mean then so is $X + Y$.

18. For each $c > 0$ we can define the density function f_c by

$$f_c(x) = \frac{c}{\pi(x^2 + c^2)}.$$

    Show that if f_c is the density of X then for $a > 0$, the density of aX is f_{ac}. Compute $f_c * f_d(t)$ for each positive real number t.

19. Suppose that $Q : (0, 1] \to [0, \infty)$ is a strictly decreasing continuous function with $Q(1) = 0$. Show that the following procedure recursively defines a probability mass function on the positive integers.

    - $f_1 \in (0, 1)$ is the unique solution of $Q(x) = x$.
    - Given $f_j \in (0, 1)$ and $f_1 + \cdots + f_d < 1$, f_{d+1} is the unique solution of $Q(f_1 + \cdots + f_d + x) = x$ for $x \in (0, 1 - f_1 - \cdots - f_d)$. Find the f_j if $Q(x) = 1 - x$ and if $Q(x) = (1 - x)^2/x$.

20. Suppose that the sequence of random variables $(X_n)_{n=1}^{\infty}$ satisfies

$$\Pr\left(X_n = \frac{j}{n}\right) = \frac{1}{n}$$

for $j \in \{1, \dots, n\}$. Put $F_n(t) = \Pr(X_n \leq t)$. Show that the sequence of distribution functions $(F_n)_{n=1}^{\infty}$ converges in distribution, and find a closed form expression for the limiting distribution.

21. Repeat the preceding problem with $\Pr\left(X_n = \frac{j}{n}\right) = \frac{2j}{n(n+1)}$.

22. Suppose that $(X_n)_{n=1}^{\infty}$ are iid with $\Pr(X_n = 0) = \Pr(X_n = 1) = 1/2$. Put

$$Y_N = \sum_{n=1}^{N} 2^{-n} X_n.$$

Determine if the sequence $(Y_N)_{N=1}^{\infty}$ converges in distribution and if it converges almost surely. If it converges in distribution, what is the limiting distribution?

23. Show that $\phi(t) = \cos(t)$ is a characteristic function. Show that

$$\prod_{k=1}^{n} \cos(t/2^n)$$

converges as $n \to \infty$ and find the limit.