A functional equation for sines and cosines

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We are all familiar with the functional equation

$$F(x+y) = F(x)F(y),$$

or its equivalent,

$$F(y) = F(y)F(x-y),$$

which under the additional condition that F be continuous at 0 implies that $F = A\exp(Bx)$.

Surprisingly there is a similar result for sines and cosines (circular or hyperbolic) and affine functions, based on the subtraction formula for sine functions. Roughly speaking, if S and C satisfy

$$S(u-v) = S(u)C(v) - S(v)C(u),$$

or its equivalent,

$$S(h) = S(x+h)C(x) - S(x)C(x+h) \tag{1}$$

and are well-behaved at 0, then S and C are sine and cosine functions or affine functions.

That we care about this might at first appear to be idle mathematical curiosity. However, in Abroell(1998), the problem of identifying which pairs of functions satisfy the more general equation (2) below arises in design problems in Statistics.

Theorem 1. *Suppose that U and V are two real valued functions of a single real variable. Suppose that each function is differentiable at 0, and $U'(0)V(0) - V'(0)U(0) \neq 0$. If*

$$U(x+h)V(x) - U(x)V(x+h) = U(h)V(0) - V(h)U(0) \tag{2}$$

for all x and h, then for some real number k

$$\begin{aligned} U''(x) &= kU(x) \\ V''(x) &= kV(x) \end{aligned}$$

for all x.

Proof. Put $D = U'(0)V(0) - V'(0)U(0)$, and

$$\begin{aligned} S(x) &= \frac{U(x)V(0) - U(0)V(x)}{D} \\ C(x) &= \frac{U\prime(0)V(x) - V'(0)U(x)}{D} \end{aligned}$$

It follows from (2) that S and C satisfy (1), so the Theorem follows from the following special case by observing that U and V are linear combinations of S and C. $\square$

Lemma 2. *Suppose that S and C are two real valued functions of a single real variable. Suppose that each function is differentiable at 0, $S(0) = C'(0) = 0$ and*

$S'(0) = C(0) = 1$. *If S and C satisfy (1) for all x and h, then for some real number k*

$$\begin{aligned} S''(x) &= kS(x) \\ C''(x) &= kC(x) \end{aligned}$$

for all x.

Proof. Replacing h with $-h$ and x with $x+h$ in (1) gives $S(-h) = S(x)C(x+h) - S(x+h)C(x) = -S(h)$, so we see that S is odd.

It now follows from (1) by setting $h = t+h$ and $x = -t$ that

$$\frac{S(t+h) - S(t)}{h} = C(-t)\frac{S(h) - S(0)}{h} + S(t)\frac{C(h) - C(0)}{h}.$$

This last expression converges to $C(-t)$ as h converges to 0, proving that S is differentiable with $S'(t) = C(-t)$. Since S is odd, S' is even, so C is even. Therefore, $S' = C$.

Next we show that either S is periodic or S has only one zero. If S has more than one zero, then S has a least positive zero since S is continuous and $S'(0) = 1$. Call this zero x_0. We will show that S has period $2x_0$. Since $S(-x_0) = 0$, S cannot have a period smaller than $2x_0$, else x_0 would not be its smallest positive zero.

If we set $x = x_0$ and $h = -x_0/2$ in (1) we obtain $-S(x_0/2) = S(x_0/2)C(x_0)$ so $C(x_0) = -1$. Now substituting x_0 for x in (1) shows that for any h,

$$S(h) = -S(x_0 + h). \tag{3}$$

Therefore $S(h + 2x_0) = -S(h + x_0) = S(h)$, establishing that $2x_0$ is the period of S.

In addition we see that (3) and the fact that $C = S'$ imply that

$$C(x) = -C(x + x_0). \tag{4}$$

for all x.

Now we can show that C is differentiable. Replace h with $2x$ and x with $-x$ in (1) to obtain

$$\begin{aligned} S(2x) &= S(x)C(-x) - S(-x)C(x) \\ &= 2S(x)C(x) \end{aligned}$$

Therefore C is differentiable at any value x where $S(x) \neq 0$. We assumed that C is differentiable at 0. If S has other zeroes, then S is periodic. Recall that x_0 denotes the smallest positive zero of S in this case. It then follows from (4) that C is differentiable at the zeroes of S.

Since $S' = C$ we now know that S is twice differentiable. We may rewrite (1) one last time as

$$S(h) = S(x+h)S'(x) - S(x)S'(x+h).$$

Differentiating with respect to x gives us

$$0 = S(x+h)S''(x) - S(x)S''(x+h).$$

Fix x_1 so that $S(x_1) \neq 0$. Then we have

$$S''(x_1 + h) = \frac{S''(x_1)}{S(x_1)}S(x_1 + h)$$

or, by renaming variables,

$$S''(x) = kS(x)$$

as claimed. This in turn implies that $C''(x) = kC(x)$. $\square$

References

1. Abroell, S. E. (1998) *Asymptotic Behaviour and Design of a Sieve Estimator for a Gaussian Mean Function.* PhD thesis, University of Wisconsin-Milwaukee.