Calculus Survival Facts: Functions

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Abstract

This is the bare minimum knowledge of functions you need to be successful in the first semester of calculus.

1 Relations and Functions

Let X and Y be non-empty sets.

1. $X \times Y = \{(x, y) : x \in X,\ y \in Y\}$, that is, the set of **all** ordered pairs whose first element is contained in X and whose second element is contained in Y.

2. A **relation** R is a non-empty subset of $X \times Y$. The **domain** of R is the set of all first coordinates of the elements of R and the **range** of R is the set of all second coordinates of elements of R.

3. A **function** F is a relation with the property that if $(c, d) \in F$ and $(a, b) \in F$ and $a = c$ then $b = d$. That is, if the first coordinates of two ordered pairs in F agree then they are the same ordered pair. This is the basis of the **vertical line test**.

4. If F is a function and $(a, b) \in F$ we will write $b = F(a)$. $F(a)$ is the **value of** F **at** a. Conversely, if $F(x)$ is an expression in x defined for all $x \in D$, then $\{(x, F(x)) : x \in D\}$ defines a function whose domain is D. The convention is to call this function F. For example, if $F(x) = 3x+2$ for $x \in [-1, 1]$, then F is the function $\{(x, 3x+2) : -1 leq x \leq 1\}$. We can visualize F as the line segment joining $(-1, -1)$ to $(1, 5)$.

5. If R is a relation, the **inverse relation** of R, denoted by R^{-1} is the set of ordered pairs $\{(y, x) : (x, y) \in R\}$. It is the set of ordered pairs you get by switching the places of each ordered pair in R.

6. A function is said to be **invertible** if its inverse is also a function. This is the basis for the **horizontal line test for invertible functions**.

7. Formula notation for functions. If $f(x)$ is an expression applicable to all $x \in X$, then $y = f(x)$ is shorthand for the function $f = \{(x, f(x)) : x \in X\}$. $f(x)$ is called the **rule** for f. $f^{-1} = \{(f(x), x) : x \in X\}$ is the inverse of f.

2 Exponential functions

. A function f is said to be an **exponential function** if for each x and y in its domain, $x + y$ is also in its domain and $f(x + y) = f(x)f(y)$. An example of an exponential function is the function whose domain is the non-negative rational numbers and whose rule is $f(x) = 2^x$. Note that if f is exponential then $f(0) = f(0 + 0) = f(0)f(0)$ so either $f(0) = 1$ or $f(0) = 0$. If $f(0) = 0$ then $f(x) = 0$ for all x in the domain of f since $f(x) = f(x + 0) = f(x)f(0) = 0$. Hence not every exponential function can be written in the form $f(x) = a^x$ since we have the convention that $a^0 = 1$ for every base a.

Theorem 1 *Suppose that the domain of f is all real numbers and*

- $f(r + s) = f(r)f(s)$ *for all* $r \geq 0$ *and* $s \geq 0$*;*
- $f(r)f(-r) = 1$ *for all real numbers* r*.*

Then f is an exponential function.

Reason: The fact that $f(r)f(-r) = 1$ for any real number r means that $f(r) \neq 0$ for any r. There are 3 cases to consider.

- $r \leq 0$ and $s \leq 0$. Then

  $$f(-(r+s)) = f((-r)+(-s)) = f(-r)f(-s).$$

  Now multiply the extremes of this equation by $f(r+s)f(r)f(s)$.

- $r \geq 0$, $s \leq 0$ and $r+s \geq 0$. Then

  $$f(r+s)f(-s) = f((r+s)+(-s)) = f(r).$$

  Multiply the extremes of this equation by $f(s)$.

- $r \geq 0$, $s \leq 0$ and $r+s \leq 0$. Then

  $$f(-(r+s))f(r) = f(-(r+s)+r) = f(-s).$$

  Multiply the extremes of this equation by $f(r+s)f(s)$.

3 Logarithmic functions

A function f is said to be a **logarithmic function** if for each x and y in its domain xy is also in its domain and $f(xy) = f(x) + f(y)$. The inverse of each exponential function is a logarithmic function. Note that if a function f is logarithmic then $f(1)$ must equal 0 since $f(1) = f(1 \times 1) = f(1) + f(1)$. If f is logarithmic and 0 is in the domain of f then $f(x) = 0$ for all x in the domain of f, since $f(0) = f(0 \times x) = f(0) + f(x)$.