Rational Function Inequalities (C)

A rational function inequality is one of the form

$$R_1(x) \leq R_2(x)$$

or

$$R_1(x) < R_2(x),$$

where $R_1(x)$ and $R_2(x)$ are rational functions in x. For example

$$\frac{x^2 + 2x - 1}{x^3 + 1} \leq \frac{2x + 1}{3x^2 - 12}$$

Since the difference of two rational functions is a rational function, any rational function inequality can be reduced to the form

$$\frac{P(x)}{Q(x)} \leq 0$$

or

$$\frac{P(x)}{Q(x)} > 0$$

Since a rational function will equal 0 only when the numerator is 0 and the denominator is not 0, we will concentrate on strict inequalities. The most effective method is to find a simpler rational function which is positive for exactly the same values of x as the orignal rational function. The way to do this is to eliminate any factors which are always positive. First, factor the numerator and denominator into linear and irreducible factors. Eliminate all values of x for which the rational function is undefined or equal to . The irreducible quadratic factors do not change sign, and will be positive if their lead coefficients are positive and negative if their lead coefficients are negative. Replace each positive irreducible factor by 1 and each negative quadratic factor by -1. Next, if any linear factors appear to an odd power, change the power to one, as the even powers you are ignoring are always positive. Replace all linear factors to even powers by 1. On a number line mark each of the zeros of the linear factors of the new expression, as well as all the zeros and poles of the original rational function. In each of these intervals, count the number of factors which are negative. If this number is odd, the rational function is negative in the interval, and if it is even, the rational function is positive in the interval.

Example

$$0 < R(x) = \frac{(x-1)(x-2)^2(x-3)^3(x^2+4)}{(x+1)(x+2)(-x^2+4x-5)}$$

The zeros are {1, 2, 3} and the poles are {-1, -2}. $x^2 + 4$ is an irreducible quadratic which is always positive, and $-x^2 + 4x - 5 = -((x-2)^2+1)$ is an irreducible quadratic which is always negative. Thus we look at

$$0 < Q(x) \frac{(x-1)(1)(x-3)(1)}{(x+1)(x+2)(-1)} = \frac{(x-1)(x-3)}{(x+1)(x+2)(-1)}$$

Note that there are now 5 factors, as the (-1) counts as a factor, and the intervals under consideration are $(-\infty, -2)$, (-2,-1), (-1,1), (1,2), (2,3) and $(3, \infty)$. On $(-\infty, -2)$ all five of the factors of $Q(x)$ are negative so $Q(x)$ and $R(x)$ are negative on $(-\infty, -3)$. On (-2,-1) four of the factors are negative, while $(x+2)$ is positive, so $Q(x)$ and $R(x)$ are positive, and so on.