Perfect Squares and Completing the Square (C)

A quadratic expression of the form $Az^2 + Bz + C$, where $A \neq 0$, B and C are complex numbers can often be analysed by writing is in the form

$$Az^2 + Bz + C = A((z+a)^2 \pm b^2$$

Doing so is called **completing the square**. Since we may always factor out A, we shall illustrate this in cases where $A = 1$. The idea is based on the obsevation that

$$(z+a)^2 = z^2 + 2az + a^2.$$

Notice that a is half the coefficient of z. We shall stick to examples where the coefficients are real numbers, as these occur most often, but the method is quite general.

$$z^2 + 4z + 5 = (z^2 + 2 \cdot 2z + 2^2) + 5 - 2^2 = (z+2)^2 + 1^2\,.$$

$$2z^2 + 12z + 4 = 2(z^2 + 2 \cdot 3z + 2) = 2\big((z^2 + 2 \cdot 3z + 9) + (2-9)\big) = 2\left((z+3)^2 - \left(\sqrt{7}\right)^2\right).$$

From this we see that the least value of $2z^2 + 12z + 4$ as z ranges over the real numbers occurs when $z = -3$, and this least value is -14.

Sometimes there is more that one variable, and we can complete the square variable by variable:

$$\begin{aligned} 5x^2 + 4xy + y^2 + 6x + 2y + 2 &= y^2 + (4x+2)y + 5x^2 + 6x + 2 \\ &= (y^2 + 2(2x+1)y + (2x+1)^2) \\ &\quad +5x^2 + 6x + 2 - (2x+1)^2 \\ &= (y + 2x + 1)^2 + x^2 + 2x + 1 \\ &= (y + 2x + 1)^2 + (x+1)^2 \end{aligned}$$

so we can see that the expression $5x^2 + 4xy + y^2 + 6x + 2y + 2$ is never negative, and in fact is exactly when $x + 1 = 0$ and $y + 2x + 1 = 0$, that is when $x=-1$ and $y = 1$.

Exercises

1.

In each case write as a sum or difference of squares. If there is more than one variable there will be more than one answer.

(a)

$x^2 + 12x + 2;$

(b)
$4x^2 - 4x + 11$;
(c)
$-x^2 + 9x + 2$;
(d)
$e^{2x} + 4e^x + 9$; (hint: treat e^x as the variable.)
(e)
$x^2 + 4*x + 4y^2 + 16y + 9$;
(f)
$4x^2 + 12y^2 + 6x + 2y - 5$;
(g)
$x^2 + 6xy + 2y^2 + 10x + 3y + 5$;
(h)
$x^2 + 6xy - 2y^2 + 10x + 3y + 5$;

2.
Find the minimum value of each of the following expressions as the variables range over all real numbers.
(a)
$x^2 + 10x + 2$;
(b)
$x^2 - 7x - 3$;
(c)
$x^2 + 4xy - 2y^2 + 3x + 2y + 3$;
(d)
$x^2 + 2y^2 + 3z^2 + 2xy + 2xz + 4yz + 2z + 10$.

3.
Write as a perfect square:
(a)
$$\frac{1}{x^6} + x^6 + 2$$
;
(b)
$$\left(\frac{1}{4x^6} - x^6\right)^2 + 1$$
;
(c)
$$\left(\frac{1}{12x^3} - 3x^3\right)^2 + 1$$
;

4.

Find an equivalent expression which does not involve a square root:

(a)

$$\sqrt{\left(x^{-2} - \frac{x^2}{4}\right)^2 + 1}$$

(b)

$$\sqrt{\left(2x^{-3} - \frac{x^3}{8}\right)^2 + 1}$$