Logarithms

Given a positive base b and a positive real number x we can ask for the exponent r such at $b^r = x$. In this context, the exponent r is called the **logarithm of x in the base b**, and is denoted by

$$\log_b(x) = r$$

For example,

$$
\begin{aligned}
\log_{10}(1000) &= 3 \\
\log_2 16 &= 4 \\
\log_{1/3}(9) &= -2 \\
\log_{16} 2 &= \tfrac{1}{4}.
\end{aligned}
$$

It is easy to verify that for any base b and any rational number r that $\log_b(b^r) = r$.

$$\log_b(1) = 0$$

$$\log_b(b) = 1$$

More generally, if $x > 0$ and $y > 0$ have logarithms in the base b then so does xy and for any rational number r,

$$\log_b(xy) = \log_b(x) + \log_b(y)$$

$$\log_b(x^r) = r\log_b(x)$$

It is shown in calculus how to define b^x for any real number x, how to compute such numbers to any desired accuracy, and that the addition property of exponents extends to all exponents. This establishes that all positive real numbers have logarithms with respect to each and every base. Since

$$a^{\log_a(x)} = x = b^{\log_b(x)}$$

for $a > 0$, $b > 0$, and $x > 0$ we have the base conversion formula

$$\log_b(x) = \frac{\log_a(x)}{\log_a(b)}.$$

Thus logarithms need only be tabulated in one base. The four most important bases are 2 (binary), 10 (decimal), 16 (hexadecimal) and e, the so called natural base. Although the number e is irrational ($e \approx 2.71828459045$), it is the base where it is simplest to obtain accurate approximation of the logarithm of any number. $\log_e$ is usually denoted by $\ln$, which is read as **natural log**. For conversion purposes is it useful to know that

$$
\begin{aligned}
\ln(10) &\approx 2.30259 \\
\ln(2) &\approx 0.69315 \\
\ln(16) &= 4\ln 2 \\
\ln(16) &\approx 2.772589
\end{aligned}
$$

Exercises

1.

    Find the indicated logarithm.

    (a)

    $\log_{10}(0.0001)$

    (b)

    $\log_{10}(1000)$

    (c)

    $\log_{16}(64)$

    (d)

    $\log_9(243)$

    (e)

    $\log_{1/9}(243)$

2.

    Write each expression as $x\log_b(a)$. There may be more than one way to do so. Then use a calculator to give a decimal approximation for each of your answers.

    (a)

    $\log_8(9) + \log_8(1/3)$

    (b)

$2\log_2(11) + \log_2(1/3)$

(c)

$3\log_3(2) - 4\log_3(7)$

(d)

$\log_3(7) + \log_9(7)$

(e)

$\log_3(11) + 2\log_4(2)$

.

3.

Write each of the following as a base 2 logarithm, and then find a decimal approximation with a calculator.

(a)

$\log_8(9) + \log_8(1/3)$

(b)

$\log_{16}(27) - \log_{1/2}(12)$

(c)

$\log_{32}(9) + 3\log_{1/4}(13)$