Hidden Quadratic Equations (C)

Sometimes quadratic equations come in disguise like

$$u^4 - 3u^2 + 4 = 0 \; or \; \frac{u+2}{u^2+1} = 4$$

or even

$$\frac{e^x - e^{-x}}{2} = 4$$

(2)

which can be seen to be a quadratic equation in e^x by multiplying through by $2e^x$.

Hidden quadratic equations such as (2) are most easily solved by substitution. For example, in (2) x only appears as e^x ($e^{-x} = 1/e^x$), we let $w = e^x$ to get

$$\frac{w - (1/w)}{2} = 4$$

(3)

Multiplying (3) through by 2w yields $w^2 - 1 = 8w$ or $w^2 - 8w - 1 = 0$, with the side condition that $w > 0$. Ignoring the side condition for the moment and applying the quadratic formula gives

$$w = \frac{8 \pm \sqrt{64+4}}{2} = \frac{8 \pm \sqrt{68}}{2} = 4 \pm \sqrt{17}$$

Since $w > 0$ we have $e^x = w = 4 + \sqrt{17}$, so $x = \ln(4 + \sqrt{17})$.

Exercises In each case, solve for x. Remember, x could be a complex number.

1.

$x^4 + 2x^2 + 1 = 0$;

2.

$$\frac{x^2 + 9x + 1}{x^2 + 5x + 3} = \frac{1}{3}$$

3.

$$\frac{x^2 + 9x + 1}{5x + 3} = 5x - 3$$

4.

$e^x + e^{-x} = 8$.

5.

Find the length of the equal sides of an isosceles triangle if the area of the triangle is 5 and the remaining side has length 4.