Geometric Progessions (C)

If r is a complex number and N is a positive integer, then the sum

$$\sum_{k=0}^{N-1} r^k \equiv 1 + r + \cdots + r^{N-1}$$

is called a **geometric progression** with N terms. It is easy to verify by long division that if $r \neq 1$ then

$$\sum_{k=0}^{N-1} r^k = \frac{r^N - 1}{r - 1} = \frac{1 - r^N}{1 - r}.$$

Of course, if $r = 1$ we have

$$\sum_{k=0}^{N-1} r^k = N.$$

Geometric progressions often arise as the soution of factoring problems involving differences of powers:

$$\frac{x^4 - 16}{x - 2} = 8\frac{\left(\frac{x}{2}\right)^4 - 1}{\frac{x}{2} - 1}$$

and by letting $r = x/2$ we see that

$$\begin{aligned}
\frac{x^4 - 16}{x - 2} &= 8\frac{r^4 - 1}{r - 1} \\
&= 8(r^3 + r^2 + r + 1) \\
&= 8\left(\frac{x^3}{8} + \frac{x^2}{4} + \frac{x}{2} + 1\right) \\
&= x^3 + 2x^2 + 4x + 8
\end{aligned}$$

The general formula for N an positive integer and z and a distinct complex numbers is that

$$\frac{z^N - a^N}{z - a} = \sum_{k=0}^{N-1} z^k a^{N-1-k},$$

which is easily verified by the same technique as in the example. Well-known special cases are

$$\frac{z^2 - a^2}{z - a} = z + a \quad \text{or} \quad z^2 - a^2 = (z - a)(z + a);$$

$$\frac{z^3 - a^3}{z - a} = z^2 + az + a^2 \quad \text{or} \quad z^3 - a^3 = (z - a)(z^2 + az + z^2);$$

and

$$\frac{z^3 + a^3}{z + a} = \frac{z^3 - (-a)^3}{z - (-a)} = z^2 - az + a^2 \quad \text{or} \quad z^3 + a^3 = (z + a)(z^2 - az + z^2).$$

Exercises

1. Find $\sum_{k=0}^{10} 2^k$
2. Find $\sum_{k=0}^{11} \left(\frac{1}{2}\right)^k$
3. Find $\sum_{k=0}^{10} 2^{-k}$
4. Find $\sum_{k=3}^{11} \left(\frac{-1}{2}\right)^k$
5. If x^6 - 64 is divided by x-2, what is the result?
6. If x - 2 is divided by $\sqrt{x} - \sqrt[3]{2}$, what is the result?