General Equation of a Line (C)

Every line in the plane is described by an equation of the form

$$Ax + By = C$$

where at least one of A and B is not zero. Here we are assuming x denotes the first coordinate of a point P in the plane, and y denotes the second coordinate of this point.

Each line has many such equations. In fact, if

$$Ax + By = C$$

is an equation of a line, so is

$$ADx + BDy = CD$$

for any non-zero real number D. To effect a standardization, one can require that A and B are always chosen so that

$$A^2 + B^2 = 1$$

and that A is not negative. If A is zero, make B equal to 1. If we make this choice, then there is a geometric interpretation which makes it easy to see why all lines can be described by such equations. Suppose that $P = (x_0, y_0)$ is a point on the given line. Draw a circle of radius 1 about about P, and then draw the diameter of this circle which is perpendicular to the given line. Then one of the endpoints of this diameter has coordinates $(A + x_0, B + y_0)$ where A and B meet the requirements outlined above. From the Pythagorean Theorem, if (x, y) is another point on the line we must have

$$\left(\sqrt{A^2 + B^2}\right)^2 + \left(\sqrt{(x - x_0)^2 + (y - y_0)^2}\right)^2 = \left(\sqrt{(A + x_0 - x)^2 + (B + y_0 - y)^2}\right)^2$$

which simplifies to

$$0 = A(x-x_0) + B(y-y_0)$$

or

$$Ax + By = Ax_0 + By_0$$

so. $C = Ax_0 + By_0$.

To find a form of the general equation of a line given two points we proceed as in the following example:

Given: The line passes through (8, 4) and (3,11).

Find: A, B, and C so that $Ax + By = C$ that describes this line.

Solution: We must have

$$8A + 4B = C = 3A + 11B$$

or

$$5A = 7B$$

We also want $A^2 + B^2 = 1$, or $49A^2 + 49B^2 = 49$. Since $25A^2 = 49B^2$ we have $49A^2 + 25A^2 = 49$ or $74A^2 = 49$.

From here we can find A, then B, and finally C, since $C = 8A + 4B$.