Exponents

If b is a complex number and $n > 1$ is an integer, the symbol

$$b^n$$

represents the product of n factors of b. Thus

$$\begin{aligned} 3^3 &= 27 \\ (1/2)^4 &= 1/16 \\ (-3)^5 &= -243 \end{aligned}$$

The number b is called the **base** of b^n and the number n is called the **exponent** of the expression, or the **power of** b, or the **logarithm of** b^n **in the base** b if $b > 0$.

We if $b \neq 0$ we **DEFINE**

$$b^0 = 1$$

as it would be nonsensical to speak of zero factors, and for any complex number, we **DEFINE**

$$b^1 = b$$

since it would be nonsensical to speak of the product of a single factor.

If n is a negative integer and $b \neq 0$ is a complex number we **DEFINE**

$$b^n = \frac{1}{b^{-n}} = \frac{1}{b^{|n|}}$$

since it would make no sense to speak of a product of a negative number of factors. For example,

$$(1+i)^{-1} = \frac{1}{1+i} = \frac{1}{2} - \frac{1}{2}i.$$

It is straightforward to check that if $a \neq 0$, $b \neq 0$ and m and n are any integers then

$$a^n b^n = (ab)^n,$$

$$b^n \times b^m = b^{m+n}$$

$$\frac{b^n}{b^m} = b^{n-m},$$

and

$$(b^n)^m = b^{nm}.$$

For example,

$$(2+i)^3(2-i)^3 = 5^3,$$

$$(9+7i)^3 \times (9+7i)^{-4} = (9+7i)^{-1} = \frac{1}{9+7i} = \frac{9}{130} - \frac{7}{130}i,$$

$$\frac{10^5}{10^{-2}} = 10^7$$

and

$$(2^3)^2 = 64 = 2^6.$$

We now restrict our attention to positive bases. If d is a positive integer and $b > 0$ we define

$$b^{1/d}$$

to be the **POSITIVE** solution to the equation $x^d = b$. For example,

$$8^{1/3} = 2$$

since $2^3 = 8$ and $2 > 0$.

Finally if n and d are integers and $d > 0$ then we define

$$b^{n/d} = (b^{1/d})^n.$$

We have, then, for all $b > 0$ and all rational numbers r defined the symbol b^r. It can be checked in a straightforward matter that

$$a^r b^r = (ab)^r,$$

$$b^r \times b^s = b^{r+s}$$

$$\frac{b^r}{b^s} = b^{r-s}$$

and

$$(b^r)^s = b^{rs},$$

for any pair of rational numbers r and s, and any $a > 0$ and $b > 0$.

It should be noted that for some rational numbers r it is possible to extend the definition of b^r to negative bases. We shall not consider that here, and in the case of reciprocals of positive integers, that is $1/d$, we suggest the notation $\sqrt[d]{b}$ to refer to solutions to $x^d = b$ when $b \leq 0$ provided such solutions exist.

Exercises

1.

Write each expression in the form b^r:

(a) $6^{1/3}6^{3/5}$

(b) $(1/3)^6(3/2)^6$

(c) $6^{1/3}6^{-3/5}$

(d) $\left(6^{1/3}\right)^{-3/5}$

(e) $\frac{7^{-3/5}}{7^{-4/9}}$

(f) $3^{1/3}5^{1/3}$

2.

Combine exponents where possible.

(a) $(a^2b^3)^5$.

(b) $(a^2b^{-1})^3(a^2 + b^3)$.

(c)

$$\frac{(x^3y^2z^{-3})^2}{(xy^2z^3)^2}$$
.

(d)

$$\frac{(x^{1/3}y^2z^{-3})^2}{(xy^{-1/2}z^3)^2}$$
.

(e)

$$\frac{(x^{-1}y^2z^{-1/3})^5}{(x^{-1}y^2z^3)^{1/7}}$$
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