## Compound Interest (C)

Suppose that the amount $P$ is invested for one period at the rate of $r \%$ for the period. At the end of the period an amount $r P$ has been earned, giving the total amount

$$
P+r P=(1+r) P
$$

on account. If this amount is reinvested under the same terms for a second period, the amount $r(1+r) P$ will be earned, giving the total amount

$$
P(1+r)+r(1+r) P=(1+r)^{2} P
$$

on account. In fact, if the terms are offered over $N$ consecutive periods, the holdings at the end of the $N^{\text {th }}$ period are

$$
P(1+r)^{N} .
$$

It is typical to give the rate $R \%$ on an annualized basis, even if the interest is credited more frequently. If $R \%$ is the nominal annual interest rate, and interest is credited $k$ times during the year, then the effect is to have a rate $r \%=R / k \%$, and to consider the year as $k$ consecutive periods. In this case, the value of an initial deposit of $P$ by the end of the year is

$$
P\left(1+\frac{R}{k}\right)^{k}
$$

and at the end of $x$ years is

$$
P\left(1+\frac{R}{k}\right)^{k x}=P\left[\left(1+\frac{R}{k}\right)^{k}\right]^{x} .
$$

It is, therefore, important to study the behavior of

$$
\left(1+\frac{R}{k}\right)^{k}
$$

for larger and larger values of $k$. It will be shown below by using the Binomial Formula that

$$
\left(1+\frac{R}{k}\right)^{k} \approx 1+R+\frac{R^{2}}{2}+\frac{R^{3}}{6}+\cdots+\frac{R^{k}}{k(k-1)(k-2) \ldots(3)(2)(1)} .
$$

This approximation improves dramatically as the ratio $R / k$ decreases to 0 . In calculus one shows that as $k$ increases the quantity

$$
\left(1+\frac{R}{k}\right)^{k}
$$

approaches

$$
e^{R}
$$

## Exercises

1. 

How much does $\$ 100$ grow to in 2 years if it is invested at nominal ${ }^{5 \%}$ per year and interest is awarded quarterly? Compare with the amount if interest is awarded daily. What rate would have to be offered if interest were only awarded yearly to give the same final amounts in each case?
2.

Write out

$$
\left(1+\frac{R}{k}\right)^{k}
$$

and

$$
1+R+\frac{R^{2}}{2}+\frac{R^{3}}{6}+\cdots+\frac{R^{k}}{k(k-1)(k-2) \ldots(3)(2)(1)},
$$

when $k=5$ and graph the resulting expressions in $R$, along with $e^{R}$.

