Binomial Formula (C)

We will have on many occasions the need to expand expressions of the form

$$(a + b)^N$$

where N is a positive integer. There is a simple formula for this, known as the binomial formula, which says that

Theorem 2864 (Binomial Formula)

If N is a positive integer and a and b are complex numbers then

$$(a + b)^N = \sum_{k=0}^{N} \binom{N}{k} a^k b^{N-k}$$

where

$$\binom{N}{k} = \frac{N!}{k!(N-k)!}.$$

It is not surprising that the terms in $(a + b)^N$ involve powers of a and b. The remarkable thing is the coefficients. The theorem becomes transparent when one realizes that $\binom{N}{k}$ counts the number of ways of choosing k things out of N things. Thus, to obtain $a^k b^{N-k}$ one had to choose k a's out of the N factors in $(a+b)^N$, and there are $\binom{N}{k}$ ways to do this.

Here is an example:

$$(a + b)^4 = \binom{4}{0} a^0 b^4 + \binom{4}{1} a^1 b^3 + \binom{4}{2} a^2 b^2 + \binom{4}{3} a^3 b^1 + \binom{4}{4} a^4 b^0$$

so

$$(a+b)^4 = b^4 + 4ab^3 + 6a^2b^2 + 4a^3b + a^4.$$

Remark: The binomial coefficients

$$\binom{p}{k}$$

can be given a meaning even if *p* is not a positive integer, so long as *k* is positive integer.

We put

$$\binom{p}{0} = 1$$

and

$$\binom{p}{1} = p$$

For $k \geq 2$

$$\binom{p}{k} = \frac{p(p-1)\cdots(p-(k-1))}{k!}.$$

This conforms to the case where *p* is a positive integer since

$$p! = p(p-1)\cdots(p-(k-1))(p-k)!$$

For example

$$\binom{\frac{1}{3}}{2} = \frac{\frac{1}{3}\,\frac{-2}{3}}{2!}$$